Orbital Dynamics of the
Laser Interferometer Space Antenna

Matt Reagor[1]

University of Notre Dame, Notre Dame, IN 46556

Peter L Bender

*JILA, University of Colorado and National Institute
of Standards and Technology, Boulder, CO 80309-0440, USA*

ABSTRACT

The Laser Interferometer Space Antenna (LISA) is a proposed NASA/ESA space based gravitational wave observatory. It will constist of three satelites in a nearly equilateral triangle constellation of arm lengths $\approx 5 \times 10^6$ km flying in a heliocentric orbit at approximately 1AU from the Sun and about 20 deg behind the Earth. Gravitational waves will be measured as small variations in arm length. Since the craft are not attached to one another, they must rely on special charectoristics of their orbits to remain in a fixed constellation with fixed arm lengths (Bender et al 1997). My REU research has been to study how pertubations due to the presence of the Earth will affect the arm lengths of the constellation, as well as the constellation shape itself. Understanding these pertubations is essential as the mission has fairly tight constraints for variations in arm length, as well as variations in shape (Sweetser 2005).

Subject headings: REU, LISA, classical gravity, orbital dynamics

1. Introduction

The nominal orbit behaves in a very interesting way before one even considers pertubations due to the Earth. Each craft's orbit has an inclination of about 1 deg with respect to the ecliptic plane (the plane of Earth's orbit), and each reaches perihelion (is closest to the Sun) at the same time that the craft is at the highest point in its orbit with respect to the ecliptic due to this inclination. However, the craft are $2\pi/3$ out of phase with respect to one another for this behavior. In other words, each craft reaches the maximum of its orbit 1/3 of a year after the preceeding craft's maximum. The nominal orbit is shown without pertubations from the Earth below (Fig1).


Out[65]=
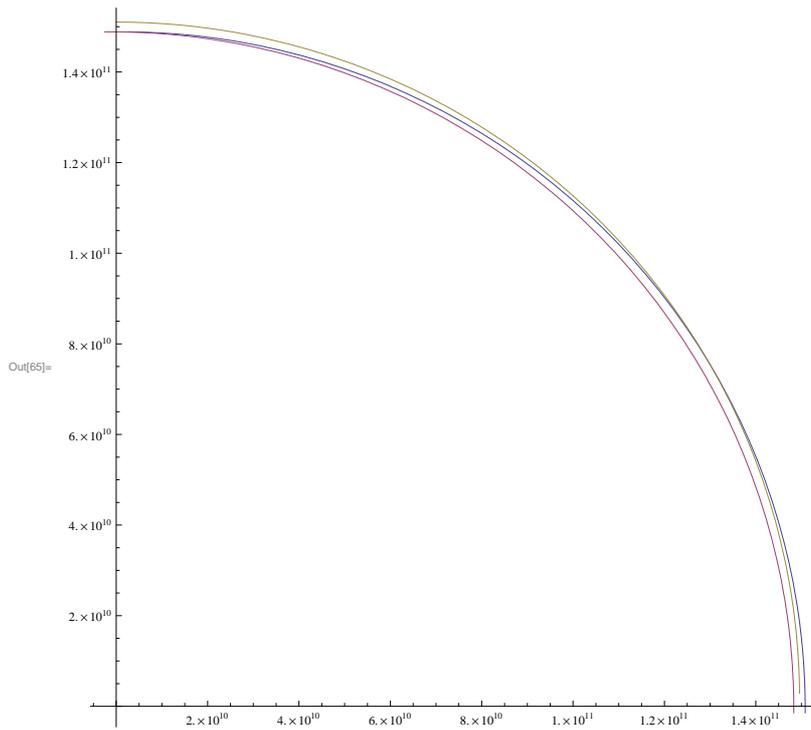
Fig. 1.— The Nominal Orbit

the rotating frame. Also, the craft travels faster than its mean motion near perihelion, catching up and surpassing the rotating frame before falling back to aphelion. The repition of this motion traces out an ellipse within a rotating frame. This ellipse under rotation depends on the eccentricity of the orbiting body.

