

Solitary Waves: A Study of Atomic-Molecular BECs

by

Robert McRae

Introduction

To form a typical Bose-Einstein condensate (BEC), bosonic atoms are cooled until they condense into the same quantum mechanical state. However, in some atoms, such as ^{87}Rb , a Feshbach resonance occurs that allows for the creation of diatomic molecules [1]. In this case, we can have a coupled atomic-molecular condensate. In particular, there exist solitonic, or solitary wave, solutions to the coupled differential equations that describe this atomic-molecular system. Furthermore, it is possible to obtain a purely molecular condensate. The question, then, is which possibility, the atomic-molecular soliton or the molecular condensate, will be lower in energy and thus will be preferentially formed? The aim of this study is to explore this question by considering the energies and forms of these condensates and the conditions under which each one can form.

Equations for the System

The energy of the coupled atomic molecular system (in one dimension) is given by the following functional:

$$E = \int_{-\infty}^{\infty} \frac{1}{2} \left(\frac{\partial \Psi_A}{\partial x} \right)^2 + \frac{1}{4} \left(\frac{\partial \Psi_M}{\partial x} \right)^2 + \frac{g_A}{2} |\Psi_A|^4 + \frac{g_M}{2} |\Psi_M|^4 + g_{AM} |\Psi_A|^2 |\Psi_M|^2 + v |\Psi_M|^2 + \lambda (\Psi_M \Psi_A^{*2} + \Psi_M^* \Psi_A^2) dx \quad (1)$$

Here Ψ_A is the wavefunction for the atomic condensate, while Ψ_M is the wavefunction for the molecular condensate. The energy functional can easily be generalized to three dimensions by replacing the derivative terms with $|\nabla \Psi|^2$. In this energy functional, the derivative terms represent the kinetic energy contribution; technically, the coefficients should be $\hbar^2 / 2m$ for the

atomic condensate and $\hbar^2 / 4m$ for the molecular condensate (note that we use $2m$ instead of m for the molecular condensate, since it is a diatomic molecule with twice the mass of an atom). However, the units of the wavefunctions and of all the other coefficients can easily be redefined so that they absorb the \hbar^2 / m . The quartic terms in the energy functional represent atom-atom, atom-molecule, and molecule-molecule interactions. We assume that g_A and g_M will both be positive, that is, that atom-atom and molecule-molecule interactions will be repulsive. We allow g_{AM} to be negative as long as the sum total of interactions is repulsive. v relates to the binding energy of the molecules, that is, the energy difference between a diatomic molecule and two free atoms; it is called the detuning, and can be adjusted experimentally with a magnetic field [1]. The cubic terms in the energy functional are the Feshbach coupling terms, and they relate to interconversions between atoms and molecules; we assume λ to be positive.

$|\psi_A|^2$ and $|\psi_M|^2$, as typical quantum mechanical wavefunctions, relate to the probabilities of finding atomic or molecular condensate at a specific location. However, since a condensate will consist of many particles, $|\psi_A|^2$ and $|\psi_M|^2$ may be described as density functions. When these functions are integrated over all space, they give the total number of particles in the condensate. Thus we have the following number functional:

$$N = \int_{-\infty}^{\infty} (|\psi_A|^2 + 2|\psi_M|^2) dx \quad (2)$$

Note that since each molecule consists of two atoms, the coefficient of $|\psi_M|^2$ is 2, so that the integral gives the total number of atoms in the condensate, whether bound or free.

The problem is to find wavefunctions for the atomic and molecular condensates that will yield energy minima, subject to the constraint that particle number must remain constant. We may then define a new functional:

$$H = E - \mu N \quad (3)$$

Here, μ , called the chemical potential, is the Lagrange multiplier for the constraint of constant number. Thus, we wish to minimize H , and this can be done by taking the functional derivatives of H and setting them equal to zero:

$$\frac{\delta H}{\delta \psi_A^*} = 0, \quad \frac{\delta H}{\delta \psi_M^*} = 0 \quad (4)$$

After these differentiations are carried out the result is the following equations:

$$\begin{aligned} \frac{1}{2} \frac{d^2 \psi_A}{dx^2} &= g_A \psi_A^3 + g_{AM} \psi_A \psi_M^2 + 2\lambda \psi_A \psi_M - \mu \psi_A \\ \frac{1}{4} \frac{d^2 \psi_M}{dx^2} &= g_M \psi_M^3 + g_{AM} \psi_A^2 \psi_M + \lambda \psi_A^2 - (2\mu - \nu) \psi_M \end{aligned} \quad (5)$$

Here, after differentiating, we have assumed that the solutions are real. These are the two coupled differential equations that must be solved to obtain the condensate wavefunctions.

