Stirling’s formula, \( n \)-spheres and the Gamma Function

We start by noticing that
\[
\int_0^\infty x^n e^{-x} \, dx = \lim_{a \to 1} (-1)^n \frac{\partial^n}{\partial a^n} \int_0^\infty e^{-ax} \, dx = \lim_{a \to 1} (-1)^n \frac{\partial^n}{\partial a^n} a^{-1}
\]
and hence
\[
n! = \int_0^\infty x^n e^{-x} \, dx \tag{1}
\]
Let us make a remark in passing. Note that in general the gamma function is defined by
\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \tag{2}
\]
and hence a simple integration by parts (with \( u = t^x \) and \( dv = e^{-t} \, dt \)) shows that
\[
\Gamma(x+1) = x \Gamma(x) \tag{3}
\]
Since it is also easy to see that \( \Gamma(1) = 1 \), it follows that equation (1) is just
\[
n! = \Gamma(n+1).
\]
We will also have use for the basic result
\[
\int_{-\infty}^\infty e^{-ax^2} \, dx = \left( \frac{\pi}{a} \right)^{1/2} \tag{4}
\]
(To prove this, multiply the integral by itself (letting \( x \to y \)), resulting in a double integral with integrand \( e^{-a(x^2+y^2)} \). Change to polar coordinates with \( dx \, dy \to r \, dr \, d\theta \), and integrate \( r \) from 0 to \( \infty \), and \( \theta \) from 0 to \( 2\pi \).)
Define
\[
F = x^n e^{-x}
\]
so that \( \ln F = n \ln x - x \). Then \( \ln F \) has a maximum at \( x_0 \) given by
\[
0 = \frac{d \ln F}{dx} \bigg|_{x=x_0} = \frac{n}{x} - 1
\]
so that \( x_0 = n \). Expanding \( x \) in a neighborhood of \( n \) we write \( x = n + \xi = n(1 + \xi/n) \) where \( \xi \ll n \). Using \( \ln(1+\xi) = \xi + \xi^2/2! + \cdots \) we have
\[
\begin{align*}
\ln F &= n \ln n(1 + \xi/n) - n(1 + \xi/n) \\
&= n \ln n + n \left( \frac{\xi}{n} - \frac{1}{2} \frac{\xi^2}{n^2} \right) - n - \xi \\
&= n \ln n - n - \frac{1}{2} \frac{\xi^2}{n}
\end{align*}
\]
and therefore

\[ F = n^n e^{-n} e^{-\xi^2/2n} \]  

(5)

Observe that if \( n \) is very large, then \( \sqrt{n} \ll n \). For example, if \( n = 10^{24} \), then \( \sqrt{n} = 10^{12} \ll n \). Also, if say, \( \xi = 10^{16} \), then \( \xi \ll n \) while \( \xi \gg \sqrt{n} \) (or alternatively, \( \xi^2 = 10^{32} \gg n \)). Since it is also obvious that \( F \) has a maximum at \( \xi = 0 \) and is very small if \( |\xi| \gg \sqrt{n} \), this max is very sharp. As an example, if \( n \) is only equal to 100,

\[ F(\xi = 0) = n^n e^{-n} = 3.7 \times 10^{156} \]

while letting \( \xi = 50 \gg \sqrt{100} \) gives a damping factor of

\[ e^{-\xi^2/2n} \approx e^{-25} = 3.7 \times 10^{-6} \].

And for typical numbers like \( n = 10^{24} \) and \( \xi = 10^{16} \), this damping factor is on the order of the fantastically small number \( e^{-10^{24}/2} = 8.03 \times 10^{-21714725} \).

We now want to use (5) in (1). We first change variables in (1) from \( x \) to \( \xi = x - n \) so that the integral becomes

\[ \int_{0}^{\infty} dx = \int_{-n}^{\infty} d\xi \approx \int_{-\infty}^{\infty} d\xi \]

since if \( \xi < -n \), then \( F \) is very small. (Again, if \( n = 10^{24} = \xi \), then \( e^{-\xi^2/2n} = e^{-10^{24}/2} \ll 1 \).) Therefore, equation (1) becomes

\[ n! \cong \int_{-\infty}^{\infty} n^n e^{-n} e^{-\xi^2/2n} d\xi = n^n e^{-n} \int_{-\infty}^{\infty} e^{-\xi^2/2n} d\xi \]  

