Forward

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• projects and research made by students through courses and research done with faculty; and
• articles and essays written about research and fields of engineering mathematics which interest CU students.

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We are pleased to present the first official issue of CJAM, which consists of articles submitted by undergraduate students throughout the 2016-2017 academic year. This issue introduces readers to some undergraduate research projects at CU Boulder. We hope you enjoy reading this issue as much as we enjoyed making it! - CJAM

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NCAA Basketball Team Rankings

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Abstract
Every year, the NCAA selects 68 Division-I Men's basketball teams to compete in the final tournament. In this paper, the Colley Method is examined as a way of selecting teams that should progress to the tournament. The results from solving the Colley System are compared to the rankings provided by the NCAA tournament selection committee. Additionally, the results are also compared to another popular rating method—the ELO system. After sorting and processing over 11,000 games worth of data with a VBA Macro and computing the rankings by importing the data into MATLAB, it was found that the Colley method produced the same top 6 teams (in a slightly different order) as the NCAA committee. Furthermore, these top 6 teams matched 5 of the top 6 teams put forward by the ELO system. Despite all systems having similar top teams, the ELO ratings deviate from the NCAA rankings more than the Colley ratings. This shows that streaks create weakness in the ELO system. Additionally, it is found that the Colley Method is a reliable, computationally-inexpensive way to rank teams similarly to how they are ranked by the NCAA.

*The authors gratefully acknowledge Dr. Danielle Lyles for helpful discussions.
1 Introduction

Each year, March Madness entertains and excites millions of fans while 68 NCAA Division-I Men's basketball teams battle on the court for a shot at the NCAA Championship. This two-week spectacle of basketball displays some of the best action a person can catch. However, out of the 351 teams in Division-I, only 68 teams make it to the tournament. How does the tournament selection committee accurately choose these teams?

Currently, the selection committee ranks teams based on record, strength of schedule, significant wins/losses, and previous tournament history. Strength of schedule means taking into account which teams a certain team has played and what the ranking of the opponent team is. This also ties into significant wins/losses. For example, if the top ranked team in the nation with a record of 10–0 loses to a team that has a record of 0–10, the committee will put more weight on this loss. Previous tournament history means that the rankings of a certain team in a certain region can be affected by the team's all-time record at a certain stadium—one of which they may play at during the tournament. Basing the selection solely on wins and losses can be difficult because one team only plays about 35 games per season. Therefore, a certain team is unable to play all other teams. Being able to see trends and information about teams, conferences, and players have helped the ranking process become more accurate for the NCAA. However even though there are many factors that the selection committee uses in their rankings, this paper attempts to show that two different ranking systems, depending only on wins and losses, return similar results as the complicated NCAA selection.

One example of a method to rank teams is the Colley Method. This method involves a simple linear system:

\[ C \vec{r} = \vec{b} \]  

In this linear system, \( C \) is the Colley Matrix, \( \vec{b} \) is a vector describing a certain team's win-loss relationship, and \( \vec{r} \) a the ranking vector. Since the desired result is having rankings for teams, solving this system for \( \vec{r} \) involves:

\[ \vec{r} = C^{-1} \vec{b} \]  

Then, the output \( \vec{r} \) vector can be sorted to give final rankings of the teams observed. To look at the effectiveness of the Colley Method, this system was run in MATLAB on both small and large data sets with game results ranging from select schools to the entire Division-I bracket. Then, these sets were run with the ELO rating system and compared to the Colley Ratings. Finally, a complete, Division-I data set\(^2\) that included results from all games played during the 2016-2017 season was run using the Colley Method and the ELO system. The results were then compared to the overall rankings given by the NCAA tournament selection committee. This provided insight into the effectiveness and accuracy of using the Colley Method to rank teams.

2 The Colley Method

As stated above, the general form of the Colley Method is solving the linear system \( C \vec{r} = \vec{b} \) where \( C \) is the Colley Matrix, \( \vec{b} \) is a performance vector, and \( \vec{r} \) is the desired ranking vector. To look deeper into this, first observe the general breakdown of \( C \), the Colley Matrix:
This matrix is an $n \times n$ matrix that corresponds with a system of $n$ teams. Entries along the diagonal are equal to $2 + t_i$ where $t_i$ is the total number of games team $i$ has played. All off-diagonal entries $c_{ij}$ are equal to $-n_{ij}$ where $n_{ij}$ is the number of games team $i$ has played team $j$. The result of this composition is that $C$ is a symmetric, positive definite matrix. This means this system is non-singular and therefore always has a solution. By always having a solution, the Colley Method will always output rankings for every team. Additionally for any row $i$, the sum of the $n_{ij}$ entries equals $t_i$.

$$t_i = \left\{ \sum_{j=1}^{n} n_{ij} \mid i \neq j \right\}$$

This is a great way to check if the Colley Matrix has been set correctly. Furthermore since the matrix is symmetric, column $i$ behaves the same way. Next, the $\vec{b}$ vector is composed the following way:

$$\vec{b} = \begin{bmatrix} 1 + \frac{1}{2}(w_1 - l_1) \\ 1 + \frac{1}{2}(w_2 - l_2) \\ 1 + \frac{1}{2}(w_3 - l_3) \\ \vdots \\ 1 + \frac{1}{2}(w_n - l_n) \end{bmatrix}$$

Each element, $b_i$, in $\vec{b}$ is made up by the expression $1 + \frac{1}{2}(w_i - l_i)$ where $w_i$ is the total number of wins and $l_i$ is the total number of losses for team $i$. The reason $\vec{b}$ is created like this is because it makes the average values over all the entries equal to 1.

$$\frac{1}{n} \sum_{i=1}^{n} b_i = 1$$

By making $\vec{b}$ this way, the average value over the entries of the ranking vector is 0.5. The interpretation of this value is as follows. All possible $r_n$ values range from 0 to 1 in which the average value of the rank of a team as we observe more and more teams is a value of 0.5. Furthermore, this statistic comes from the values of $r_n$ when there are no games played. In general, this comes from the following\(^1\):
\[ r = \frac{1 + n_w}{2 + n_{tot}} \]  

(5)

In which \( n_w \) is the number of wins and \( n_{tot} \) is the total number of games. At the beginning of the season, no teams have played and therefore all the rankings are 0.5—also meaning that the average is 0.5. The nature of this the \( C \) matrix, in which a head-to-head matchup will affect both teams, creates the system so that the following is always true:

\[ \frac{1}{n} \sum_{i=1}^{n} r_i = \frac{1}{2} \]  

(6)

Solving this system by left-multiplying by \( C^{-1} \) outputs the values for \( \vec{r} \). In order to find the top \( n \) teams in the system, the values, \( r_i \) in the ranking vector have to be sorted.

In general, the Colley Method is unbiased as it relies only on win-loss information for each team. It does not factor in unusual point differences. However, it only partially accounts for strength of schedule. Since the off-diagonal entries of the matrix can flag how many times a team plays another, it accounts for the overall comparison between those two teams based on the overall win-loss information for a team.

3 The ELO System

In order to have another comparison for the Colley method and the teams chosen by the NCAA committee to play in the tournament, ELO rankings were determined for all the teams. The ELO system, which gained much respect starting in the 1930’s, is similar to the Colley Method because it relies on win-loss information. The important difference between these methods is that the ELO system is based on game-by-game results and the Colley Method observes a whole set of games together. ELO is the choice system for most chess associations around the world and is often seen in sports-analytics systems such as FiveThirtyEight\(^5\), ESPN for NCAA results and statistics, and NBA rankings from 1998 to 2013\(^6\). Therefore, it should prove a reasonable standard for comparison.

The ELO system works by first assigning a player or team a standard number of points upon entry into the system—common standards being 1000 and 1500, in which each team or player starts with the same value. From this point on, the system will calculate an expected win chance or result between two teams using the following equation:

\[ E_A = \frac{1}{R_B + R_A} \frac{1}{1 + 10 \frac{400}{R_B + R_A}} \]  

(7)

\( E_A \) is a number between 0 and 1 which represents the chance of team A winning the game. For example, 0.75 would represent a 75% chance of team A winning. The same is then done for team B so the sum of the two expected outcomes is 1. It is also worth noting that the ‘400’ in equation (7) is a constant that standardizes the system, giving significance to point differentials. When a game is played, each team is given a score, \( S_A \), between 0 and 1 representing their performance.
during the match or game. Afterward, the ranking is adjusted to reflect the difference between the expected and actual scores. The adjustment follows the equation below:

\[ R'_{A} = R_{A} + K(S_{A} - E_{A}) \]  

(8)

\( K \) in the equation is a standardization constant assigned to every system. This helps determine the significance of a win or loss; a higher \( K \) means points are more fluid and a win or loss is more significant in changing the ELO score. Standard \( S_{A} \) scores tend to be 1 for wins, 0 for losses, and 0.5 for draws. Alternatively for basketball, a function of the point differential could be introduced in order to take account for the magnitude of the win or the loss. However, this gets complicated because of potential outlying point differentials. In NCAA basketball, there are not any draws. Therefore, we will be simply looking at who won and who lost. The end effect is that the increase or decrease in A's score is compensated by the opposite change in B’s score. This maintains the number of points in the system.

Similar characteristics of the ELO system and the Colley Method are as follows:

- A team does not have to play all teams in the system to receive an accurate rank
- Accounts for team-by-team match-ups
- Maintains a constant average rating for the teams in the system
- All Division-I teams

One major difference between these methods is that the ELO system updates game-by-game, giving present performance, while the Colley Matrix allows an overview of a season to be analyzed at once, giving overall strength.