It is clear from looking at the equations that the condensate system has three distinct phases. First, it is possible that both condensate wavefunctions are equal to zero. This is a trivial solution, and is not of particular interest. Both the energy and particle number for this solution are zero, and thus it will only be favored over solutions that have positive energy. Second, one can see that by substituting $\Psi_A = 0$ into the first and second equations, it is possible to have $\Psi_A = 0$ and $\Psi_M \neq 0$. This is the molecular condensate phase. Thirdly, it is possible to have both wavefunctions nonzero.

Molecular Condensate Phase

If we let $\Psi_A = 0$, then equations (5) reduce to the following single equation:

$$\frac{1}{4} \frac{d^2 \psi_M}{dx^2} = g_M \psi_M^3 - (2\mu - \nu) \psi_M \quad (6)$$

This equation has the same form as Newton's second law for an anharmonic oscillator. If both sides of the equation are multiplied by $d\Psi_M/dx$ and integrated, the following equation is obtained:

$$\frac{(d\Psi_M/dx)^2}{2} - g_M \Psi_M^4 + 2\mu_M \Psi_M^2 = E \quad (7)$$

$$\mu_M = 2\mu - \nu$$

The constant of integration E may be thought of as the classical energy. This equation has the form Energy = Kinetic energy + Potential energy, or $E = K + U$. The problem of finding the condensate wavefunction is now reduced to finding the equations of motion of a classical particle in a quartic potential well. Specifically, the potential energy has the following form:

$$U = -\frac{g_M}{4} \Psi_M^4 + \frac{\mu_M}{2} \Psi_M^2 \quad (8)$$

Recall that we have assumed g_M to be positive; however, we allow μ_M to be either positive or negative. But if μ_M is negative, the shape of potential energy will simply be a downwards-opening parabolic shape (though steeper than a parabola, since this is a quartic polynomial).

Such a potential energy shape will allow only one (unstable) solution at $\Psi_M = 0$; we thus obtain on

For small Ψ_M , the potential can be approximated as a simple harmonic oscillator potential, since the quadratic term dominates the quartic term. However, as Ψ_M increases, the oscillating wavefunction becomes more and more anharmonic. If we set one of the maxima of the potential as the initial amplitude of the wavefunction, there are two possibilities. We could have a solution that is constant in all of space. In this case the solution will be:

$$\Psi_M = \pm \sqrt{\frac{\mu_M}{g_M}} \quad (9)$$

One can also think of a particle starting at one of the potential maxima and rolling, infinitesimally slowly, off the maximum, into the minimum, and up to the maximum on the other side. To obtain the solution in this case, equation (7) can be solved by separation of variables.

The solution is as follows:

$$\Psi_M = \pm \sqrt{\frac{\mu_M}{g_M}} \tanh(\sqrt{2\mu_M} x) \quad (10)$$

This wavefunction is known as a dark soliton and is illustrated in Fig. 2.

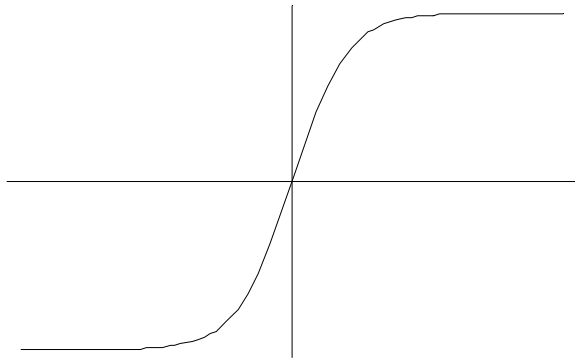


Fig. 2. Dark soliton solution of equation (7).

