Physics 7440 Lecture Notes Week 14

Quantum Hall effect
(based on http://www.pha.jhu.edu/~qiuym/qhe/)

Basics of Hall Effect

In the Drude theory of the electrical conductivity of a metal, an electron is accelerated by the electric field for an average time $\tau$, the relaxation or mean free time, before being scattered by impurities, lattice imperfections and phonons to a state which has average velocity zero. The average drift velocity of the electron is

$$\mathbf{v}_d = -\frac{e\mathbf{E}}{m} \tag{1}$$

where $\mathbf{E}$ is the electric field and $m$ is the electron mass. The current density is thus

$$\mathbf{j} = -ne\mathbf{v}_d = \sigma_0 \mathbf{E} \tag{2}$$

where

$$\sigma_0 = ne^2 \tau / m \tag{3}$$

and $n$ is the electron density.

In the presence of a steady magnetic field, the conductivity and resistivity become tensors

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \tag{4}$$

and

$$\mathbf{j} = \sigma \cdot \mathbf{E} = \rho \cdot \mathbf{j}.$$  

Still assuming that the relaxation time is $\tau$, the Lorentz force must be added to the force from the electric field in Eq. (1),

$$\mathbf{v}_d = -e \left( \frac{\mathbf{E} + \mathbf{v}_d \times \mathbf{B}}{c} \right) \frac{\tau}{m} \tag{5}$$

In the steady state, $\mathbf{j} = 0$. We will always assume that the magnetic field is in $z$ direction. Then in $xy$ plane

$$\sigma_0 \mathbf{E}_x = \omega_0 \tau j_y + j_x$$
$$\sigma_0 \mathbf{E}_y = -\omega_0 \tau j_x + j_y \tag{6}$$

where $\omega_0$ is defined in Eq. (3),

$$\omega_0 = \frac{eB}{mc} \tag{7}$$

is the cyclotron frequency. From Eq. (6), we can easily get

$$\rho_{xx} = \rho_{yy} = 1 / \sigma_0, \quad \rho_{xy} = -\rho_{yx} = \omega_0 \tau / \sigma_0$$

$$\sigma_{xx} = \sigma_{yy} = \frac{\epsilon_0}{1 + (\omega_0 \tau)^2}, \quad \sigma_{xy} = -\sigma_{yx} = -\frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2} \tag{8}$$

Eqs. (8) directly leads to the relation between conductivity and resistivity

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}, \quad \sigma_{yy} = -\frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \tag{9}$$

We can see that if $\rho_{xy} \neq 0$, the conductivity $\sigma_{xx}$ vanishes when the resistivity $\rho_{xx}$ vanishes. On the other hand,

$$\sigma_{xy} = -\frac{ne^2 \tau}{B} + \frac{\sigma_{xx}}{\omega_0 \tau} \tag{10}$$
Therefore when \( \sigma_{xx} = 0, \sigma_{xy} = \sigma_{xy} E_y \), where \( \sigma_{xy} \) is given by the first term in Eq. (10), i.e. Hall conductivity

\[
\sigma_H = \sigma_{xy} = -\frac{ne^2}{B}
\]

(11)

In the experiment we can let \( E_y = 0 \),

\[
\sigma_{yx} = \frac{j_y}{E_x} = -\sigma_H
\]

(12)

The above discussion is the classical result. In quantum mechanics, the Hamiltonian is (\( E \) is along x direction)

\[
\mathcal{H} = \frac{1}{2m} \left( \mathbf{p} + \frac{eA}{c} \right)^2 + eEz
\]

(13)

For this problem it is convenient to choose the Landau gauge, in which the vector potential is independent of \( y \) coordinate

\[
\mathbf{A} = (0, Bz, 0)
\]

(14)

This choice allows us to choose a wavefunction which has a plane-wave dependence on the \( y \) coordinate

\[
\psi(x, y) = e^{-i\mathbf{k}_0 \cdot \mathbf{y}} \phi(x)
\]

(15)

Substituting Eq. (15) into Eq. (13), the Schrödinger equation becomes

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 (x - l_0^2 k_y)^2 + eEz \right] \phi(x) = \mathcal{E} \phi(x)
\]

(16)

where

\[
l_0 = \left( \frac{\hbar c}{eB} \right)^{\frac{1}{2}}
\]

(17)

is the classical cyclotron orbit radius.