4 Examples and Numerical Results

The data for all 2016-2017 NCAA Division-I is available for download from Spreadsheet Sports\(^2\). This spreadsheet shows results and information for every game for the season. In order to extract the necessary data, a VBA macro script parsed over 11,000 games down certain selected sets of games. Finally this data was then exported into .csv files and pushed into MATLAB for processing. Several sets of data were extracted from this large spreadsheet:

- 4-team sets—ex: Duke, North Carolina, Louisville, Florida State
- Conference sets—ex: ACC
- Notable teams—ex: Kansas, Duke, UCLA, Arizona, Gonzaga, Villanova, North Carolina, Oregon

Once the data is sorted using a VBA macro, MATLAB processes the data. To begin, each team is assigned a number from one to the total number of teams that are analyzed. Next, MATLAB reads the data row by row, recording the first team's number, the second team's number, and whether team one won or lost (which was represented as a 1 for win and 0 for loss). Using this data the Colley Matrix, \( C \), and performance vector, \( \vec{b} \), positions for team one and two can be
iteratively updated for each game that is played. Once the entire set of data is complete, the linear system can easily be solved for the rank of each team. Finally, MATLAB sorts the ranks from highest to lowest as well as the team names corresponding to each rank to tabulate the results. The ELO ranking system operates in a slightly different way where the ranks of each team are updated after every game from the data.

In order to ensure that our algorithms and code were correct, we chose a small sample of 4 teams from the ACC conference to test. The four teams that were selected were Florida State, Duke, Louisville, and North Carolina. There are two reasons we chose to look at these four teams to show the Colley Method. First, these teams are in the same conference and end up playing each other at least two times in a season. Second, it will be easy to see the accuracy of the method because the NCAA rankings show the final relative rankings of these teams. The Colley Matrix for the system including just these four teams is as follows: *NOTE: The vector with team names is just a "flag" vector which is used to visualize how the data for each team is organized in the Colley Matrix.*

\[
C = \begin{bmatrix}
Florida State & 6 & -1 & -1 & -2 \\
Louisville & -1 & 7 & -1 & -3 \\
North Carolina & -1 & -1 & 6 & -2 \\
Duke & -2 & -3 & -2 & 9
\end{bmatrix}
\]

In this Colley Matrix, the first row and column represents Florida State, the second row and column represents Louisville and so on. Additionally, the performance vector, \( \vec{b} \) is computed and shown below.

\[
\vec{b} = \begin{bmatrix}
Florida State \\
Louisville \\
North Carolina \\
Duke
\end{bmatrix}
\begin{bmatrix}
1 \\
1.5 \\
0 \\
1.5
\end{bmatrix}
\]

Finally, solving the system of linear equations using equation (2) gives the ratings for each team. This result is shown below:

\[
\vec{r} = \begin{bmatrix}
Florida State \\
Louisville \\
North Carolina \\
Duke
\end{bmatrix}
\begin{bmatrix}
0.51 \\
0.58 \\
0.36 \\
0.55
\end{bmatrix}
\]

This is only a small example of the full extent of the Colley Method. However, this is useful because these ranks are the same rankings based on head-to-head record. To further test this method, the entire Division-I 2016-2017 season data was examined. Although the process is the same, the data set for the entire D-I was significantly larger. After running the same MATLAB algorithms, the following ranks for the top 20 teams were found.
The average of the ranking values is 0.5 and it is quite clear to see that the top 20 teams in the nation have rankings high above the average.

Since the data was organized by the date the game was played, the ELO rating for each week was easily computed in MATLAB. As with the Colley Method, the data is read in row by row (or game by game) but now the ranking of each team is instantly updated, with the performance value being determined as a 1 or 0, according to whether the team won or lost. Once every game has processed, the ranks and teams are sorted and tabulated. Just like the Colley Method, only the top 20 teams are presented below for brevity.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Villanova</td>
<td>1</td>
</tr>
<tr>
<td>Kansas</td>
<td>2</td>
</tr>
<tr>
<td>Kentucky</td>
<td>3</td>
</tr>
<tr>
<td>Arizona</td>
<td>4</td>
</tr>
<tr>
<td>Gonzaga</td>
<td>5</td>
</tr>
<tr>
<td>North Carolina</td>
<td>6</td>
</tr>
<tr>
<td>Oregon</td>
<td>7</td>
</tr>
<tr>
<td>Duke</td>
<td>8</td>
</tr>
<tr>
<td>UCLA</td>
<td>9</td>
</tr>
<tr>
<td>Louisville</td>
<td>10</td>
</tr>
<tr>
<td>Baylor</td>
<td>11</td>
</tr>
<tr>
<td>Florida</td>
<td>12</td>
</tr>
<tr>
<td>Southern Methodist</td>
<td>13</td>
</tr>
<tr>
<td>Florida State</td>
<td>14</td>
</tr>
<tr>
<td>Butler</td>
<td>15</td>
</tr>
<tr>
<td>Cincinnati</td>
<td>16</td>
</tr>
<tr>
<td>Purdue</td>
<td>17</td>
</tr>
<tr>
<td>Saint Mary's (CA)</td>
<td>18</td>
</tr>
<tr>
<td>Notre Dame</td>
<td>19</td>
</tr>
<tr>
<td>West Virginia</td>
<td>20</td>
</tr>
</tbody>
</table>

Fig. 1: Top 20 Teams From Colley Method
Comparing both ranking methods we can see that similar teams seem to stay near the same rank for the most part. There are some notable exceptions though such as Southern Methodist which is ranked 3 with ELO but 13 with the Colley Method. Furthermore, Southern Methodist was ranked 21st by the NCAA.

5 Conclusion

To start, here is a list of the top 10 teams, ranked by the NCAA Selection Committee\textsuperscript{3}, and how they compare to our Colley rank and ELO rank:

<table>
<thead>
<tr>
<th>Rank</th>
<th>ELO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1387</td>
</tr>
<tr>
<td>2</td>
<td>1382</td>
</tr>
<tr>
<td>3</td>
<td>1380</td>
</tr>
<tr>
<td>4</td>
<td>1373</td>
</tr>
<tr>
<td>5</td>
<td>1367</td>
</tr>
<tr>
<td>6</td>
<td>1330</td>
</tr>
<tr>
<td>7</td>
<td>1327</td>
</tr>
<tr>
<td>8</td>
<td>1326</td>
</tr>
<tr>
<td>9</td>
<td>1325</td>
</tr>
<tr>
<td>10</td>
<td>1324</td>
</tr>
<tr>
<td>11</td>
<td>1295</td>
</tr>
<tr>
<td>12</td>
<td>1294</td>
</tr>
<tr>
<td>13</td>
<td>1291</td>
</tr>
<tr>
<td>14</td>
<td>1289</td>
</tr>
<tr>
<td>15</td>
<td>1279</td>
</tr>
<tr>
<td>16</td>
<td>1278</td>
</tr>
<tr>
<td>17</td>
<td>1272</td>
</tr>
<tr>
<td>18</td>
<td>1270</td>
</tr>
<tr>
<td>19</td>
<td>1269</td>
</tr>
<tr>
<td>20</td>
<td>1265</td>
</tr>
</tbody>
</table>

Fig. 2: Top 20 Teams From ELO Ranking
<table>
<thead>
<tr>
<th>Team</th>
<th>NCAA Rank</th>
<th>Colley Rank</th>
<th>ELO Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>Villanova</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Kansas</td>
<td>2</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>North Carolina</td>
<td>3</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>Gonzaga</td>
<td>4</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Kentucky</td>
<td>5</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Arizona</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Duke</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Louisville</td>
<td>8</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>Oregon</td>
<td>9</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>Florida State</td>
<td>10</td>
<td>14</td>
<td>30</td>
</tr>
</tbody>
</table>

The first thing to note is that the top six NCAA Ranked teams, in a slightly different order, are the same six teams from the Colley rankings. Additionally, the ELO system ranks several teams quite differently than the Colley Method, potentially due to win/loss streaks. For the top ten NCAA teams, the average difference between the Colley rank and NCAA rank is 1.7 while the average difference between the ELO rank and NCAA rank is 5.6. Even if the outliers—Louisville and Florida State—are thrown out, the average difference between the ELO rank and NCAA rank is greater than that of the Colley rank and NCAA rank.

According to the NCAA⁴, here are some selection criteria for ranking the 68 teams that make the tournament:

- Rating Percentage Index (RPI): win/loss for team, its opponents, and its opponents opponents’
- Conference Results
- Comparing common opponents
- Box Score and key-role players
- Strength of schedule
- Road record
- Historical Statistics
- Regional Committee Input

All of these criteria are not accounted for in the Colley Method. However, if you compare the accuracy of the Colley Method with the final NCAA Ranks, they are quite similar. From testing the Colley Method on the Division-I NCAA Men’s basketball teams, it is clear that using the Colley Method is a simple and computationally inexpensive way to rank teams. Although the ELO rating system is widely used and has been around for many years, the Colley method provides a simpler manner in team ranking that does not weigh results according to present strength. Instead, it averages strength over the entire season. In general, the Colley Method got closer to the NCAA ranks than the ELO system. This makes the system a great method for determining ranks at the the end of the season once all of the games have been played since it only considers the net wins.
and losses. Also, the Colley Method should only be hesitantly applied if team ranks are desired according to how they are doing at a particular stage of the season.

Unfortunately, any computer-based ranking system is unable to factor in everything that the NCAA Selection Committee does in order to select teams for the final tournament. However in looking at the Colley Method, it is seen that the availability of data can help closely rank teams with a computer. While most fans find magic in the chaos that is March Madness, our team found magic in the powerful information that can be found by using the Colley Method, which involves solving the simple linear system: $C \vec{r} = \vec{b}$.
SUPPLEMENTARY MATERIAL

Code can be found at github.com/bakerscott/CU-CJAM_ColleyRankings

6 References


Abstract
This paper was a Finalist (top 1%) to the 2017 COMAP Interdisciplinary Contest in Modeling (8,100 teams). Our team chose Problem F, in which we were tasked with creating a plan for the government and societal structure on a Mars colony that optimizes GDP and happiness with emphases in equality, education, and income. The problem becomes interesting when trying to quantify happiness and model utopia.