It is possible also to have $\Psi_M = 0$ as a solution for this potential, since there is a critical point of the potential energy at $\Psi_M = 0$. These three solutions are the only pertinent analytical solutions

for this phase. If the amplitude of Ψ_M ever becomes greater than $\sqrt{\frac{\mu_M}{g_M}}$, then the corresponding

solution will diverge. For any maximum amplitude between zero and $\sqrt{\frac{\mu_M}{g_M}}$, it becomes

necessary to use numerical solutions.

The energy functional for the molecular condensate takes the form:

$$E = \int_{-\infty}^{\infty} \frac{1}{4} \left(\frac{\partial \Psi_M}{\partial x} \right)^2 + \frac{g_M}{2} |\Psi_M|^4 + v |\Psi_M|^2 dx \quad (11)$$

For each of the possible solutions of equation (6), except $\Psi_M = 0$, this integral diverges.

However, we can think of the integrand of equation (11) as an energy density. For the constant

solutions of equation (9), the energy density is constant in space and equal to $\frac{4\mu^2 - v^2}{2g_M}$. It is

clear that the energy density of this condensate is negative, and thus favored over $\Psi_M = 0$, when

$|\mu| < \frac{1}{2}|v|$. However, we have also argued that $\mu_M = 2\mu - v > 0$. Thus we find that $\mu > \frac{1}{2}v$. The

preceding two inequalities are satisfied only when v is negative. For the hyperbolic tangent dark soliton solution, the energy density is not constant in space and is more difficult to compute.

However, it is possible to calculate an average of the energy density over a given interval, and this average approaches the energy density for the constant solution as x approaches infinity; this is to be expected, since the hyperbolic tangent function approaches a constant as x approaches infinity.

Similarly, the integral for particle number diverges for most solutions of equation (6).

However, the integrand of the particle number integral is simply the density of the condensate at a particular region in space. Thus, for the constant solutions of equation (9), the density is equal

to $\frac{2(2\mu - v)}{g_M}$. The possible values for the particle density are limited by the restrictions on μ and

v given in the preceding paragraph. Since $\frac{1}{2}v < \mu < \frac{1}{2}|v|$, the possible values of the number density range between 0 and $4|v|/g_M$. As with the energy density, the average number density for the dark soliton solution approaches the density for the constant solution as x approaches infinity.

Atomic-Molecular Condensate Phase

The atomic-molecular condensate phase is much more difficult to study than the molecular condensate phase, because both coupled equations of (5) must be solved. As a start, one may consider solutions of (5) that are constant in space. In this case, the equations of (5) become:

$$\begin{aligned} 0 &= g_A \Psi_A^3 + g_{AM} \Psi_A \Psi_M^2 + 2\lambda \Psi_A \Psi_M - \mu \Psi_A \\ 0 &= g_M \Psi_M^3 + g_{AM} \Psi_A^2 \Psi_M + \lambda \Psi_A^2 - (2\mu - v) \Psi_M \end{aligned} \quad (12)$$

These equations are, however, still complicated coupled algebraic equations. It is not difficult to solve the first equation of (12) for Ψ_A , but when the result is substituted into the second equation, one must still solve a cubic equation. Although cubic equations are in general solvable, the general solutions are complicated enough that the results are not enlightening. It is possible to obtain manageable solutions with an appropriate choice of values for the various coefficients of the polynomial terms, but it is difficult to say how valuable these solutions are for studying the condensate system.

Another way to simplify equations (5) is to assume that $\Psi_M = k \Psi_A$ for some constant of proportionality k . If this value for Ψ_M is substituted into equations (5), the following equations result:

$$\frac{d^2\psi_A}{dx^2} = 2(g_A + g_{AM}k^2)\psi_A^3 + 4\lambda k\psi_A^2 - 2\mu\psi_A \quad (13)$$

$$\frac{d^2\psi_A}{dx^2} = 4(g_M k^2 + g_{AM})\psi_A^3 + \frac{4\lambda}{k}\psi_A^2 - 4(2\mu - \nu)\psi_A$$