(6)

or, using (4)

\[ n! \cong \sqrt{2\pi n} n^n e^{-n} \]  

(7)

Taking the logarithm yields the formula

\[ \ln n! = n \ln n - n + \frac{1}{2} \ln 2\pi n \]  

(8)

Finally, we note that for \( n \) large, \( \ln 2\pi n \ll n \), and hence the last term in (8) is completely negligible. (If \( n = 10^{24} \), then \( \ln n \simeq 55 \).) We are then left with the usual form of Stirling’s equation

\[ \ln n! \cong n \ln n - n \]  

(9)

As a measure of the accuracy of this formula, let us look at a comparison of equations (8) and (9) with the exact result:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \ln n! )</th>
<th>( n \ln n - n + (1/2) \ln 2\pi n )</th>
<th>( n \ln n - n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15.1044</td>
<td>15.0961</td>
<td>15.0259</td>
</tr>
<tr>
<td>( 10^2 )</td>
<td>363.739</td>
<td>363.739</td>
<td>360.517</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>5912.13</td>
<td>5912.13</td>
<td>5907.76</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>( 1.28155 \times 10^7 )</td>
<td>( 1.28155 \times 10^7 )</td>
<td>( 1.28155 \times 10^7 )</td>
</tr>
</tbody>
</table>

It is clear that for the numbers of interest in statistical mechanics, equation (9) is quite accurate.

2
As another application of equations (2) and (4), we derive an equation for the volume of an \( n \)-dimensional sphere of radius \( R \).

First of all, since \( \exp\left(-\sum_{i=1}^{n} x_i^2\right) = \prod_{i=1}^{n} \exp(-x_i^2) \), it is clear from (4) that

\[
I_n = \int_{-\infty}^{\infty} \exp\left(-\sum_{i=1}^{n} x_i^2\right) dx_1 \cdots dx_n = \pi^{n/2}.
\]

Note that \( dx_1 \cdots dx_n \) is the volume element \( dV_n \) in the cartesian coordinates for \( \mathbb{R}^n \). We now want to change to spherical coordinates where \( x_1^2 + \cdots + x_n^2 = R^2 \). Since an \( n \)-dimensional volume must go like \( R^n \), we can write \( V_n = C_n R^n \) for some constant \( C_n \), and hence \( dV_n = d(C_n R^n) = nC_n R^n-1 dR \). Then

\[
I_n = \int_{-\infty}^{\infty} \exp(-R^2) dV_n = nC_n \int_{-\infty}^{\infty} e^{-R^2} R^{n-1} dR.
\]

Now let \( t = R^2 \) so that \( dt = 2R dR \) or \( dR = \frac{1}{2} t^{1/2} dt \). Using equations (2) and (3) this gives us

\[
I_n = \frac{n}{2} \int_{-\infty}^{\infty} e^{-t(n-1)/2} t^{-1/2} dt
\]

\[
= C_n \frac{n}{2} \int_{-\infty}^{\infty} e^{-t/2} t^{-1} dt
\]

\[
= C_n \frac{n}{2} \Gamma\left(\frac{n}{2}\right)
\]

\[
= C_n \Gamma\left(\frac{n}{2} + 1\right).
\]

But we already showed that \( I_n = \pi^{n/2} \) and therefore \( C_n = \pi^{n/2}/\Gamma\left(\frac{n}{2} + 1\right) \) so that the volume of the \( n \)-sphere is given by

\[
V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n \quad (10)
\]

Let us write this in another form. If we let \( t = u^2 \) in (2) we find another useful representation for the gamma function:

\[
\Gamma(x) = \int_0^\infty e^{-u^2} u^{2x-2} 2u du = 2 \int_0^\infty e^{-u^2} u^{2x-1} du
\]

or

\[
\Gamma(x) = 2 \int_0^\infty e^{-t^2} t^{2x-1} dt \quad (11)
\]

From equation (4) we clearly have (since the integrand is symmetric)

\[
\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \left(\frac{\pi}{a}\right)^{1/2}
\]
and hence $\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$.