To develop these societal priorities into a working system, our team built a model to optimize the combination of Gross Domestic Product (GDP) and subjective well-being (happiness) of the Martian colony with variable income tax rates. The first task was to derive equations for both GDP and happiness based on related quantifiable measures. Current research gave us several trends to implement in our happiness equation, namely that while making lots of money is not what it takes to make people happy, income relative to others and government spending do increase people's life satisfaction. Other literature makes the general case that income has diminishing marginal returns in satisfaction. To account for these factors, we incorporated the Gini Coefficient (a score of income inequality), government spending, and the logarithm of disposable income into our happiness equation. We then wrote the happiness and GDP equations in terms of individual income (taken from Census data) and a general tax rate equation, which we wrote as a function of the logarithm of each individual's income.

After running a program to find the constants for our tax equation that optimized GDP and happiness while keeping personal income tax rates under 60% (the top income tax rate on Earth), we discovered that the relationship between our GDP and happiness scores reflects that described in prominent real-world research on the same correlation. Extending the model to accommodate a more diversified (global) population revealed that other common evaluations, such as the Human Development Indices, also reflect well on the accuracy and progressive nature of our model. The results of our model verify that it is possible to establish a sustainable society with both a basic income and universal higher education, which we recommend be initiated on Mars.

†The authors gratefully acknowledge Anne Dougherty and Engineering Honors Program (EHP).
1 Logistics of a New Society

The objective of the UTOPIA: 2100 project is to create a better quality of life than is currently present on Earth by optimizing GDP and happiness. This means all citizens will have access to the necessities of life and some enriching activities, education and social equality will be prioritized, income inequality will be minimized without sacrificing the incentives for innovation and entrepreneurship, and fiscal expenditures will not exceed revenue. We believe this will result in prosperity and sustainability in the Martian colony for at least the next 100 years.

1.1 Basic Income

The International Coalition on Mars (ICM) government’s first priority is to care for its citizens. On the naturally inhospitable planet of Mars, we will provide a basic income (living wage) to ensure that all citizens of the colony have hospitable lives. The basic income will account for all of the necessities mentioned below, and in doing so, will decide the value of the Martian Dollar (MD) in our colony [39, 23]. Our basic income is set at 20,800 Martian Dollars a year distributed weekly (equivalent to 10 MD per hour, forty hours a week), and will account for the following:

- Housing, food and water, clothing, toiletries, and other basic necessities. The ICM owns a store that will distribute these at the appropriate prices so that even the unemployed can afford to clothe and clean themselves and their living areas.

<table>
<thead>
<tr>
<th>Basic Need</th>
<th>Stipend (MD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Housing</td>
<td>2.9</td>
</tr>
<tr>
<td>Food</td>
<td>1.8</td>
</tr>
<tr>
<td>Child Care/ Education</td>
<td>1.6</td>
</tr>
<tr>
<td>Transportation</td>
<td>1.5</td>
</tr>
<tr>
<td>Health Care</td>
<td>0.9</td>
</tr>
<tr>
<td>Clothing</td>
<td>0.6</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td>0.7</td>
</tr>
</tbody>
</table>

- Note about health insurance: we decided to keep the insurance industry private on Mars, but will provide a small stipend within the basic income for health insurance, as this benefits the entire population by both reducing disease and increasing overall happiness. Another ICM team is working on the details of this model.

1.1.1 Basic Income for Minors

Each individual on Mars will receive her own basic income at eighteen or at such time that she becomes an emancipated adult. However, after proof of pregnancy, parents will receive the child's basic income while the child is still a minor for up to two successful pregnancies to increase the family’s overall income. Parents will receive no additional basic income for more than two children in order to discourage unsustainable population growth. (Note: basic income will be provided for children born from the same pregnancy - i.e. twins, triplets, etc.)
1.1.2 Early Childcare and Parental Leave

Companies are required to provide six months paid leave to both parents when a child is born, as the extended leave encourages cognitive development in the child and reduces the risk of postpartum depression in the mother [36]. Both parents are required to take at least part of their paid parental leave and encouraged to take all of it. The child's basic income will cover early childcare including day cares and babysitting services. At the age of three, the child will enter free preschool, provided as part of their educational right.

1.2 Education

Education will be a right for all citizens on Mars. In order to lead fulfilling lives and contribute effectively to our 22nd-century society, our students will need strong backgrounds in critical thinking and problem-solving, and deep understanding of social and economic issues. Our early curriculum will emphasize innovation, sustainability, financial and traditional literacy, and spreading awareness about discrimination. In many ways we will also model our education system after that of Switzerland, which promotes creativity and a love of learning. Students will go on a number of field trips each year which will expose them to different careers and fields of knowledge. At some point during their secondary education, students will take an aptitude test, which, when combined with the student's stated preferences, will recommend a list of career options that would best suit both their skills and their desires. However, students will not be required to pursue one of the selected careers.

1.2.1 Higher Education

The Martian government will cover the cost of four years in university for every citizen who graduates high school. However, it will be completely up to each individual to decide what level of formal education he/she would like to obtain. If the individual decides to enter the workforce directly after high school, his entire educational stipend will be deposited in his bank account. If the individual decides to receive a two-year degree or attend a two-year trade school, he will receive the remaining money. The top-ranking individuals in university will also be given the option to continue on to some sort of graduate schooling, paid for on the basis that these individuals will have higher paying jobs in the future and will therefore pay higher taxes back into the system. If an individual or his family can afford further education (beyond what the government is willing to provide for initial education or structural unemployment), he will be allowed to do so. (Note: in addition to providing a higher education for all those who wish to pursue one, the four-year stipend conditional on high school graduation will aid high school retention rates.)

1.2.2 Structural Unemployment

If an individual becomes unemployed by the advancement of new technology that renders her present skills useless, she will have the option of pursuing a trade degree at no additional cost.
1.3 Property on Mars

The government will own all land and buildings within the colony. All domestic and commercial inhabitants will have to pay rent to the government, and may apply to alter their building's structure. The government should approve all applications that display creativity without compromising utility, within code parameters. Investors may also invest in land outside the present habitable environment, in anticipation of future development as population grows and/or more Earth inhabitants arrive.

1.3.1 Housing

The government on Mars will guarantee a certain amount of living space for each of its citizens, which shall be arranged in double and single apartments. The apartments will be around 400 square feet in area, self-sustaining (i.e. containing a kitchen, bathroom, and washer-dryer), and will have 2 opposite retracting walls as well as temporary dividing walls between apartment rooms, allowing multiple apartments/rooms to be joined to create larger living spaces if needed. However, since space in the Martian colony will be limited, each apartment or group of apartments will have a government-imposed minimum number of occupants. The limited space will also discourage accumulation of excess possessions, and thus encourage industries to focus on the quality of goods over quantity. Early education will help install the practices of upgrading material goods and recycling the old (as opposed to amassing both new and old) as the societal norm. This will have a positive environmental influence as well.

The basic income will cover the rent for single apartments with absolutely basic furnishing. Upon her eighteenth birthday every citizen will be issued an apartment in a random location within the colony, at the cheapest rental price. At this time she may choose to move into that apartment or search for another. This element of random chance will help to mix up traditional socioeconomic lines and promote social equality. The "owner" - person with the right to the base-level rent of an apartment, may charge higher rent to another person if he happens to be randomly issued an apartment in a more desirable area. However, no person will be allowed to inhabit two apartments at once (given a grace period of three months).

1.4 System of Government

The government on Mars will be structured similarly to the current system of the United States of America, with three branches: Executive, Legislative, and Judicial. For simplicity's sake, we assume that unless explicitly stated otherwise, we will follow their Constitution exactly. To change the Mars Constitution, we will require a 70 percent majority in the popular vote.

1.4.1 The Executive Council

Instead of a President, the ICM will be headed by an Executive Council consisting of one or two individuals from each "neighborhood", or section of our colony. Citizens will be selected at random to serve on the Executive Council, in a fashion similar to United States jury duty. Each individual will serve for six months, during which time she cannot be fired from her current job by her employer. All new laws/major government initiatives must be approved by the Executive Council.
1.4.2 Fiscal Balance

The ICM government will assume no debt. In other words, its spending will always equal or exceed its revenue. Building up government assets in the years when revenue exceeds expenditures will serve as an insurance policy for the colony; these funds will be used to alleviate inexorable emergencies.

1.4.3 Currency

To maximize social equality at the start of this new society, there will be a new currency, the Martian Dollar (MD). This ensures that people will come to Mars with nothing more than their prior knowledge. To define the conversion between USD and MD for our readers’ reference, we assume that the basic income (arbitrarily set at 10 MD) would provide the same services in the United States, where the hourly basic income would be 15.12 USD [23]. Thus, the conversion factor is 1.512 USD per MD. In our model, we adjusted all salary data from the US Census accordingly, so the salaries we used in our analysis and model were all in MD.

1.4.4 Inflation

Because the Martian economy will be, for all intents and purposes, isolated from all Earthly economies, economic growth on Mars will mean inflation of the MD. However, if the government keeps track of real GDP and ensures that inflation matches pace with increased production, purchasing power should not be reduced.

1.4.5 Government Oversight

To promote social equality in the workplace and guarantee everyone equal opportunity for advancement in all fields, we decided that there would be a good deal of government oversight in the corporate world. The oversight is not meant to control industry, but simply to monitor for cases of discrimination based on gender, race, religion, or sexuality. For instance, women typically leave the workforce (especially in the STEM industry) not because of motherhood, but because they aren’t given the same opportunities as their male counterparts. This oversight will help to condition more comfortable office environments and thereby increase the retention of women and minorities in fields in which they have traditionally been underrepresented.

1.5 Assumptions

- The Universal Declaration of Human Rights will have power on Mars, because fundamental human rights are an immeasurable requirement for subjective well-being [38].

- Mars has stable resources for Population Zero including food, water, and building materials. All infrastructure required for the colony (i.e. roads, city buildings, apartments, and agriculture) is already present on Mars.

- People will act similarly despite the planet of their origin. Thus, the percentages of people at each level of education will remain the same, people will save and invest their money
similarly, and the relationship between their salaries and a living wage will be the same as it was on Earth.

- Manufacturing, waste management, transportation, plant agriculture, and low-level maintenance will all be automated, eliminating the need for people in these vocations [37].