For this proportionality assumption to work, the coefficients of corresponding polynomial terms in each equation must be equal. Thus we have the following equations:

$$2(g_A + g_{AM}k^2) = 4(g_M k^2 + g_{AM})$$

$$4\lambda k = \frac{4\lambda}{k} \quad (14)$$

$$-2\mu = -4(2\mu - \nu)$$

It is then a simple matter to reduce this set of equations to the following conditions:

$$k = \pm 1$$

$$\mu = \frac{2}{3}\nu \quad (15)$$

$$g_{AM} = g_A - 2g_M$$

Thus, if the molecular wavefunction is proportional to the atomic wavefunction, it must be equal or opposite.

Let us assume, then, that $\Psi_M = -\Psi_A$. This is not qualitatively different from the assumption that $\Psi_M = \Psi_A$ because the only difference will be the sign of Ψ_A , and no odd power of Ψ_A appears in the energy or particle number integrals. The first of equations (13) then becomes

$$\frac{d^2\psi_A}{dx^2} = 2(g_A + g_{AM})\psi_A^3 - 4\lambda\psi_A^2 - 2\mu\psi_A \quad (16)$$

This equation, like (7), has the form of Newton's second law for a classical particle moving in a quartic potential. Equation (16) may be integrated to yield:

$$\frac{(d\psi_A/dx)^2}{2} - \frac{g_A + g_{AM}}{2}\psi_A^4 + \frac{4\lambda}{3}\psi_A^3 + \mu\psi_A^2 = E \quad (17)$$

In this case, the potential energy function is as follows:

$$U = -\frac{g_A + g_{AM}}{2}\Psi_A^4 + \frac{4\lambda}{3}\Psi_A^3 + \mu\Psi_A^2 \quad (18)$$

As was the case with the molecular condensate, μ may be either positive or negative. If μ is positive, the potential energy function for small Ψ_A will be another anharmonic oscillator, like the molecular condensate potential. However, due to the presence of the cubic term, the potential energy will not be symmetric about the origin. For large Ψ_A , the quartic term will dominate, and the potential will go to negative infinity. (Fig. 3)

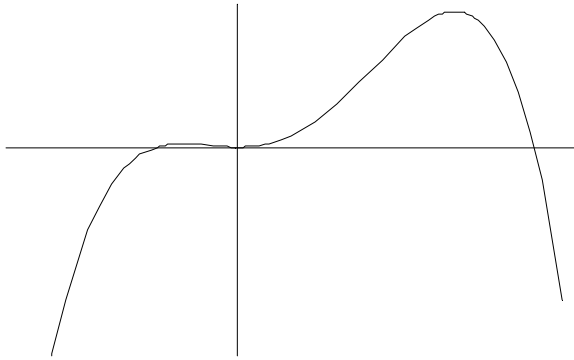


Fig. 3. Potential energy profile for positive μ .

This potential energy profile will allow for a number of bound solutions, in addition to the constant solutions that occur at the critical points. A typical bound solution is shown in Fig. 4.

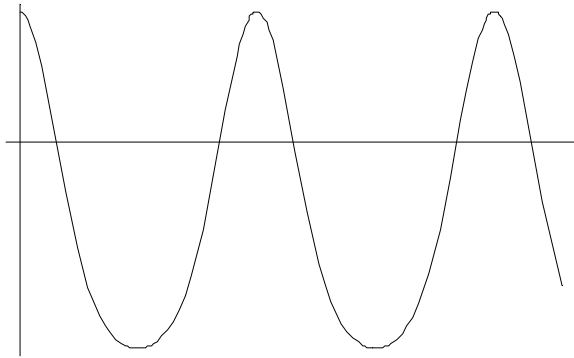


Fig. 4. Bound solution of equation (17).