Since all three LISA spacecraft have equal eccentricities, the ellipse they trace out within S' is the same when viewed from above, accept the craft are out of phase with respect one another by $2\pi/3$. The same nominal orbit from Fig1 is presented below within S' (Fig2).
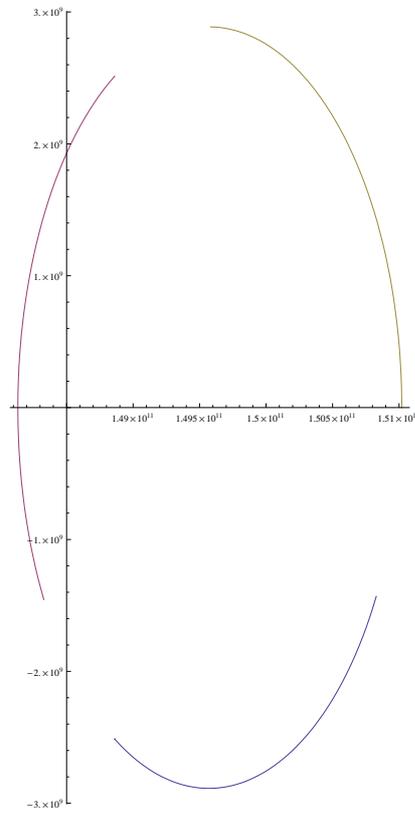
Fig. 2.— The Nominal Orbit within S'

When one considers the inclination of the orbit within S', the usefulness of the nominal orbit becomes much more clear. For each craft's orbit, perihelion is in phase with the maximum displacement from the ecliptic in the Z direction. Approximately one degree of inclination may have

seemed random at first, but this value for inclination was chosen specifically to be the $\sqrt{3} \times e$, where e is the eccentricity of the craft. Therefore, the maximum displacement from ecliptic, which occurs at perihelion, cooresponds to a value which, when added to the displacement in the X direction at that time equals the maximum displacement in the Y direction on the ellipse. This clever choice for the inclination means that the craft actually follow a circle (to first order approximation) in three dimensions within S'. Figure 2 is an ellipse due to the fact that this circle is tilted by about 60 deg with respect to the ecliptic. Therefore, under first order approximations, the craft are following a three-dimensional circle and are $2\pi/3$ out of phase with respect to one another within S'.

This is an ideal orbit for gravitaional wave observations because it provides a means to keep the arm lengths and constellation shape constant to first order - three points on a circle $2\pi/3$ out phase with respect to one another form an equilateral triangle of constant arm length as the three points travel around the circle.

2. Numerical Approach to the Unperturbed Orbit

For a second order (in eccentricity) approximation, the motion of one of spacecraft within S' can represented by the equations (in astronomical units, t in years):

$$\chi[t] = 1 - e\text{Cos}[2\pi t - \Omega - \omega] - (1/2)(e^2)(1 - \text{Cos}[4\pi t - 2\Omega - 2\omega]) - (1/2)(\epsilon^2)\text{Sin}[2\pi t - \Omega]^2$$

$$\psi[t] = 2e\text{Sin}[2\pi t - \Omega - \omega] + (1/4)(e^2)\text{Sin}[4\pi t - 2\Omega - 2\omega] - (1/4)(\epsilon^2)\text{Sin}[4\pi t - 2\Omega]$$

$$\text{z}[t] = \epsilon\text{Sin}[2\pi t - \Omega] - (3/2)(\epsilon)(e)\text{Sin}[\omega] + (1/2)(\epsilon)(e)\text{Sin}[4\pi t - 2\Omega - \omega]$$

(1)

where (χ, ψ) are the (x,y) coordinates under the rotation of S', e is the eccentricity of the craft, ϵ is the inclination of the craft's orbit with respect to the Sun, and finally, Ω and ω are constants that allow for the $2\pi/3$ phase separation between craft.