- The government is responsible for maintenance of public property, including schools, infrastructure, parks, and buildings. Individuals are, however, responsible for maintaining their own living space. Major utilities (namely water and energy) will be privatized with strict government regulation.

- GDP and happiness are of equal importance, and thus weighted equally in our model.

1.6 Definitions

- Subjective Well-Being: describes how happy individuals are at a moment in time and how satisfied they are with their lives as a whole [33]. Happiness in this report is synonymous to life satisfaction because life satisfaction is the subject of most existing research that attempts to quantify happiness; the majority of economists just use the term "happiness."

- \( P \): Population
- \( G \): Government Spending
- \( R \): Property Rent
- \( \phi \): Gini Coefficient
- \( Y_i \): Individual Income
- \( T_i \): Individual Tax Rate
- \( H \): Subjective Well-Being
- \( \alpha \): Total Government Revenue
- \( \beta \): Basic Income (per person)

2 Population Zero

2.1 Analysis of Demographics

The 10,000 individuals chosen to populate the colony on Mars were selected from the PUMS Census dataset [8] under the initial parameters that they had an education level of twelfth grade or higher and a yearly income above 12,000 MD. From there we decided that each individual should be between the ages of 18 and 45 to optimize our work and child-bearing force. We used Python to filter the data based on our parameters, and MATLAB to analyze the remaining data for trends.
Of our 10,000 citizens, 4,253 are women and 5,748 are men. They have an average age of 34 years, which is preferable both because it is well above the age for brain maturation and because it is generally a period of stability in an individual’s life. Most of the population has at least a Bachelor’s Degree of some sort, which indicates both that they value education and that they will be well equipped to step into our 22nd-century society. The state that will send the most people to Mars in Population Zero is California, followed closely by Texas. These and other Census analyses indicate this is a fairly accurate distribution of individuals from around the United States.

The top five ancestral ethnicities of our citizens are German, Irish, African American, Mexican, and Italian. Thus Population Zero represents the "melting pot" that defines this nation, bringing with them their culture and traditions. This diversity will enrich our society in addition to contributing to a healthier gene pool.

2.2 Fostering Innovation and Social Equality

Innovation and creativity will be vital to both the initial success and the long-term sustainability of our Martian colony. Thus, they will be core focuses of our educational system. Students will have an "open study" class during which they will be encouraged to pursue a subject of their choice, whether it be music, welding, circuitry, language, or literature. They will be encouraged to think about sustainability and their impact on society from a young age, encouraging an innovative and entrepreneurial attitude. This class will foster students’ love of learning and require them to develop skills in research and imagination.

The government will also encourage entrepreneurship and investment through the absence of capital gains taxes for the first twenty years in the colony. Since people won’t have to pay taxes on the profits they gain from investments during that period of time, they will be encouraged to invest in new Martian businesses.

Social equality is fostered through the provision of the basic income and the random assignment of living spaces, which will create diverse and constantly changing communities of all socioeconomic statuses. Discrimination will be discussed in school starting at a young age and strictly discouraged, so young generations of Martians will grow up in a society where racism and sexism have been all but eradicated.

3 Development of the Model

The Laboratory of International Financial and Exploration Policy (LIFE) has tasked our policy modeling team with the design of a governmental and economic system for Population Zero that will establish an achievable utopia by maximizing both Gross Domestic Product (GDP) and each individual’s happiness, which we have defined as subjective well-being. We were asked to consider income, education, and social equality in our model. The measures that will indicate the success of our model include a balanced government budget with income tax rates no higher than those found on Earth and a reduction of the Gini Coefficient in our population.
3.1 Building the Government Budget Distribution

3.1.1 Accounting for Education

In order to calculate our fiscal needs for education and basic income, we used U.S. Census data to select a representative first-wave population, the demographics of which were discussed in more detail in section 2. The 10,000 people are all between the ages of 18 and 45, have received education to at least a high school graduation equivalent, and have incomes greater than 12,000 MD (18,144 USD) annually. With regard to how this impacts our Martian demographic, our age filter ensures that most everyone will be either single or in a couple (no kids); our education and income filters ensure that everyone will be working and earning at least half of their government-provisioned basic income. However, instead of creating an artificial proportion of innovators and producers, we expect that the workers will naturally arrange themselves into the most economically-beneficial distribution. This is one case where the ICM government will not intervene, and let the market find an efficient equilibrium.

After researching the average costs of a year in public grade school, a trade/community college (two-year program), an undergraduate four-year program, and a graduate program, and assuming that all individuals who have not finished their high school degree will do so, we calculated how much it would cost to provide all the education outlined in our Education Policy section. Along with the tuition for students in the traditional higher education age range (18-22, plus those in graduate school), this includes the cost of retraining people whose jobs disappeared, using the average rate of job disappearance in the U.S. due to the adoption of new technologies since the year 1960. It also includes the cost of grade school/pre-school for the new generation of children to be born on Mars in the first decade, assuming a replacement-level fertility rate of 2.1 children per woman[35] and that nine percent of women will be pregnant in any given year[5]. Given our initial population and these statistics, we were able to assume the number of children in our society by year 5.

3.1.2 Calculating the Basic Income

The basic income expenditure is the product of the hourly basic income (ten Martian Dollars per hour) times forty hours per week times fifty-two weeks per year times the size of our population (10,000 plus the number of children we expect to be born in the first five years).

3.1.3 Expenditure Categories

Based on the budget structures of governments from around the world and the special characteristics of our Martian society (such as the basic income, lack of international involvement or defense, and large emphasis on education and innovation), we created seven expenditure categories: Basic Income (which covers basic housing, food, clothing, transportation, health care, and some miscellaneous items), Education, Government Personnel, Scientific Research and Development, Infrastructure, Agricultural Subsidies, and Environment, Community and Culture.
3.1.4 Analysis of Current World Budgets

Because Switzerland has an educational system that is ranked number two in the world [34] and is the first country to have held a vote for the adoption of a basic income, making it the country with the closest-aligned goals to our Martian society, we decided the weighting of our expenditure categories (as a percentage of the annual government budget) by first looking at the breakdown of Switzerland’s federal budget [28], and then slightly altering the percentage in each category in order to fit our needs. Once we had both the education and basic income expenditures in Martian Dollars (MD), we adjusted the budget percentages for these two categories until the percentage ratio in our budget matched the ratio of the expenses. Then, we applied the same MD value of each percentage to the rest of the expenditure categories and found the total annual government spending budget: 484.4 million MD.

<table>
<thead>
<tr>
<th>Expenditure Category</th>
<th>Percentage</th>
<th>MD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic Income</td>
<td>60%</td>
<td>291,347,140</td>
</tr>
<tr>
<td>Education</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Government Personnel</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Infrastructure Construction and Maintenance</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Science, R&amp;D</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Agricultural Subsidies</td>
<td>4%</td>
<td>19,374,705</td>
</tr>
<tr>
<td>Environment, Community, and Culture</td>
<td>4%</td>
<td>19,374,705</td>
</tr>
<tr>
<td><strong>Total Budget</strong></td>
<td><strong>100%</strong></td>
<td><strong>484,367,648</strong></td>
</tr>
</tbody>
</table>

3.2 Creating a Tax Code

The main modeling task was to maximize both GDP and subjective well-being, both of which are dependent on taxation. In order to mitigate income inequality, we attempted to individualize the tax rate by basing it on the natural log of a person’s income. This idea arose from the attempt to “linearize” the Lorenz curve, which is an exponential function. The equation is as follows:

\[ T(Y_i) = a \times \ln((Y_i)^b) \]  

Note: sales taxes are considered regressive because they affect people with lower incomes higher by proportion of their income, so there will be no sales taxes on Mars.

Businesses in the U.S. can expect to spend about eight percent of their revenue on property rent [26], and approximately ten percent of investments in the US are in real estate [14]. Also, 29% of each individual’s basic income goes towards the base-level rent on their apartment, which the government owns. Assuming a ten percent saving rate (and pre-tax savings), our equation for property rent going to the government is:

\[ R = 0.09 \times \sum_{i=1}^{P} (Y_i) + (0.29\beta \times P) \]  

We assume that running a balanced budget or a slight surplus will make our Martian colony most sustainable over the next 100 years. Therefore, our budget constraint is set less-than-or-equal
to the total revenue from taxes plus rent from property ($\alpha$). Also, because income taxes only account for 53.8% of U.S. tax revenue [27], we divide the taxes term in our budget equation by 0.538 to obtain a fairly accurate prediction of total revenue from all taxes. Originally, we didn't include corporate taxes in an attempt to encourage entrepreneurship in our new society, so the coefficient was 0.636 instead of 0.538; however, after running our model, we discovered that we would need to include corporate taxes to account for our expenditures.

$$G = 484.4 \text{ million} \leq \sum_{i=1}^{P} \frac{Y_i * T_i}{0.538} + R = \alpha$$

(3)

Gross Domestic Product (GDP) can be calculated in two ways, the expenditures approach or the income approach [6]. We chose to use the expenditures approach because of the focus in our model on government spending to provide a basic income and high-standard education. The equation for calculating GDP in the expenditures approach is:

$$GDP = C + I + G + (X - M)$$

(4)

Where $C$ is gross private consumption expenditures, $I$ is gross private investment, $G$ is government purchases, and $(X - M)$ is exports minus imports, or net exports. Because there are no separate countries on Mars (yet), there are no exports or imports, so we leave these terms out of our GDP equation. Also, in order to optimize our tax function coefficients, we wrote $C$ and $I$ in terms of individuals' disposable income post-tax (assuming a ten percent savings rate). Note: because the consensus among available sources is that approximately 17 percent of U.S. GDP comes from business investment [3], we divide the GDP equation by 0.83.

$$GDP = \frac{\sum_{i=1}^{P} (0.9 * (Y_i - Y_i * T_i) + \beta) + G}{0.83}$$

(5)

### 3.2.1 Deriving an Equation to Measure Happiness

Quantifying and measuring subjective well-being was a problem in itself. As the name implies, subjective well-being is inherently subjective. However, research in the realm of happiness economics correlates relative income and government spending with overall life satisfaction.