If μ is negative, the potential energy function will approximate a downward-facing parabola for small Ψ_A . If μ and the coefficient of the quartic term are not too large, the cubic

term will dominate for moderately positive Ψ_A , and the potential energy function will reach a minimum. However, the quartic term will eventually dominate, and the potential energy function will go to negative infinity. (Fig. 5)

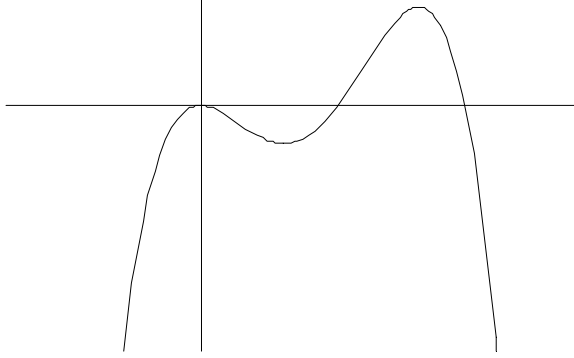


Fig. 4. Potential energy profile for negative μ .

This potential energy function also allows for a number of bound solutions, besides the constant solutions. In particular, this potential energy function allows for a bright soliton solution. One can think of a particle resting at the origin at $x = -\infty$, rolling off towards the minimum infinitesimally slowly, hitting the spot where the potential energy crosses the horizontal axis at $x = 0$, and rolling back to the origin at $x = \infty$. This solution corresponds to a total (classical) energy of zero. Thus, equation (17) becomes

$$\frac{(d\psi_A/dx)^2}{2} = \frac{g_A + g_{AM}}{2}\psi_A^4 - \frac{4\lambda}{3}\psi_A^3 - \mu\psi_A^2 \quad (19)$$

Equation (19) has been solved for the case of no interactions (all g terms equal to zero) [1]. The result is the following:

$$\psi_A = \frac{3|\mu|}{4\lambda} \operatorname{sech}^2 \left(\sqrt{\frac{|\mu|}{2}} x \right) \quad (20)$$

If the interaction terms are not neglected, the equation becomes much more complicated, but it can still be solved (by separation of variables). The result is the following:

$$\psi_A = \frac{a}{b} \sec \left(2 \tan^{-1} \left(\sqrt{\frac{c+a}{c-a}} \coth \left(\frac{1}{2} \sqrt{\frac{c^2-a^2}{b}} x \right) \right) \right) + \frac{c}{b}$$

$$a = \sqrt{\frac{16\lambda^2}{9} + 2\mu(g_A + g_{AM})} \quad (21)$$

$$b = g_A + g_{AM}$$

$$c = 4\lambda/3$$

In terms of v , (20) becomes $\psi_A = \frac{|v|}{2\lambda} \operatorname{sech}^2 \left(\sqrt{\frac{|v|}{3}} x \right)$; (21) maintains the same form when

expressed in terms of v , but we have $a = \sqrt{\frac{16\lambda^2}{9} + \frac{4v}{3}(g_A + g_{AM})}$. Fig. 6 illustrates bright soliton

solutions to equations (5). The positive solution is the atomic condensate wavefunction, while the negative solution is the molecular condensate wavefunction.

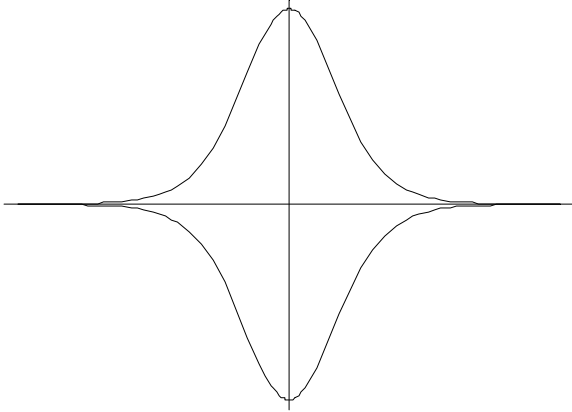


Fig. 6. Bright soliton solutions of equations (5).

For equation (21) to work as a solution to equations (5), the conditions of equations (15) must be met; especially, the three g parameters must satisfy the specified relationship; otherwise, consistent solutions such that $\Psi_M = \pm\Psi_A$ will not be obtained. Furthermore, equation (21) assumes that the constant a is real. Thus the following inequality must be satisfied:

$$\frac{16\lambda^2}{9} > -\frac{4v}{3}(g_A + g_{AM}) \rightarrow v > -\frac{4\lambda^2}{3(g_A + g_{AM})} \quad (22)$$

The preceding inequality gives the lower bound for v . The upper bound for v is zero; because μ is negative and $\mu = 2v/3$, v must also be negative. If v were equal to its lower bound, the bright soliton solution would be a constant; the potential energy function for this situation is shown in Fig. 7. The solution will be located at the maximum that is found to the right of the origin.