Eq. (16) can be easily solved by transformation to a familiar harmonic oscillator equation. The eigenvalues and eigenstates are

\[
\mathcal{E}_i(\mathcal{B}) = (i + 1/2) \hbar \omega + eB(l_0^2 k_y - e\xi / 2m \omega^2)
\]

\[
\psi_i(x, y) = e^{-(x - x_0)^2 / 2l_0^2} \exp[-(y - y_0)^2 / 2l_y^2] \frac{H_i[(x - x_0)/l_0]}{l_0}
\]

(18)

where \( i = 0, 1, 2, 3, \ldots \). The different oscillator levels are also called Landau Levels. The electric field simply shifts the eigenvalues by a value without changing the structure of the energy spectrum. From Figure 1 we can see that in two-dimensional systems, the Landau energy levels are completely separate while in three-dimensional systems the spectrum is continuous due to the free movement of electrons in the direction of the magnetic field.

From the wave functions, we can calculate the mean value of the velocities

\[
< \mathbf{v}_y > = \frac{1}{\alpha} \int \psi_i^* \left( \frac{\partial}{\partial y} + \frac{eEz}{\alpha} \right) \psi_i \, dx = -Be/B
\]

\[
< \mathbf{v}_x > = \frac{1}{\alpha} \int \psi_i^* \frac{\partial}{\partial x} \psi_i \, dx = 0
\]

(19)
Thus $j_x = -neEc/B$, which is the same as Eq. (11) of the classical result. The current along the direction of electric field ($x$) is zero at Landau levels.

Fig. 1 Schematic diagram of the density of states of two and three-dimensional electron systems

**Two Dimensional Electron Systems**

At present, there are mainly three types of 2-D electron systems acheived in experiment

- MOSFET-Metal Oxide Semiconductor Field Effect Transistor
- Superlattice
- Liquid Helium Surface
In MOSFET, inversion layers are formed at the interface between a semiconductor and an insulator or between two semiconductors, with one of them acting as an insulator. The system in which the Quantum Hall Effect (QHE) was discovered has Si for the semiconductor, SiO$_2$ for the insulator. Figure 2 is a schematic side view of a silicon MOSFET showing the aluminum gate, the SiO$_2$ insulator and the p-type Si crystal substrate. The principle of the inversion layer is quite simple. It is arranged that an electric field perpendicular to the interface attracts electrons from the semiconductor to it. These electrons sit in a quantum well created by this field and the interface. The motion perpendicular to the interface is quantized and thus has a fundamental rigidity which freezes out motional degrees of freedom in this direction. The result is a two-dimensional system of electrons. Further, the wavelengths of these electrons are long so that an effective mass approximation with parabolic bands is quite good. The total self-consistent potential seen by the electrons is conveniently described by the picture of "band bending". That is to say, the periodic lattice potential gives rise to energy bands, and the slowly varying electric potential then is regarded as bending these bands. Figure 3 gives the schematic diagram for this process. (Charge density $\sim 10^{13}$ cm$^{-2}$)

Another type of two-dimensional electron system is formed in the heterostructures of two semiconductors. Using molecular beam epitaxy (MBE) technique, people can grow two semiconductors alternately to form a one dimensional sandwich like structure. Each layer has a width of about several nanometers. They are called superlattice. They can also be grown by metal-organic chemical vapor deposition (MOCVD). For example, in GaAs-Ga$_{1-x}$Al$_x$As superlattice, a certain controlled number of layers of GaAs is followed by an almost perfectly matched sequence of layers of GaAlAs. The GaAlAs is deliberately doped n-type, which puts mobile electrons into its conduction band. These electrons will migrate to fill the few holes on the top of the GaAs valence band but most of them will end up in states near the bottom of the GaAs conduction band. However, there is a
positive charge left on the donor impurities which attracts these electrons to the interface and bends the bands in the process. This is the source of the electric field in this system. The transfer of electrons from GaAlAs to GaAs will continue until the dipole layer formed from the positive donors and the negative inversion layer is sufficiently strong. This dipole layer gives rise to a potential discontinuity which finally makes the Fermi level of the GaAs equal to that of the GaAlAs. Figure 4 shows the band structure. (Charge density \(\sim 10^{11} \text{cm}^{-2}\) )

![Fig.4 Electron energy level diagram of a GaAs-AlGaAs heterostructure device](image)

Two-dimensional electron system can also be formed in the surface of liquid Helium. There exists a potential barrier of about 1eV in the surface of liquid Helium which prevents electrons from transmitting into the liquid. On the other hand, the mirror potential attracts the electrons in the surface, resulting a 2D electron system. (Charge density \(\sim 10^9 \text{cm}^{-2}\))

Quantum Hall effect has been observed in the first two types.