From these higher order approximations, we can take the orbit out of the rotating frame for the purpose of finding accurate starting conditions for a numerical integration. To do this, we must define an angle, Θ that represents the craft's true anomoly (angular distance from Θ_0=0). Such an angle is given (in radians) by:

$$\Theta[t] = \text{Tan}^{-1}[\psi[t]/\chi[t]] + 2\pi t \tag{2}$$

With an equation for the true anomoly, we can find an accurate transformation of coordinates back into S:

$$x[t] = \sqrt{\chi[t]^2 + \psi[t]^2} \times \mathrm{Cos}[\Theta[t]]$$
$$y[t] = \sqrt{\chi[t]^2 + \psi[t]^2} \times \mathrm{Sin}[\Theta[t]]$$
$$z[t] = z[t]$$

(3)

For the first few weeks of this program, I worked on a numerical integration in Fortran based on starting conditions such that:

$$x_o[0] = x[0]$$
$$y_o[0] = y[0]$$
$$z_o[0] = z[0]$$
$$x_o'[0] = x'[0]$$
$$y_o'[0] = y'[0]$$
$$z_o'[0] = z'[0]$$

(4)

The program calculated positions as a funciton of time given initial positions and velocities of all three spacecraft as well as calculated the distances between them throughout the course of ten years for the purpose of studying how stable the nominal orbit was, as well as how accurate the second order approximations were. The program had much success verifying that the nominal orbit was stable and that the second order approximations were good to about 1×10^5 m, which equates to about 1×10^{-4} percent.

I was able to verify my Fortran program by turning approximately 300 lines of code into a nine coupled differential equations in Mathematica, which could be numerically solved via

NDSolve. Since Mathematica made it much easier to adjust starting conditions and made work with the output much more intuitave, I decided to continue the remainder of my research within Mathematica.

3. Pertubations Due to the Earth (Numerical Analysis)

The next step in my research was to include pertubations due to the Earth in the numerical analysis. This was fairly straight forward, as it meant only adding additional acceleration terms to the coupled differential equations in Mathematica, as well as adding additional acceleration terms to the integration in Fortran (in order to verify the Mathematica results).

Pertubations were mainly studied within a rotating frame since a non-rotating frame would make extracting valuable data nearly impossible. This was accomplished by doing the computations in S and transforming the results back to S'. Below are the results for the numerical analysis of the pertubation due to the Earth (Fig.3-4). Pertubations were measured by subtracting the second order analytic approximation of the non-perturbed orbit from the perturbed numerical analysis for the S' coordinates ($\chi[t]$, $\psi[t]$).
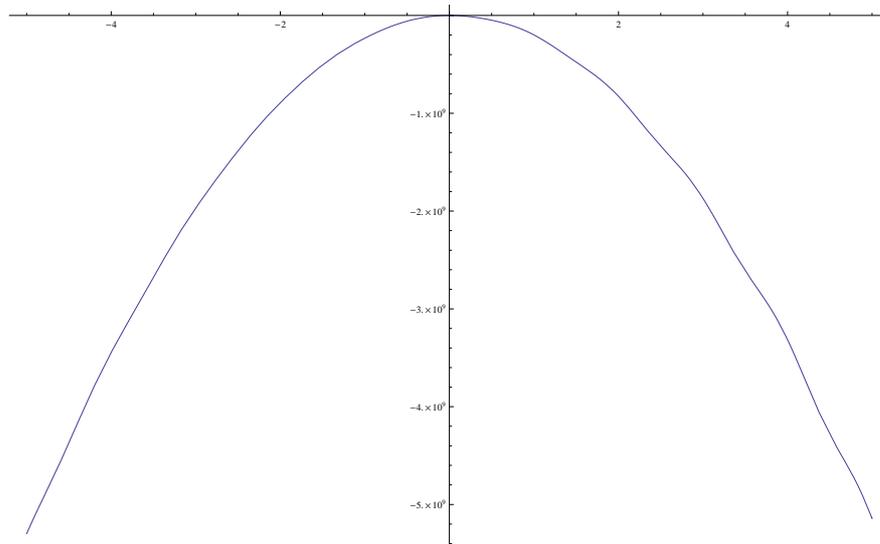
Fig. 4.— Pertubation in the ψ direction for one of the spacecraft

As you can see, the main effect is a linear term in the χ direction and a quadratic term in the ψ direction. These main effects can be understood fairly well analytically.