The Easterlin Paradox [12] in happiness economics theorizes that an increase in income does not make people happier. For the most part, this is true: beyond a certain income, money does not necessarily buy happiness. However, modern researchers have side-stepped this paradox by focusing on the relative income hypothesis [25]. This states that the satisfaction derived from a given consumption level depends on its magnitude in society; in effect, people want to "keep up with the Joneses." We accounted for this trend in our model by using the Gini Coefficient, which measures income inequality. The lower the Gini Coefficient, the higher the income equality and, therefore, the overall life satisfaction.

A study by the National Bureau of Economic Research refutes the theory that income itself does not matter when it comes to subjective well-being [33]. The study correlates the log of an individual's absolute income with his/her overall life satisfaction. This implies that a 20% rise in income will provide the same benefit to subjective well-being, regardless of the initial income.
Using the combination of this functional form and the Gini Coefficient, our model accounts for the impact of economic equality and income on subjective well-being.

The same study indicates a correlation between government spending per individual and happiness, citing that countries with higher GDP-per-capitas also have higher overall life satisfaction rates, which is measured through surveys such as the Cantril Self-Anchoring Scale [7].

Taking all these factors into account, our equation for Happiness is as follows:

\[ H = \sum_{i=1}^{p} \left( \frac{G_i}{P_i} + \ln(Y_i - Y_i^* T_i + \beta) \right) * (1 - \phi) \] (6)

Note: because money is the one quantitative measure we have, our Happiness score ends up in terms of MD.

The happiness score is inversely related to income inequality, which is modeled using the Gini Coefficient, calculated using the Lorenz curve, \( L(x) \). \( L(x) \) is the relationship between the cumulative percent of a population’s income and the cumulative percent of the population. Using the income of our population of 10,000 before taxes, the curve looks like this:

![Lorenz Curve](image)

The Gini coefficient is calculated using this curve and the line of perfect equality:

\[ \phi = 1 - 2 \int_0^1 L(x) \, dx = \frac{A}{A + B} \] (7)

Where \( A \) is the region between \( L(x) \) and the line of perfect equality, and \( B \) is the region below \( L(x) \).

### 3.3 Multi-Object and Multi-Variable Constrained Optimization

After deriving our government budget and GDP and Happiness equations, we ran through 5,000,000 combinations of our two tax equation variables (\( a \) and \( b \)) in Python (see Appendix for pseudocode) until we found the mean and standard deviation (respectively) of all the GDP and
Happiness scores. Then we ran through the program again and calculated z-scores for each GDP and Happiness score using the following equation:

$$Z = \frac{x - M_x}{SD_x} \quad (8)$$

Where \(x\) is the GDP or Happiness score, \(M_x\) is the mean of the distribution and \(SD_x\) is the standard deviation of the distribution. By manipulating the data in this way, each new distribution of z-scores is given a mean of zero and a standard deviation of one [2]. Calculating the z-scores allowed us to compare the GDP and Happiness scores with equal weight, even though they come from different distributions. Once we had both sets of z-scores, we added them together and found the maximum z-score sum.

### 3.4 Initial Results

This optimization model we programmed allowed us to find both the ideal coefficients for our tax function and a Pareto set, or the set of all maximum GDP + Happiness pairs that fit the constraint of the government budget (greater than or equal to our expenditures). After gathering the Python data for each run of the model, we confirmed the calculations in MATLAB.

(Note: the following is the last of the "initial" results, because in earlier runs of the program we neglected to include the basic income along with post-tax disposable income in our GDP and Happiness equations.)

When we ran the optimization program without setting an upper limit on tax rate, the optimal tax function ended up with \(a = 1.29\) and \(b = 5.97\):

$$T(Y_i) = 1.29 \times ln((Y_i)^{5.97}) \quad (9)$$

Note: the initial model stopped at the first maximum value pair, whereas in the final model we found two. However, without an upper bound on tax rate, the final model would take too long to run and wouldn’t be able to operate within the time limit of the contest.

In this case, the highest tax rate was 99% and the lowest tax rate was 72% of income, which are both too high and would leave people with barely any disposable income. When we set an upper bound on tax rate at 60% (the highest income tax rate in 2015 was 57% in Sweden), the optimal tax function ended up with \(a = 2.11\) or 2.19 and \(b = 2.19\) or 2.11:

$$T(Y_i) = 2.11 \times ln((Y_i)^{2.19}) \text{ OR } T(Y_i) = 2.19 \times ln((Y_i)^{2.11}) \quad (10)$$

In this case, the maximum tax rate was 59.9% and the minimum tax rate was 43.4% of income. The government revenue from taxes and rent, \(\alpha\), was 485.3 million MD. Thus, all ICM expenditures are covered, with a slight surplus, which will go into the emergency fund. The Gini Coefficient fell from 0.39 to 0.35, and people with higher salaries still take home higher incomes than people with lower salaries. The following graph illustrates the relationship between income and tax rate using our optimized tax equation.
When finding the optimal combination of GDP and Happiness, we used z-scores to give both distributions a mean of zero and a standard deviation of one. We abandoned this approach when we discovered that the z-scores for GDP and Happiness were exactly the same in each a and b pair, thinking the phenomena had to be the result of an error. However, when we graphed the actual scores of GDP and Happiness, we found that the relationship was also a straight line. This result is not at all intuitive, considering our equations for GDP and Happiness:

\[
GDP = \frac{\sum_{i=1}^{P} (0.9 \times (Y_i - Y_i \times T_i) + \beta) + G}{0.83}
\]

\[
H = \sum_{i=1}^{P} (G/P + ln(Y_i - Y_i \times T_i) + \beta) \times (1 - \phi)
\]

Viewing the following graphs (our results) led us to the realization that the logarithmic function of disposable income in the Happiness equation is linearized by the Gini Coefficient term (which is based on the exponential Lorenz curve).
The maximum GDP and Happiness occur at the top right end of the line (where the top income tax rate is 60%). Also, considering the axis intercept of the equation for the line on this graph, \( H = GDP - 7.9 \times 10^8 \), we conclude that it is possible to have GDP without Happiness. This makes sense if one imagines a society where there is production of goods but a terrible standard of living.

Our model’s relationship of GDP and Happiness matches that discovered by other researchers. The following graph [38] was developed for the Journal of Economic Perspectives in 2008. It uses GDP per capita data from a variety of countries and mean "happiness level," as determined by the Cantril Self-anchoring Scale [7], which is widely regarded as one of the most accurate surveys for the measurement of subjective well-being. The correlation between GDP and happiness displayed on this graph looks distinctly linear.

The similarity between the correlation of the GDP and happiness data generated by our equations and that collected by this Journal leads us to believe that our equations and therefore our model are fairly accurate to present-day happiness economics.

4 Extending the Model - Sensitivity to Scale

In order to ensure that our model is running smoothly, we recommend that it be re-evaluated every five years. Because it is the first of its kind (on Earth, Mars, or elsewhere as far as we know), frequent monitoring of our system will be crucial, especially in the first few decades. One important aspect of this model is that almost no dynamic analysis is required. The existing system (of governance, taxes, basic income, education, etc.) will simply grow to accommodate new migrants.

4.1 Immigrants from Earth

4.1.1 Infrastructure

As more people come in waves from Earth, we expect that the governments on Earth will pay to have cities built on Mars that will accommodate the growing population. Once they are settled,
working, and paying taxes in their new Martian society, our model can provide for any number of citizens. We predict that in the long-term, the colonies on Mars will grow to become a series of neighboring city-states, each with its own system of government and economy that is similar to our original model.

4.1.2 Increasing Diversity

In the Population Zero model, we assumed that all 10,000 citizens moving to the colony were between the ages of 18 and 45, had at least a high school education, and made more than 12,000 MD (18,144 USD) per year. When our model is scaled upwards, taking in more and more citizens, this will cease to be true. However, the extra expenditures of providing both basic income and education for more children and basic income for the severely disabled and the elderly who are too old to work will be covered by our original budget.

In the original ICM budget, we accounted for enough births in the first decade that 40% of our population was assumed to be too young to work, and in need of education/childcare. In the U.S. and other countries of similar development level, only 24% of the population is too young to work. (The reason for this discrepancy is that all of the 10,000 people coming in the first wave are in the fertile age range, 18-45, and we assumed females on Mars would have an average of 2.1 children.) So when in future years we add the 12.6% of the population that are severely disabled and the 8% that are too old to work, taking out the 7.8% of the severely disabled who are younger than 18 (1% of the total population), the initial estimate of the total percentage of people who will not be contributing to tax revenue but who will be receiving basic income is 43.6%. However, when we account for the fact that only 24% of the non-workers will be needing education, the drop in educational cost more than covers the increase in living wage from the additional 3.6%.

Also, as technology and medicine continue to advance, we anticipate that people will live longer and therefore be able to work longer, and that some disabilities and diseases will eventually be combated and perhaps eliminated. These advancements will allow more people to experience a higher quality of life and will increase our government revenue.

Initially, the small amount of surplus generated from the larger drop in educational costs compared to income tax revenue reduction (a sum of about 1% of the annual government budget) will be spent helping to provide unskilled workers from Earth with skills that they can use to be employed on Mars, where the vast majority of unskilled work is automated. Further down the road, this annual surplus will be put into the ICM emergency fund. All in all, our economic model is still functional even as the general population from Earth dilutes the work force.

4.1.3 Savings

Different populations of people (especially those with different geographic, ethnic and religious backgrounds) will have very different ideas about how much of their income to save each year. Among Earth nations, the average saving rate ranges from 17% of income to negative savings (i.e. debt greater than savings). With a 10% saving rate (as in our original model), the GDP of the colony is 1.056 billion MD. With a 17% saving rate (the average saving rate of Switzerland [31]), the GDP is 1.039 billion MD. Finally, with a 5% saving rate (the average saving rate of the U.S. [31]), the GDP is 1.069 billion MD. Thus, saving rate does have a significant effect on GDP - specifically, for every additional percentage of their income that individuals save, the GDP will
drop about 0.24%. However, given the distribution of Earth nations’ average saving rates (greater and less than 10%), it is likely that some households saving more will be counterbalanced by other households saving less, so there will not be a significant change in GDP due to saving rate as the population diversifies.