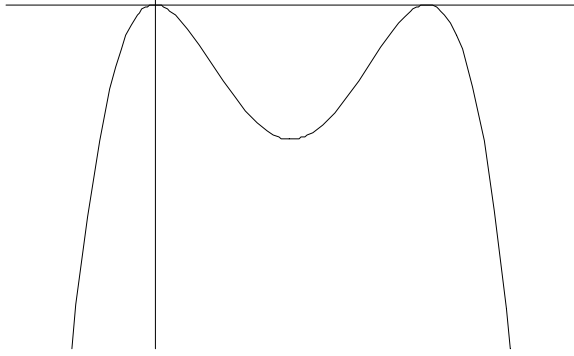


Fig. 7. Potential energy function at critical v .

Comparison of Atomic-Molecular and Molecular Condensate Energies

Although it is very difficult in general to find the constant solutions of (12), it is not necessary to find the exact solutions to compare energies of atomic-molecular condensates with molecular condensates. For constant solutions, the energy functional (equation (1)) becomes,

$$E = \frac{g_A}{2} |\psi_A|^4 + \frac{g_M}{2} |\psi_M|^4 + g_{AM} |\psi_A|^2 |\psi_M|^2 + v |\psi_M|^2 + \lambda (\psi_M \psi_A^{*2} + \psi_M^* \psi_A^2) \quad (23)$$

Now let us denote the total density of the condensate as ρ^2 . Then we have,

$$\psi_A^2 + 2\psi_M^2 = \rho^2 \rightarrow \psi_A^2 = \rho^2 - 2\psi_M^2 \quad (24)$$

This value for the square of the atomic wavefunction may be substituted into equation (23). We know that $\psi_M \leq \rho / \sqrt{2}$. Thus, we can define a new variable, x , such that $\psi_M \leq \rho x / \sqrt{2}$. Thus $-1 \leq x \leq 1$. Thus, the energy function is now a quartic polynomial function of x . The problem is to find the value of x that will minimize this polynomial. Actually finding this value of x requires finding the critical points of the polynomial and comparing the values of $E(x)$ at these points

with the values of $E(x)$ at -1 and 1, the endpoints. It can be shown that if the coefficient of the quartic term of the polynomial $\left(\frac{\rho^4}{4}\left(2g_A + \frac{g_M}{2} - 2g_{AM}\right)\right)$ is positive, then the minimum of the energy will occur at the endpoints when

$$v \leq \rho^2 \left(g_{AM} - \frac{g_M}{2} \right) - 2\sqrt{2}\lambda\rho \quad (25)$$

This suggests that the pure molecular condensate only appears when v is sufficiently small. If the coefficient of the quartic term is negative, however, it is much more difficult to find the conditions for the presence of the molecular condensate.

The energy of the bright soliton will be finite, since this solution goes to zero at $\pm\infty$. Its energy can be calculated from equation (1). For the case of no interactions (equation (20)), the energy turns out to be:

$$E = -\frac{8|v|^{5/2}}{5\sqrt{3}\lambda^2} \quad (26)$$

The particle number corresponding to the bright soliton is:

$$N = \frac{\sqrt{3}|v|^{3/2}}{\lambda^2} \quad (27)$$

Now, suppose that we also have a molecular condensate of constant density that exists only over a finite length L . We require that $L\Psi_M^2 = N$, the same particle number as the atomic-molecular condensate. The energy of the molecular condensate is then found by evaluating the integral of (11) from 0 to L . The result is,

$$E = -\frac{\sqrt{3}|v|^{5/2}}{2\lambda^2} \quad (28)$$

This energy is somewhat greater than the energy of the atomic-molecular soliton, which means that the atomic-molecular soliton is favored over the molecular condensate. Similar comparisons can be done when interactions are included, and the result is the same: the atomic-molecular soliton is favored over a molecular condensate.