4. Pertubations Due to the Earth (Analytical Approach)

The LISA mission would greatly benifit from an analytical understanding of the pertubations due to Earth's gravitational pull. Not only could this analytical understanding serve as a check for numerical results, but optimization efforts for the mission's orbit could be much more effective if those efforts had an analytical backing to serve as a sort of compass for their work. Pertubations have been studied analytically in the past (Sweetser 2004). However, presently there is no analytical description that can fully account for the numerical results presented above.

In order to study pertubations from an analytical stand point, one must use a set of differential equations known as the Hill's equations (Kaplan 1976):

$$\ddot{r}[t] - (2n)\dot{h}[t] - (3n^2)r[t] = \text{Pertubation Acceleration in } \chi \text{ Direction}$$

$$\ddot{h}[t] + (2n)\dot{r}[t] = \text{Pertubation Acceleration in } \psi \text{ Direction}$$

$$\ddot{z}[t] + (2n^2)z[t] = \text{Pertubation Acceleration in Z Direction}$$

(5)

The Hill's equations are solved within S', where n is the mean motion of the orbiting body (in our case $n = 2\pi$), r[t] is pertubation measured in the χ direction due to accelerations along that direction, and h[t] is pertubation in the ψ direction measured along the circumference of the orbit (i.e. h[t]/(a) = the angular displacement, where a is the semi-major axis of the orbit). Thus, contributions from h[t] perturb the χ direction as well as the ψ direction, yielding a total pertubation of:

$$\Delta\chi[t] = r[t] + a(\text{Cos}[h[t]/a]\text{-}1)$$

$$\Delta\psi[t] = a(\text{Sin}[h[t]/a])$$

(6)

In solving the Hill's equations, we can, to high order approximation, ignore accelerations due to the Earth in the χ direction. To second order approximatinos, for the reference point of the constellation (the point in the center of the equilateral triangle), the acceleration felt due to the Earth can be treated as a constant, β, minus a quadratic term, κt^2, that accounts for the drop off in acceleration due to the horseshoe nature of the constellation's behavior during a perturbed orbit (Fig4). This yields a solution to the Hill's equation:

$$h[t] = (4\kappa/n^2)t^2 - (3\beta/2)t^2 - (\kappa/4)t^4$$

$$r[t] = (2\beta/n)t - 4\kappa t + (2\kappa/3n)t^3$$

(7)

This solution to the Hill's equation is plotted against the pertubations of the numerical integration for the reference point's orbit below (Fig.5-6):
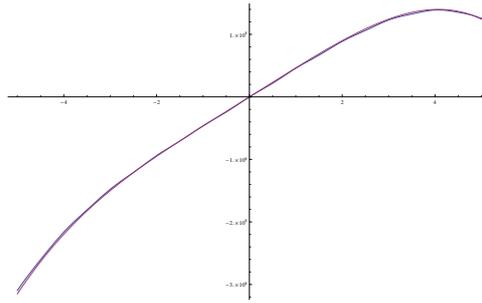
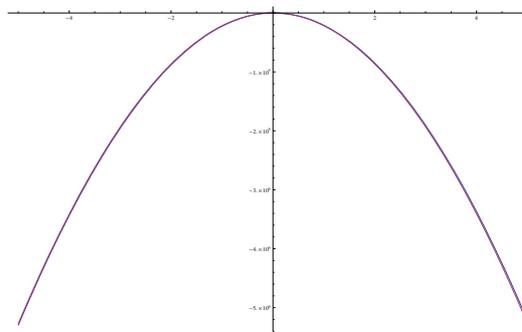Fig. 5.— Pertubation in χ direction for the reference point (analytic and numeric)

Fig. 6.— Pertubation in ψ direction for the reference point (analytic and numeric)

The analytic results compliment the numerical integration well. The maximum deviation between the numeric and analytic results is about 1 percent (which occurs at t = 5 for the ψ pertubation).