**The Integer Quantum Hall Effect**

In our previous discussion, we deal with resistivities and conductivities, but note that the conductance and resistance are the fundamental quantities of interest both experimentally and theoretically. If it were the conductivity rather than the conductance which were quantized then precision measurements would be impossible since one would have to invoke assumptions of a homogeneous medium with a well-defined geometry in order to infer the microscopic conductivity from the macroscopic conductance. It is a remarkable feature of the QHE that this is not necessary.

According to Streda, whenever the Fermi level lies in a gap the Hall conductance will be given by
For a two dimensional system, the density of states at the absence of magnetic field is $g(E)=\frac{m}{2}\pi\hbar^2$. After applying a magnetic field, the energy states contract into separate Landau levels. Each Landau level is degenerate, including states. If the electrons completely occupy all the $i$ levels-leaving all other levels empty, then the charge density

$$n = \frac{i e B}{\hbar c}$$  \hspace{1cm} (21)

Eq. (20) and Eq. (21) may be combined to yield

$$R_H(i) = 1/G_H = \frac{h}{e^2 i}$$  \hspace{1cm} (22)

Note that the Hall resistance is actually inversely proportional to the charge density. Eq. (22) is only correct in certain specific $n$ values. For the inversion layer of Si-MOSFET, $n$ is proportional to the gate voltage $V_G$. So Hall resistance should be inverse proportional to the gate voltage. But in 1980, K. von Klitzing etc. discovered quantized Hall plateaus. Using a Hall voltage method suitable for precision measurements they obtained good steps in high-mobility Si-MOSFET devices and found that $R_H(i)=h/e^2 i$ to an accuracy of at least 5 parts-per-million(ppm). Figure 6 is the experimental setup.

![Fig.5 IQHE observed in Si-MOSFET](image)
The experiments show that between two adjacent Landau levels, the Hall resistance has fixed values and the longitudinal resistance $R_{xx}$ vanishes, which means that the electrons are localized in this region. Localization is a key point to interpret IQHE.

Due to impurity, the density of states will evolve from sharp Landau levels to a broader spectrum of levels (Figure 7). There are two kinds of levels, localized and extended, in the new spectrum, and it is expected that the extended states occupy a core near the original Landau level energy while the localized states are more spread out in energy. Only the extended states can carry current at zero temperature. Therefore, if the occupation of the extended states does not change, neither will the current change. An argument due to Laughlin (1981) and Halperin (1982) shows that extended states indeed exist at the cores of the Landau levels and if these states are full, (i.e., the Fermi level is not in the core of extended states) then they carry exactly the right current to give Eq. (22).
The existence of the localized states can explain the appearance of plateaus. As the density is increased (or the magnetic field is decreased) the localized states gradually fill up without any change in occupation of the extended states, thus without any change in the Hall resistance. For these densities the Hall resistance is on a step in the Figure and the longitudinal resistance vanishes (at zero temperature). It is only as the Fermi level passes through the core of extended states that the longitudinal resistance becomes appreciable and the Hall resistance makes its transition from one plateau step to the next.

Finally, at finite temperature there is a small longitudinal resistance due to hopping processes between localized states at the Fermi level.

**Tunneling between bands induced by applied electric or magnetic fields**

Applying a time-independent electric or magnetic field will cause electrons to move along trajectories within bands. When the electric field is applied to an electron at the bottom of a band, it will keep accelerating until the zone boundary where it will be Bragg-reflected by the lattice. Then it will be forced to decelerate until the bottom of the band, etc. (see Fig. 113 in Ziman and the corresponding discussion) Under the influence of the magnetic field the electron will also stay in the same band moving along constant-energy contours.

However, this is not the full picture, because if the fields are large enough, tunneling between bands becomes possible, i.e. electron can jump across band gaps.

When the electron reaches the zone boundary under the influence of the electric field it will be Bragg reflected and start moving in the opposite direction as the entire crystal.
recoils. This process is equivalent to the electron bouncing off a potential barrier. In this case the wavefunction will extend into the barrier region, but instead of a plane wave-like solution it will be decaying exponentially. In such a case, the electron can penetrate some distance d into the forbidden region. If the applied field is such that \( d = \frac{\epsilon_{\text{gap}}}{eE} \), then it will be possible for the electron to actually tunnel across the bandgap into the upper band. If the field is applied along x, then in the forbidden region the wavefunction will have the form \( \psi(x) = \psi_0(x)e^{-\beta x} \). Based on the W.K.B. method, the transition probability is

\[
P = \exp\left(-2 \int_{x_1}^{x_2} \beta(x) dx \right).
\]

Here the region between \( x_1 \) and \( x_2 \) is the “forbidden” region. To see under which circumstances this tunneling can occur we need to find \( \beta \).