Numerical Solutions

None of the solutions of (5) that have been presented so far are general solutions to the system of equations. In fact, there are only a few special cases in which it is possible to obtain analytical solutions to the equations. There are several methods by which numerical solutions can be obtained. The technique that has been used to study equations (5) first of all notes that the equations can both be written in the following manner: $L(u) = N(u, v)$; here, u and v are wavefunctions, L is a linear operator, and N is a non-linear operator. Specifically, equations (5) may be represented as follows:

$$\begin{aligned} \left(\frac{1}{2} \frac{d^2}{dx^2} + \mu \right) \psi_A &= g_A \psi_A^3 + g_{AM} \psi_A \psi_M^2 + 2\lambda \psi_A \psi_M \\ \left(\frac{1}{4} \frac{d^2}{dx^2} + (2\mu - \nu) \right) \psi_M &= g_M \psi_M^3 + g_{AM} \psi_A^2 \psi_M + \lambda \psi_A^2 \end{aligned} \quad (29)$$

The linear operators may both be represented as matrices; let us denote these matrices as L_A and L_M , respectively. The non-linear operators may be denoted as $N_A(\Psi_A, \Psi_M)$ and $N_M(\Psi_A, \Psi_M)$.

Equations (29), then, become,

$$\begin{aligned} L_A \cdot \psi_A &= N_A(\psi_A, \psi_M) \\ L_M \cdot \psi_M &= N_M(\psi_A, \psi_M) \end{aligned} \quad (30)$$

Here, the wavefunctions are both discretized, that is, they are both column vectors of values at discrete points along the x-axis.

What is done first is to make an initial guess as to the shape of the wavefunctions; for instance, if one is looking for bright soliton solutions to the equations, it is convenient to choose as an initial waveform a triangular shape, or perhaps a Gaussian distribution. To obtain the next approximation to the exact solution, one first multiplies both sides of the equations in (30) by the inverses of the linear operators:

$$\begin{aligned} u_{A,n+1} &= L_A^{-1} \cdot N_A(\psi_{A,n}, \psi_{M,n}) \\ u_{M,n+1} &= L_M^{-1} \cdot N_M(\psi_{A,n}, \psi_{M,n}) \end{aligned} \quad (31)$$

The u functions are not the actual wavefunctions. It would indeed be possible to have these u functions be the next approximations to the wavefunctions, but if this is done, these solutions will tend to either diverge or go to zero as n increases. We assume, then, the following:

$$\begin{aligned} \psi_{A,n+1} &= \alpha \cdot u_{A,n+1} \\ \psi_{M,n+1} &= \beta \cdot u_{M,n+1} \end{aligned} \quad (32)$$

α and β are simply constants; they function as rescaling parameters. To determine values for α and β , we impose the following conditions:

$$\begin{aligned} \int_{-\infty}^{\infty} (L_A \cdot (\alpha \cdot u_{A,n+1})) u_{A,n+1} dx &= \int_{-\infty}^{\infty} N_A(\alpha \cdot u_{A,n+1}, \beta \cdot u_{M,n+1}) u_{A,n+1} dx \\ \int_{-\infty}^{\infty} (L_M \cdot (\beta \cdot u_{M,n+1})) u_{M,n+1} dx &= \int_{-\infty}^{\infty} N_M(\alpha \cdot u_{A,n+1}, \beta \cdot u_{M,n+1}) u_{M,n+1} dx \end{aligned} \quad (33)$$

When these integrals are evaluated, the result is a pair of coupled algebraic equations in α and β . These equations can then be solved, either analytically or numerically.

The process described above is then repeated for as many times as needed. The goal is to obtain a solution that does not change between consecutive iterations. This numerical method can work very nicely to give solitonic solutions of equations (5). In fact, it can be used to

produce solitonic solutions in which Ψ_A and Ψ_M are not proportional to each other. Fig. 8 shows solitonic wavefunctions for which $\mu \neq 2\nu/3$.