If $n$ is even, then $\Gamma\left(\frac{n}{2} + 1\right)$ isn’t a problem. But if $n$ is odd, then we observe that

$$\Gamma\left(\frac{n}{2} + 1\right) = \frac{n}{2} \Gamma\left(\frac{n}{2}\right) = \frac{n}{2} \left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2} - 1\right)$$

$$\Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2} - 2\right) = \ldots$$

$$\Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2} - 2\right) \cdots \left(\frac{n}{2} - \frac{n-1}{2}\right) \Gamma\left(\frac{1}{2}\right)$$

Note there are $(n-1)/2 + 1 = (n+1)/2$ terms in the product, and hence we have

$$\Gamma\left(\frac{n}{2} + 1\right) = \frac{n!!}{2^{n-1} \sqrt{n}} \quad \text{for } n \text{ odd} \quad (12)$$

where the double factorial is defined by $n!! = n(n-2)(n-4) \cdots (1)$.

Using equation (12), we can now rewrite equation (10) for $n$ odd as

$$V_n = \frac{2^{n+1} \pi^{n-1}}{n!!} R^n.$$

The double factorial can be written in another form as follows:

$$n!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdots (n-4) \cdot (n-2) \cdot n$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdots (n-4) \cdot (n-3) \cdot (n-2) \cdot (n-1) \cdot n}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (n-3) \cdot (n-1)}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdots (n-4) \cdot (n-3) \cdot (n-2) \cdot (n-1) \cdot n}{(2 \cdot 1)(2 \cdot 2)(2 \cdot 3)(2 \cdot 4) \cdots (2 \cdot \frac{n-3}{2})(2 \cdot \frac{n-1}{2})}$$

$$= \frac{n!}{2^{n-1} (n-1)!}.$$

Then we also have (for $n$ odd)

$$V_n = \frac{2^n \pi^{n-1} (\frac{n-1}{2})!}{n!} R^n.$$

Finally, as a general comment, for any nonnegative integer $m$ we note that $2m+1$ is odd and hence

$$\frac{(2m+1)!!}{2^m m!} = m = 1, 2, 3, \ldots$$
As another application of these techniques, we evaluate the so-called Bose-Einstein integral

\[ I_\nu(z) = \int_0^\infty \frac{x^\nu}{z^{-1}e^x - 1} \, dx \]  

(13)

where \( 0 \leq z \leq 1 \) and \( \nu \in \mathbb{R} \).

Recall that if \( w \in (0, 1) \) then

\[
\frac{1}{1 - w} = 1 + w + w^2 + \cdots = \sum_{n=0}^{\infty} w^n.
\]

We have for the integrand of equation (13)

\[
x^\nu z^{-1}e^{-x} - 1 = x^\nu z e^{-x} \sum_{n=0}^{\infty} (ze^{-x})^n = x^\nu \sum_{n=1}^{\infty} z^n e^{-nx}
\]

and hence the integral becomes

\[
\int_0^\infty \frac{x^\nu}{z^{-1}e^x - 1} \, dx = \int_0^\infty x^\nu \sum_{n=1}^{\infty} z^n e^{-nx} \, dx = \sum_{n=1}^{\infty} z^n \int_0^\infty x^\nu e^{-nx} \, dx.
\]

Now let \( t = nx \) so that \( dx = dt/n \) and \( x^\nu = t^\nu/n^\nu \), and therefore

\[
I_\nu(z) = \int_0^\infty \frac{x^\nu}{z^{-1}e^x - 1} \, dx = \sum_{n=1}^{\infty} \frac{z^n}{n^{\nu+1}} \int_0^\infty t^\nu e^{-t} \, dt.
\]

From equation (2) we see that the integral is just \( \Gamma(\nu + 1) \), and the sum we will denote by \( g_{\nu+1}(z) \), i.e.,

\[
g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{\nu}} \quad 0 \leq z \leq 1.
\]

In the particular case that \( z = 1 \) we have \( g_\nu(1) = \zeta(\nu) \), the Riemann zeta function.

In any case, we have

\[
I_\nu(z) = \int_0^\infty \frac{x^\nu}{z^{-1}e^x - 1} \, dx = g_{\nu+1}(z)\Gamma(\nu + 1)
\]

and, in particular,

\[
\int_0^\infty \frac{x^\nu}{e^x - 1} \, dx = \Gamma(\nu + 1)\zeta(\nu + 1)
\]