We first write down the Schrödindger equation for an electron in a periodic potential as we did in Week 7 for the nonperturbative case:

\[
\left( \frac{\hbar^2 K^2}{2m} + \sum_U U e^{i g r} \right) \psi_k = \epsilon_k \psi_k.
\]

and try the solution of the form: \( \langle r | \psi_k \rangle = \sum_g \alpha_g e^{i(k-g)r} \)

When \( k \) is in the vicinity of the zone boundary with the wavevector \( \frac{1}{2} \mathbf{G} = \pi/a \), we can consider only the Fourier component of the periodic potential. Then as before we get a system with 2 equations:

\[
\begin{align*}
(\epsilon_k^0 - \epsilon_k) \alpha_k + U_G \alpha_{k-G} &= 0; \\
(\epsilon_{k-G}^0 - \epsilon_k) \alpha_{k-G} + U_{-G} \alpha_k &= 0;
\end{align*}
\]

which is equivalent to:

\[
\begin{pmatrix}
(\epsilon_k^0 - \epsilon_k) & U_G \\
U_{-G} & (\epsilon_{k-G}^0 - \epsilon_k)
\end{pmatrix}
\begin{pmatrix}
\alpha_k \\
\alpha_{k-G}
\end{pmatrix}
= 0;
\]

We now substitute in:

\[
\epsilon_k^0 = \frac{\hbar^2}{2m} k^2
\]

and write down the equation for the determinant of the matrix equal to zero:

\[
\left( \frac{\hbar^2}{2m} k^2 - \epsilon \right) \left( \frac{\hbar^2}{2m} (k-G)^2 - \epsilon \right) = |U_G|^2
\]

We can write \( k = \frac{1}{2} \mathbf{G} + \kappa \), \( \epsilon = \epsilon_0 + \epsilon_1 \) with \( \epsilon_0 = \frac{\hbar^2 G^2}{2m} 4 \).
Then: 
\[
\left\{ \frac{\hbar^2}{2m} \left( \frac{1}{2}G + \kappa \right)^2 - \frac{\hbar^2 G^2}{2m 4} - \varepsilon_i \right\} \left\{ \frac{\hbar^2}{2m} \left( \kappa - \frac{1}{2}G \right)^2 - \frac{\hbar^2 G^2}{2m 4} - \varepsilon_i \right\} = \left| U_G \right|^2
\]

For \( k \) near the zone boundary \( \frac{1}{2}G \gg \kappa \) and we can write

\[
\left( \frac{\hbar^2 \kappa G}{2m} - \varepsilon_i \right) \left( \frac{\hbar^2 \kappa G}{2m} + \frac{\hbar^2 \kappa^2}{2m 4} - \varepsilon_i \right) = \left| U_G \right|^2
\]

This expression tells us how the electronic wavefunction decays inside the forbidden region. If the electron energy increases linearly with how far it travels into the forbidden region due to the electric field, \( \varepsilon_i \) will be proportional to distance \( x \) from the center of the potential barrier whose total width is \( d \).

Then you will show in the homework that:

\[
\kappa^2 = \frac{2m \varepsilon_i^2 - \left| U_G \right|^2}{\hbar^2 4\varepsilon_0}
\]

and \( P = \exp \left( \frac{\pi^2 \varepsilon^2_{gap}}{4 \varepsilon_0 e E a} \right) \)

Thus the probability for tunneling into the upper band is appreciable if the electric field is so strong that the electron can gain the fraction \( \varepsilon_{gap}/\varepsilon_0 \) over a distance comparable with 1 lattice spacing. This phenomenon is called Zener breakdown. A similar effect is observed in semiconductor tunnel junctions.

A similar phenomenon called magnetic breakdown can occur in high magnetic fields. To understand how it works we can start from high magnetic fields and treat the lattice potential as a small perturbation. Free electrons will always be confined to closed circular orbits, but when the gap opens up at the Fermi level, the electron cannot “jump” across the gap without gaining energy, so it gets reflected at the zone boundary and continues along an open orbit (see Ziman Fig. 174).

We know that when a electron moves in a magnetic field, it actually feels the effect of an electric field derived from the magnetic field due to a Lotentz transformation. This
Electric field is: \( E = \frac{vH}{c} \). It can cause tunneling in the same way as discussed above if the condition \( \frac{eEaE_0}{\varepsilon^2_{gap}} > 1 \).