As discussed above, our model will still be economically viable with an enlarged population. Logistically, however, the migration of large numbers of Earth inhabitants to Mars will only be feasible if the construction of the additional infrastructure can keep ahead of the arrivals. In order to expand the livable environment, manufacture new cities, and get the migrants settled (registering with the government and finding employment), the likelihood of success would be higher if the migrations occurred in phases. One notable strength of our model in admitting new waves of migrants is that in order to receive their basic income, each family will have to register with the government upon their arrival to Mars, which will help to keep the colony organized. Also, the basic income will help to reduce financial stress while new migrants find employment.

### 4.2 Impending Destruction of Earth

The importance of our model increases dramatically with the news that a planet-sized comet is on course to collide with Earth. To model the impact on our Martian economy and well-being of taking in all of Earth’s population, we removed all filters from the Census data and collected the information of 10,000 completely random people from the U.S. This represents the fact that in future years, the selection of persons to move to Mars will occur via a family lottery, and we will accept citizens of all different socioeconomic classes, from around the world, into our society with each wave of migration.

#### 4.2.1 Discovery of Census Data Inconsistency

When we tried running these new salaries through our tax-optimization model, we found no coefficients that fit our budget constraint. Upon reanalyzing the Census data, we realized that the sub-population of people not making any money was 42% (our prediction based on research and calculation was 43.6%), but the minimum age in the population was 15, meaning that if all ages of children were included in the data set, the percentage of people not making any money (or generating any tax revenue) would be much higher, and inconsistent with the conclusions of the available research.

While we were unable to investigate this Census inconsistency further within the time frame of the contest, we accounted for it in our extended model by imposing a limit on the number of people making no money in our population that matches what we would expect given the research (accounting for the fact that 5/6 of students below the age of 18 are not included in the Census data).

#### 4.2.2 Global Results

After accounting for the new ratios of people with each level of education and ensuring that the proportion of non-workers reflected the expected value, our government budget constraint decreased from 484.4 million MD to 362.4 million MD. This confirms the prediction from the
previous section (that there will be a significant decrease in education costs and basic income expense because children are already included in the sample population of 10,000).

With the tax rate limit set at 60%, our program returned no solution points. This meant that it was not possible to generate enough tax revenue to cover our budget without taking more than 60% of at least some citizens’ salaries - the salaries were too low. After using a shortcut to locate the approximate value, we upped the limit to 75%. With the new budget and tax rate constraints and the salaries from the "global" (un-filtered) population, our optimal tax equation is:

\[ T(Y_i) = 1.05 \times \ln((Y_i)^{5.5}) \]  

(11)

With this tax equation, the GDP was 801 million MD and the Gini Coefficient fell from 0.517 to 0.455. The highest tax rate is 74.97% and the lowest tax rate is 51.76%. However, if our high-level education system results in citizens making higher salaries on Mars, these tax rates could be lowered. In the current model, we assume that the proportions of our population with each degree will match those currently observed in the U.S. Census data, despite the fact that the ICM government will be footing the bill. Our model accounts for the cost of paying for everyone's higher education, but not the additional tax revenue collected from a population with higher salaries.

4.3 The Human Development Index and the GNI

We used the Human Development Index (HDI) to account for income, social equality, and education on Mars. The HDI incorporates 3 indexes (relating to income, education, and health) that speak for the level of development in a region relative to others. For example, in a data set of the average incomes of nations around the world, the Gross National Income Index (GNI) is calculated by comparing the GNI per capita of a specific country to the maximum and minimum values in the data set [18].

\[ GNI = \frac{\text{GNI per capita of Country X} - \text{minimum}}{\text{maximum} - \text{minimum}} \]  

(12)

Where maximum and minimum, respectively, refer to the maximum and minimum values of GNI per capita in the data set. Just as for income, there exists a metric for education and health care, but as our team is not focused on health care, only the GNI and the Education Index (EI) apply to our model.

For this report, we calculated the GNI, EI and \( \phi \) (the Gini Coefficient, based on the Lorenz curve) of the colony on Mars to compare with all of the countries’ data on Earth and discovered that we ranked number one in both GNI and EI on a scale from 0 to 1, meaning our colony has the highest GDP per capita and average years of education relative to all countries on Earth.

The Gross National Income (GNI) is the sum of the GDP and the cash flow from all non-resident citizens. We will operate under the assumption that the ratio of GDP to GNI is the same in our society as it is in the United States (0.99 [15]) in order to facilitate comparison with other countries. The optimized GDP per capita of Population Zero on Mars is 105,000 MD and the GNI per capita is 104,000 MD, which translates to about 157,000 USD. Even when the Martian population is altered to reflect the demographics of the world, the average GDP per capita is about 75,000 MD (113,000 USD), which still gives us a GNI index of one. This means that we rank
first in the GNI Index for the population selected by lottery from Earth as well as for Population Zero.

The Education Index (EI) is dependent on the average years of education received by an individual in a particular country. The highest average length of schooling for any country on Earth is 12.6 years [24]. However, that same measure for our colony on Mars is 18.1 years, which also contributes to a GNI Index score of 1.

When examining the GDP per capita, we noticed that it is significantly higher on Mars than it is here on Earth. If the GDP per capita is plotted as a function of education, it yields the following graph:

\[
\text{GDP Per Capita} = 2469 \times \exp(0.2185(x)) \quad (13)
\]

There is a distinct correlation between GDP per capita and the average years of schooling of a society, supported by \( R^2 = 0.311 \). If the average years of schooling for the colony is plugged into Equation 13, the average GDP per capita is 126,000 USD. In the previous section, we predicted that the average 18.1 years of education on Mars would yield 113,000 USD (GDP per capita). The close proximity of these values (126,000 and 113,000 USD) shows that our model's GDP calculation conforms with the real world data about the various correlations between GDP and other variables such as education and income.

Even though our model of Mars is performing extremely well within the GNI and EI Indexes, the Gini Coefficient remains relatively high. Compared to other countries on Earth, the Gini Index of the colony (calculated using the GNI method), is 0.27. (Note: our GNI and EI Index scores were 1.0.) This means that the income equality of the colony is on the lower side of the spectrum. One reason for the model's low Gini index yet high GNI and Education indexes is that even though the average income in our Martian population is high, the range between the highest and the lowest salaries is massive, with fewer salaries in between. We observe the same phenomenon with years of education [16]. As mentioned previously, this team consciously chose to leave the Gini Coefficient relatively high in order to encourage hard work and innovation in the colony, while maximizing Happiness. This is a delicate balance to maintain.
5 Possible Sources of Error

5.1 United States Influence

The largest source of error in our model as we scale the Mars population to include all of the Earthly humans is that nearly all assumptions about the population demographics and tax revenue distributions (between personal income, business income, and employment/payroll) are based off of the United States Census and tax revenue structure. Populations from different countries and continents will have significantly different proportions of certain races and levels of education than the population described by our random sample from the U.S. Census. This could have a large impact on the amount of tax revenue (from salaries) in the first generation, and the demand for higher education in further generations, based on different cultural values. A revised version of our model would include census data from around the world to account for the true demographics of Earth.

In order to fit the global population (with lower salaries) into our model, we would need to reconstruct our annual government budget so that the basic income and education categories were weighted more heavily (smaller percentages of the budget would go in the other categories). In this way, the overall budget would get smaller, and we would not need to take in as much money through taxes. In order to calculate the new budget, we would use this equation:

$$Budget = 100 \times (\frac{\beta \times P - E}{G_B - G_E})$$ (14)

Where $E$ is the cost of education for the given population, $G_B$ is the percentage of the budget that we want to allocate to basic income, and $G_E$ is the percentage of the budget that will be allocated to the education category. Note:

$$G_E = \frac{G_B}{(\frac{\beta \times P}{E})}$$

Also, because of difficulty obtaining data on corporate revenues (U.S. or otherwise), we based our business income tax revenue percentage off of the U.S. tax revenue breakdown (assuming that business and employment revenues would account for 46.2% of our total tax revenues). In essence, this meant that we were combining the high corporate tax rates found in the U.S. with the high personal income tax rates found in more socialist countries like Norway, Denmark and Switzerland. Taking so much in the form of taxes from our citizens and corporations on Mars will be necessary to maintain our lofty goals, but we cannot be sure that it will not negatively impact our citizens’ drive and innovation (despite our investment in education and R&D) if no nation here on Earth has ever asked their people to pay as much. If Martian citizens were to lose their motivation to work as hard and think as creatively, our entire society would suffer. This is a potential weakness in our plan. On the other hand, Switzerland, Denmark and Sweden (all with some of the highest income taxes in the world) appear in the top ten list of the Global Entrepreneurship Index.[17]
5.2 Basic Income Weighting in the Budget

The government budget is heavily weighted in the Basic Income category, moderately in Education, Scientific Research and Development, Government Personnel, and Infrastructure, and lightly in Agricultural Subsidies and Environment, Community and Culture. Our rationale for this budget breakdown (discussed in detail in the Appendix) is based on the ideas that the basic income will help smooth out a lot of Earth society problems by eradicating poverty, that the emphasis on education and innovation will lead to better personal and entrepreneurial decision-making, and that since the government owns all property on Mars, they have a substantial responsibility to maintain it. Since the government budget was built with special regard to the basic income and education expenditures, lowering the percentage of the budget in the basic income category also lowers the percentage in the education category, and raises the entire government budget.

As the budget rises, so does the amount of tax revenue that must be raised. For Population Zero, we capped the top personal tax rate at 60% (the ceiling of all Earth nation income tax rates), which meant that our annual budget could not be too large. We also needed to save enough of our budget to cover the other expenditure categories. Setting our basic income allocation at 60.2% allowed us to fit these constraints. Allowing for higher rates of income tax would have allowed for a lower budget allocation (weighting of government expenditures) to basic income. For the global population, we had to raise the top income tax rate to 75% anyway in order to keep the same budget.