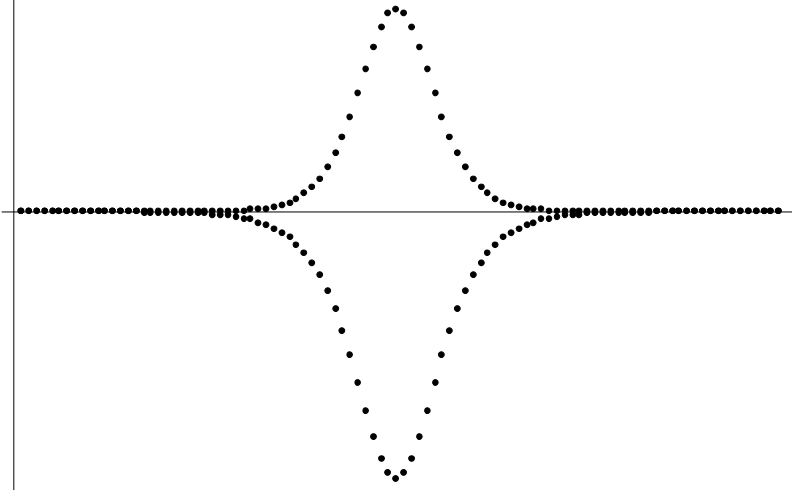


Fig. 8. Numerical soliton solution of equations (5)

The disadvantage of this numerical technique is that it requires the solution of coupled algebraic equations. In general, there will be several solutions to these equations, and some of them may be complex. In fact, the specific problem that has been encountered is the situation in which all solutions are complex except one ((0, 0) will always be a solution of these equations, but we are not looking for a solution in which both wavefunctions are equal to zero; we do not need numerical techniques to find this solution). As of yet, this problem has not been resolved.

Results and Predictions

The results of this study may be summarized as follows. First, it is possible to find constant solutions of equations (5) in three distinct phases: both wavefunctions equal to zero, both wavefunctions non-zero, or the atomic wavefunction equal to zero and the molecular wavefunction non-zero. If the detuning is greater than the critical value given in (25), the atomic-molecular mixture will be favored, as long as the quantity $2g_A + \frac{g_M}{2} - 2g_{AM}$ is positive. Second, in the case of no interactions, it is possible to find bright solitons analytically when the detuning

is negative and when $\mu = 2\nu/3$. These solitons are energetically favored over the molecular condensate phase. Finally, if interactions are present, analytic bright solitons exist if the conditions of (15) hold, and if $-\frac{4\lambda^2}{3(g_A + g_{AM})} < \nu < 0$. These solitons are also favored energetically over the molecular condensate phase. If ν is not negative, or the conditions of (15) do not hold, solitonic solutions can still be found, but they have not been studied thoroughly enough to determine which phase is favored under these conditions.

Finally, we would like to estimate the actual experimental value for the critical detuning given in (22) for ^{87}Rb . For this isotope, the Feshbach resonance occurs at a magnetic field of 685.43 G [1]. According to [1], we can calculate the interaction and Feshbach coupling parameters as follows:

$$\lambda = \frac{\sqrt{2\pi a \Delta \mu \Delta B m}}{\hbar}, a = 5.7 \text{ nm}, \Delta \mu = 1.4 \mu_B, \Delta B = 0.017 \text{ G}, m = 1.44 \times 10^{-25} \text{ kg} \quad (34)$$

$$g_A \approx g_M \approx g_{AM} \approx 4\pi a$$

Here, μ_B is the Bohr magneton. a is the scattering length for atom-atom collisions, ΔB is the width of the Feshbach resonance, and $\Delta \mu$ is the difference between the magnetic moments of the atoms and molecules. These definitions for the interaction and Feshbach parameters leads to a value of the critical detuning of 7.4×10^{-30} J. The difference in energy due to different spins of the free atoms and molecules in a magnetic field is $|E| = Bs\mu_B$, where s is the spin, in this case $1/2$. The calculated value of the detuning, then, implies a shift in the magnetic field of 0.016 G; this value is approximately equal to the width of the Feshbach resonance, ΔB .

Reference

- [1] Bartłomiej Oleś and Krzysztof Sacha, “Solitons in coupled atomic-molecular Bose-Einstein condensates in a trap,” (2006).