Since \( v \approx \frac{\hbar k_F}{m} \), the corresponding condition on the magnetic field is:

\[
\frac{evHaE_0}{c\varepsilon^2_{gap}} = \frac{e\hbar vHa aE_F}{mc\varepsilon^2_{gap}} > 1,
\]

since \( \omega = \frac{eH}{mc} \) and \( k_F a \) is on the order of 1, the condition for magnetic breakdown becomes:

\[
\frac{\hbar \omega H a E_F}{\varepsilon^2_{gap}} > 1.
\]

As a result, orbits can change as the field \( H \) increases leading to changes in the quantum oscillations in the de Haas-van Alfen effect.

### Magnetic susceptibility

Applying magnetic field to a material will induce magnetization in the material. Magnetic susceptibility \( \chi \) is given by: \( M = \chi H \), where \( \chi \) is a function of \( T, H \), relative orientation of the sample and \( H \), etc.

There are two principal contributions to \( \chi \): from the magnetization due to orbital angular momentum and due to the electronic spins.

**Diamagnetism vs. paramagnetism:** Diamagnetism, a tendency to cancel out the applied field, occurs when application of the magnetic field lowers the energy of the system (which includes the interaction with the field). Paramagnetism, a tendency to enhance the applied field, occurs when the energy is lowered.

### Orbital diamagnetism

Orbital diamagnetism can be derived from the kinetic energy term in the Hamiltonian:

\[
T = \frac{1}{2m} \left( i\hbar \nabla + \frac{e}{c} A \right)^2 = \frac{ie\hbar}{2mc} (\nabla \cdot A + A \cdot \nabla) + \frac{e^2}{2mc^2} A^2
\]

For the magnetic field along \( z \): \( A_x = -\frac{1}{2} yH \), \( A_y = -\frac{1}{2} xH \), \( A_z = 0 \),

\[
T = \frac{ie\hbar}{2mc} \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right) + \frac{e^2H^2}{8mc^2} (x^2 + y^2);
\]

The first term is the orbital angular momentum operator and gives rise to paramagnetism that will be discussed below. The second term is diamagnetic and gives a contribution for a spherically symmetric system:

\[
\varepsilon' = \left< 0 \left| \frac{e^2H^2}{8mc^2} (x^2 + y^2) \right| 0 \right> = \frac{e^2H^2}{12mc^2} < r^2 >
\]

The associated magnetic moment is then given by \( \mu = -\frac{\partial \varepsilon'}{\partial H} = -\frac{e^2H}{6mc^2} < r^2 > \), and...
\[ M = \frac{Ze^2HN}{6mc^2} \langle r^2 \rangle, \text{ i.e. } \chi = \frac{Ze^2N}{6mc^2} \langle r^2 \rangle \]

This is the expression for diamagnetism of insulators with all electron shells closed. It is independent of temperature, because the probability of an excited state is extremely low.

To calculate the susceptibility in metals we look at the first term. Note that in the cases when all shells are full, the total orbital angular momentum is zero in the ground state. If there are excited states, then it can give a contribution in second order in perturbation theory:

\[ \frac{ie\hbar H}{2mc} \left( \frac{x}{\partial y} - \frac{y}{\partial x} \right) = \frac{ie\hbar \cdot L}{2mc} \quad \text{and} \]

\[ \varepsilon' = \left( \frac{e}{2mc} \right)^2 \sum \frac{|\langle 0 | H \cdot L | n \rangle|^2}{\varepsilon_0 - \varepsilon_n} = \left( \frac{eH}{2mc} \right)^2 \sum \frac{|\langle 0 | L_z | n \rangle|^2}{\varepsilon_0 - \varepsilon_n} \]  
(Note that in the case of atoms with all shells filled the denominator is too large for this term to make a significant contribution.)

Note that \( \varepsilon' < 0 \) since \( \varepsilon_n > \varepsilon_0 \).

This term has the opposite sign of the diamagnetic term, thus it favors paramagnetism.

Applying this Hamiltonian to real solids is rather complicated, but it is possible to obtain a result for a free electron gas. The result based on thermodynamics of free electrons is Landau diamagnetism:

\[ \chi_L = -\frac{e^2k_F}{12\pi^2mc^2} \]

The reason is that magnetic field increases the energy of the system, because electrons get “sucked into” Landau levels.

It is easy to include the spin of electrons. All one has to do is to as a term proportional to \( H \cdot S \) into the Hamiltonian. Its effect in the case of free electrons is to produce spin paramagnetism with the Fermi energy for the spin up electrons different from the spin down electrons. The net result is \( M = \mu_o^eHN(\varepsilon_F) \) and \( \chi_p = \mu_o^eN(\varepsilon_F) = \frac{e^2k_F}{4\pi^2mc^2} \), (where \( P \) stands for Pauli paramagnetism)