5.3 Happiness and GDP Correlation

Although the fact that our relationship between GDP and Happiness agrees with that shown in the Journal of Economic Perspectives [38] provides strong support for our equations, it is possible that our view of a linear relationship is only a small section of a curve that is not linear in its entirety. However, given the time constraint imposed by this contest and the data input limit of our available graphing capabilities, we were unable to explore this possibility.
5.4 A Minor Assumption

When calculating the education and basic income expenditures, we estimated the number of children born by year 5 in order to create an average budget that could be used during the first ten years of the colony’s existence. To make the model more accurate, we could have calculated the number of children born each year and used a dynamic model for building our government budget.

5.5 Timing of Saving

One last source of error we identified in our equations is that we assumed all savings in our rent equation (being invested in real estate) were pre-tax. At least in the U.S., only some saving - namely, that going into retirement accounts - is pre-tax. However, on Mars, since other expenditures such as education are paid for by the government, this may be close to the reality. If we had accounted for some post-tax savings, our rent revenue would have been slightly smaller, and our tax rates would have been slightly larger to compensate and still fit our budget constraint.

6 Conclusions

We were tasked by the Laboratory of International Financial and Exploration Policy to create a model that will guide policymakers in the creation of a system of government and economy for the colony on Mars that will maximize both the GDP and the happiness, or subjective well-being, of its citizens.

To address this problem, we designed a system centered around the ideas of a basic income for all citizens, an emphasis on education and innovation, and social equality. Our model aims to provide every individual with decent living standards as well as access to quality education, to generate economic and environmental sustainability, and to discourage all forms of discrimination and close the wage gap. It also zeros in on a level of socioeconomic equality that will maximize our citizens’ financial well-being while still encouraging innovation and competition. After optimizing the tax rate equation to fit the constraint of our government budget, we discovered that such a system is economically feasible, when built from scratch. Introducing such a model into an existing government and economy would be more difficult, akin to suddenly converting all of the stoplights in the United States to roundabouts.

In order to maintain this system and achieve our goals, the ICM government will have to play a considerable role in Martian society. However, each individual will have a greater degree of choice due to its involvement. High taxes will allow for the provision of the basic income, which guarantees citizens and their families a plain living and allows them to focus on other aspects of their lives, such as learning, staying healthy, and simply spending time with the ones they love. The additional promise of a higher education regardless of socioeconomic status will also transform the human culture, resulting in deeper and more widespread understanding among our people, and more harmonious interaction as a result. Moving to Mars to escape an approaching comet or the climate-altering feedback cycles that our species has set in motion on Earth may be a last-ditch attempt at survival, but we intend to use this opportunity to establish a society the likes of which Earth never had the chance to appreciate.
6.1 Future Exploration

As the Martian population swells, governing the colony as one large city will be less feasible, so a variation on the current structure may be called for. The planet could exist as one cohesive country made of many states, or as a conglomerate of countries, similar to modern-day Earth. Our model does not presently incorporate multiple levels of government or the bureaucracy necessary to accommodate them. Planning for multiple levels of government on Mars would allow the model to more accurately account for larger amounts of people.

Also, an advanced model might account for the likelihood of more students pursuing higher education, as mentioned in the Extending the Model section. We did not account for this in our model because the records of Earth nations or states that have suddenly started paying for students’ tuition most often report financial trouble and discontinuation of the programs, so it is difficult to say how different the percentages of students pursuing each level of higher education would be on Mars compared to those on Earth. Even in Earth countries where the government is paying for students’ higher education, they do not offer to pay for university for everyone, so the enrollment numbers are still not indicative of what would appear in our Martian society.

Finally, since the colonization of Mars is set to take place in the year 2100, there is no telling how much technology will have progressed. We account for some of the effect of technological innovation by considering researchers’ predictions about what sectors of the economy will be automated by year 2100. Incorporating more predictions about the impacts of advanced health care, transportation and energy technologies on our Martian society would make the model more realistic.
A Appendix

A.1 ICM Budget Compared to Swiss Budget

Table 2. Expenditures of the Swiss Federation according to economic character
(in percentage of total federal expenditure, 2002)

<table>
<thead>
<tr>
<th>Expenditure Category</th>
<th>Percentage</th>
<th>MD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumptive expenditures, of which:</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>Personnel</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>Goods and services other than defense</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Defense</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Direct Investment</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Interest</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Current transfers, of which:</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>Revenue sharing with sub-national government</td>
<td>11.0</td>
<td></td>
</tr>
<tr>
<td>Indemnities to sub-national government</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Contributions to social insurance funds</td>
<td>25.0</td>
<td></td>
</tr>
<tr>
<td>Agricultural subsidies</td>
<td>7.0</td>
<td></td>
</tr>
<tr>
<td>Public transport subsidies</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>Subsidies for vocational training and fundamental research</td>
<td>7.0</td>
<td></td>
</tr>
<tr>
<td>Development aid</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Road maintenance subsidies</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Other subsidies</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Capital transfers</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Source: Swiss Federal Finance Administration.

<table>
<thead>
<tr>
<th>Expenditure Category</th>
<th>Percentage</th>
<th>MD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic Income</td>
<td>60%</td>
<td>291,347,140</td>
</tr>
<tr>
<td>Education</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Government Personnel</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Infrastructure Construction and Maintenance</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Science, R&amp;D</td>
<td>8%</td>
<td>38,749,411</td>
</tr>
<tr>
<td>Agricultural Subsidies</td>
<td>4%</td>
<td>19,374,705</td>
</tr>
<tr>
<td>Environment, Community and Culture</td>
<td>4%</td>
<td>19,374,705</td>
</tr>
<tr>
<td>Total Budget</td>
<td>100%</td>
<td>484,367,648</td>
</tr>
</tbody>
</table>

A.1.1 Explanation

We allocated funds to our other budget categories after ensuring we had enough for basic income and education provision. On Mars, there is no need for defense or development aid to other countries. Government personnel is reduced because there will be less crime (and therefore less need for both law enforcement personnel and prisons) and less bureaucracy with a popular vote and jury-council system. Because plant agriculture will be automated, agricultural subsidies will
be much smaller. Although this is not evident on the Switzerland expenditures chart because they have no specific category for it, Scientific R&D is higher on Mars because of our emphasis on innovation. Infrastructure spending is exactly the same percent as in Switzerland’s budget. Finally, we created our own category for Environment, Community and Culture.

A.2 ICM Taxes Vs. GDP Compared to Earth Nations

![Taxes as a Share of Gross Domestic Product](https://example.com/taxes.png)

**FIGURE 1**
Taxes as a Share of Gross Domestic Product
OECD, 2014

A.3 R&D On Earth

Top 25 nations in terms of gross R&D spending. Note: no country in the top 40 spends more on R&D as a percent of their GDP than Israel (4.15%).
A.4 Pseudo-Code for Optimizing GDP and Happiness
Algorithm 1 Pseudo-Code for Optimizing GDP and Happiness

1: procedure RENT(totalIncome, basicWage)
2: return totalIncome * 0.09 + basicWage * population

3: procedure GetData(people, a, b, wage)
4: \[ GDP = \sum_{i=1}^{P} \frac{(0.9 \times (Y_i - Y_i \times T_i) + \beta + G)}{0.83} \]
5: \[ H = \sum_{i=1}^{P} (\frac{G}{P} + \ln(Y_i - Y_i \times T_i + \beta)) \times (1 - \phi) \]
6: return [happiness, gdp]

7: procedure TestConstraint(people, a, b, GovernmentSpending)
8: totalTax = 0
9: basicWage = 10 \times 40 \times 52
10: totalIncome = 0
11: for person in people do
12: if person.tax(a, b) > 0.75 then
13: return Fail
14: totalTax += person.income \times person.tax(a, b)
15: totalIncome += person.income
16: revenue = (totalTax/0.53) + rent(totalIncome, 10000, basicWage)
17: return revenue \geq GovernmentSpending

18: procedure Optimize(fileName, GovernmentSpending)
19: wages ← salaries from fileName \quad \triangleright \text{wages is an array of salaries}
20: people ← create a Person's Class for each wage and put it into an array
21: Create empty array data
22: Create empty array aAndB
23: for a in range 1 to 6 stepping up by 0.01 do
24: for b in range 0 to 6 stepping up by 0.01 do
25: if TestConstraint(people, a, b, GovernmentSpending) passes then
26: Append GetData(people, a, b, wage) to data
27: Append [a, b] to aAndB
28: zscores ← Calculate Z Scores for 1st Column of data
29: MaxIndex = Max(zscore) \quad \triangleright \text{Max function returns the index of the maximum value}
30: return [a, b]
References


Exploring the Relationship Between Permafrost Degradation and Remotely Sensed Snow Seasonality in the North Slope of Alaska

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Abstract

Permafrost is an extensive part of the northern regions. The upper layer of the permafrost (active layer) thaws and freezes annually. Thawing permafrost can warp infrastructure and release methane bubbles so the active layer depth becomes a strong indicator of climate change. Though research has been done on the behavior of the active layer over time, there has been little research on the effects of snow seasonality on the thawing. Of the research that has been conducted, very few have quantified this relationship at different locations. Since snow acts as an insulator of various extremes (depending on the depth), it potentially influences the thickness of the active layer. The longer the snow remains, the more likely it is that the active layer will be shallower. Project Permafrost used snow cover observations from moderate resolution optical satellite based imagery to determine the length of the snow-free periods across the North Slope of Alaska. Since permafrost occurs below ground and is hard to detect directly using remote sensing, most permafrost research happens using field observations. Project Permafrost combined the satellite images of snow seasonality and active layer thickness at multiple points to assess the correlation between data.

*The author gratefully acknowledges Jeffery Thompson at Earth Lab.
1 Introduction

Permafrost, a significant part of the northern region, is a notable indicator of climate change. While the discontinuous permafrost zone (the area in which the subsurface layer is not entirely frozen) marches north, the continuous zone, becomes less dominant. As speculated by others, the variation in the duration of the snow free period is likely one of the causes of such change.