5.   Pertubations of the Spacecraft

Due to the motion of the spacecraft around the circle in S', they will be perturbed differently than the reference point. The most straightforward way to account for this is to add an additional term, c which varries once per year, namely, c(Sin[nt+ϕ]) to the Hill's equations. This additional term yields new solutions to the Hill's equations:

h[t]= $(4\kappa/n^2)t^2 - (3\beta/2)t^2 - (\kappa/4)t^4 - (ct/n)$Sin[nt+$\phi$]

r[t]= $(2\beta/n)t - 4\kappa t + (2\kappa/3n)t^3 - (2ct/n)$Cos[nt+$\phi$]

(8)

Unfortunately these new terms are 180 deg. out of phase with the effect that is seen within the numerical integration for individual spacecraft. This is can be easily seen in the χ direction below (Fig 7).
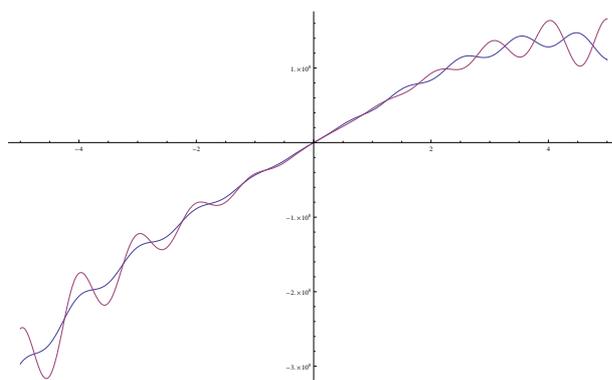
Printed by Mathematicafor Students

Fig. 7.— Analytic and numeric pertubation in the χ direction for one of the craft

Upon the end of my time in Boulder, I have yet to be able to reconcile the difference between the numeric and analytic models presented above. It is more than likely that the difference between the two models lies in the starting conditions of the numeric model. However, those starting conditions are extremely sensitive to change, which makes it incredibly difficult to pinpoint the error in them.

After exploring several options, I feel that the best candidate as a solution to this descrepancy lies within a quadratic change of phase term. In studying my numerical results, I found that the periods of the behavior within the ellipse in S' were changing slightly, quadratically with time $(2\times10^{-4} \; t^2)$. The discrepancy presented above (Fig 7) can be accounted for when one considers the displacement caused by the craft falling out of phase with the unperturbed ellipse in S'. The difference between the analytic and numeric results are plotted below against the quadratic change of phase in the χ direction (Fig 8).

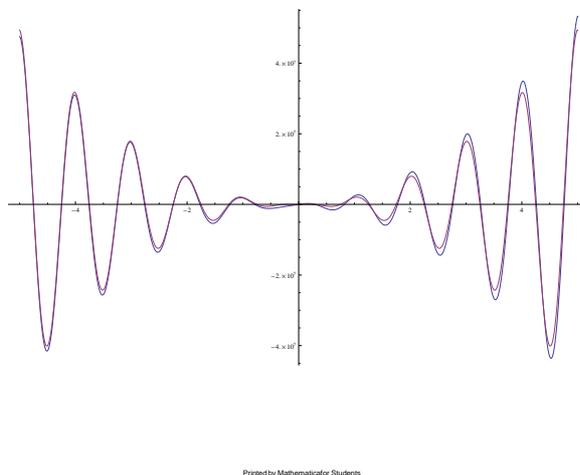Printed by Mathematica for Students

Fig. 8.— The difference between the numeric and analytic models vs. the displacement caused by a quadratic change of phase term

6.   Conclusions

I set out this summer with the fairly lofty goal of obtaining a complete numeric and analytic discription of LISA's nominal orbit. I fell short of that goal, but I am still fairly proud of what I was able to accomplish in these ten weeks. Although I was not able to verify that this quadratic change of phase term can arise from starting conditions, they are clearly the main suspect. Had I continued with this research, the next step would have been attempting to rid the numerical model of this phantom quadratic term so that the two codels would come to agreement.

REFERENCES

Kaplan, M. 1976, "Modern Spacecraft Dynamics & Control," John Wiley Sons, New York, NY

Sweetser, T. 2005, Classical and Quantum Gravity, 429, 22

Hughes, S. 2002, "Preliminary optimal orbit design for LISA," Advances in the Astronautical Sciences vol. 111, ed Culp R. D. and Jolly S. D., Univelt Inc. Publishers, San Diego, CA

Bender, P. L., & Folkner, W. L., & Hechler, F., & Sweetser, T. H., & Vincent, M. A. 1997, Classical and Quantum Gravity, 1439, 14