2 Background

A thick subsurface layer of soil that remains permanently frozen throughout the year, permafrost, and its active layer that experiences annual freeze/thaw cycles can have a particularly large impact on the environment. As the active layer melts and refreezes, it causes damage to infrastructure, construction, and challenges maintenance. [3] Permafrost also stores a large amount of carbon beneath its surface. When organic materials beneath the surface decompose and the permafrost that holds it begins to melt, carbon dioxide and methane could be released. These greenhouse gases have been shown to be contributors to climate change. [4]

Though there has been an extensive amount of research [7] concerning the causes of the fluctuation in the active layer, there has been a gap in the research pertaining to the cause(s) of the changes over time. Many have predicted that the changes in snow seasonality impact active layer dynamics, but very few have quantified this relationship at different locations. [5] It is likely that snow is a cause of the variability because it acts as an insulator. At first the snow slows the rate at which the active layer is melting, but at a certain point begins causing the melt. This presents a challenge. Not only does the snow-free period cause the active layer to melt, but the insulating characteristics of the snow also cause the active layer to melt. [6] Project Permafrost believes there could be a correlation between the duration of the snow free period and the variation of active layer depth over time.

3 Methods

In order to research this problem we chose to investigate the North Slope of Alaska since it is a fairly homogeneous environment and is in the continuous permafrost zone. We began by extracting and cleaning data. The project used Moderate Resolution Imaging Spectroradiometer (MODIS) [2] derived snow metrics to grab data for snow seasonality, and data from the Circumpolar Active Layer Monitoring Network (CALM) [1] for the collection of active layer depth data points. Project Permafrost used primarily R, and some QGIS and MATLAB for this research project. The first task was to crop the original raster stack, obtained from GINA's (Geographic Information Network of Alaska) MODIS satellite, from the entirety of Alaska to the North Slope of Alaska as seen in Figure 1.
Next, it was necessary to extract the most significant layers of the twelve layers contained in the stack. Project Permafrost extracted the start and end of the full snow season, the total number of snow days, the start and end of the longest continuous snow period, and the total number of days in that segment. The full snow season is derived by observing the first snow day and the last snow day and the continuous snow period is described as the longest period of continuous snow; a subset of the full snow period. The snow data was comprised of a time span from 2001-2016.

After extracting the data from those six rasters, the following task was to obtain and import the active layer data. The CALM data spanned a time frame of 1996-2016. There were 42 locations in the North Slope area, out of the many the CALM dataset contained.

Once all datasets had been placed into separate data frames in R, we were able to find the snow seasonality descriptors at each of the active layer points.

4 Analysis and Results

Throughout this investigation, we chose to observe the results in terms of the continuous snow period versus the full snow period. The values for the full snow period and the continuous snow period are very close, duration wise, so rather than using two similar datasets, we used the continuous snow.
Since the duration of the snow free period was unknown, Project Permafrost used two different methods to calculate it. The snow season takes place in parts of two years, so for example, if we were looking at the year 2002, the day count would begin 2001. In this example, the snow seasonality data would start in August 1, 2001 and ends July 31, 2002. The data for 2003 would start August 1, 2002 and end July 31, 2003 and so on. The two methods we used to calculate the snow free period were the "Snow Year" and the "Calendar Year". The "Snow Year" snow free period was found by:

\[
\text{Snow Free Period} = 365 - (\text{last day of snow} - \text{first day of snow})
\]

Equation (1)

The "Calendar Year" snow free period was found with a series of if/else statements:

\[
\begin{align*}
\text{if } & \text{start(year)} < 365 \text{ and end(year-1)} > 365 \\
\text{Snow Free Period} &= \text{start(year)} - (\text{end(year-1)} - 365) \\
\text{if } & \text{start(year)} < 365 \text{ and end(year-1)} < 365 \\
\text{Snow Free Period} &= \text{start(year)} \\
\text{if } & \text{start(year)} > 365 \text{ and end(year-1)} > 365 \\
\text{Snow Free Period} &= 365 - (\text{end(year-1)} - 365) \\
\text{if } & \text{start(year)} > 365 \text{ and end(year-1)} < 365 \\
\text{Snow Free Period} &= 365
\end{align*}
\]

Equation (2)

The value of the start or end of the snow period may be above 365 because their values are the number of days from January 1. Returning to the example above, if we are looking at data for 2002, the lowest value would be 213 (August 1, 2001) and the largest value would be 577 (July 31, 2002).

Choosing the "Snow Calendar" snow free period as the more accurate calculation, we began by comparing our predicted model of active layer depth versus number of snow free days, to the data. Though there appears to be a positive correlation (Figure 3b), it is much weaker than what we predicted (Figure 3a). Our prediction is linear because it is thought that the duration of the snow free period should have a 1:1 relationship with active layer depth. In order to determine our accuracy, we used the R\(^2\) value. It is a statistical measurement of how close the data are to the fitted regression line. The value determines the proportion of the variance in the dependent variable that is predictable by the independent variable. We assumed a linear least squared regression. With these facts in mind, we found that the R\(^2\) value, in Figure 3b, was around 1.37%; far from ideal.
Figure 3b was, also, a test to see if there was potentially a better fit in certain years. However, upon observing each year on separate plots, the $R^2$ value was not much better for any particular year. The values ranged from 0.9% to about 4%. Our prediction was largely exaggerated since a rate of 1 cm per every 10 days would be rather alarming. Though our prediction plot is not entirely realistic, Project Permafrost was expecting a larger correlation in the data than a percent. As it appeared in this plot, the data appeared almost random.

The following idea was to investigate whether there would be a better correlation between a different factor rather than years. We observed whether there was a correlation in active layer depth at each location.

![Predicted and Actual Results](image)

**Fig. 3:** Predicted and actual results.

There was not. In fact, the data appeared to be even more sporadic than when all of the points were viewed at once. Some of the locations, such as Franklin Bluff (Figure 4a), demonstrated a
more positively fitted correlation, though weak. Whereas other locations, such as 56 Mile, were clearly negatively correlated (weakly).

![Map showing Franklin Bluff and 56 Mile](image)

**Fig. 5:** Proximity of above locations: ≈ 1 mile.

The result may have made sense if the locations were not in relatively close proximity of one another. However, some locations, such as Franklin Bluff and 56 mile were around 1 mile apart (Figure 5). The coordinates were around (69.683, -148.717) and (69.697, -148.682) respectively.

Not every location had a complete record so each plot had a varying number of points. Some locations were not able to produce plots because of the missing data. Since the active layer data was collected using various methods (spatially oriented mechanical probing, thaw tubes, inferred from ground temperature), we hypothesized that the results might be better with one method.

![Graph showing active layer vs duration of continuous snow free period](image)

**(a)** Spatial oriented mechanical probing method.

![Graph showing active layer depth vs duration of continuous snow period](image)

**(b)** 1000x1000m grid.

**Fig. 6:** One method and grid size.

As can be seen in Figure 6a, observing data collected using one consistent method does not prove to be a better result. The $R^2$ for the data collected with the probe, is 0.9%. The spatial oriented mechanical probing method takes many sample points over a particular grid, so the data that we used was an average of those points. This could present an error, since some of the data was collected over various grid sizes. Figure 6b shows a a subset of the previous plot. It looks at a 1000x1000 m grid while still using the probing method. Looking at a constant grid size, there is a marginally better $R^2$ value, 3.88%. However, it is still a negative correlation and still poorly matches the regression line.
At this point, we had reduced our original dataset of 42 locations down to 8 locations (probe data). Perhaps there isn’t actually a strong correlation between active layer depth and snow seasonality. Returning to a simple plot of active layer depth versus time (Figure 7), we were able to report a result, though not pertaining to snow seasonality.

Figure 7 took 10 locations at which the active layer data had the most complete record (no missing data over 20 years). Each year on the plot has all 10 active layer depths. There appears to be a weak positive regression line. The dotted blue curves show the confidence interval of the regression line, and the dotted black lines are the prediction interval. Since the regression line is in such a spot within the confidence interval, there could be a positive or a negative trend in the data depending on how the regression line is tilted in the interval.

5 Conclusion

Since the slope is slightly positive in the plot of active layer depth over time, it appears that the active layer is getting deeper over time. Specifically, it is deepening at a rate of about 1.47 cm per decade. However, it could also not be changing at all depending on how the regression line is positioned within the confidence interval. Perhaps a time span of 20 years is not enough time to investigate. Some potential sources of error in this investigation could be due to factors in the active layer data as well as factors within the snow data. The active layer data is missing values in many locations and years. In particular, the depths that were inferred from ground temperature and the earlier years (before 2000) are lacking a majority of their data. It is unknown whether the data was taken at the same time each year. The documentation states that the points were collected at the end of the thaw period, however there is no documented date. Another potential error, was that the data was taken using different methods and over varying grid sizes. The data that was taken using the probe method took the average of all points collected within varying grid sizes whereas the remainder of the data was simply inferred from ground temperature. The snow seasonality imagery data could be misinterpreting clouds as snow. The MODIS satellite uses moderate resolution imagery so it is unknown whether there is confusion between the two. It is also unknown whether the imagery was compared to ground based data. The ground based
snow period could be longer than what the satellite detects depending on the resolution of the imagery. For example, the satellite may not detect snow at the end of the snow season when the snow is nearly completely melted due to its resolution. From the documentation, it is unknown whether there was filtering for view angle. As the satellite orbits the Earth, it views the same location at slightly different angles daily. If there is no filtering for view angle the pixel sizes in the images can vary by a factor of 4.

So, what is next? Project Permafrost hopes to investigate the discontinuous permafrost region as opposed to the continuous zone that was investigated in this paper. The discontinuous zone is said to be the one that is demonstrating the greater change, however, it will be more difficult to research since the region is no longer homogeneous and contains more vegetation.

6 References


7. Li, Zhiwei and Zhao, Rong et al. 2015. "InSAR analysis of surface deformation over permafrost to estimate active layer thickness based on one-dimensional heat transfer model of soils". Retrieved December 2017. (https://www.nature.com/articles/srep15542)