Realization of Classical Monodromy

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The goal of this experiment was to demonstrate and map out monodromy in the swing spring system. The experiment was successful in measuring values for the precession angle that were in agreement with the theoretical ones, usually within one degree. The experiment has not yet progressed to the point where a circle in phase space has been traversed, but work on the experiment continues on.

1 Introduction

1.1 Motivation

Monodromy is a bizarre phenomenon in classical mechanics that has well developed theory behind it, but has never been demonstrated experimentally. Experimental demonstration of monodromy was the objective of this project. Besides being just an interesting classical quirk, monodromy also has applications in molecular spectroscopy, for example, in describing the behavior of diatomic molecules in an electric field.

1.2 Purpose

The goal of this project was to build a resonant swing spring system and to demonstrate monodromy in this system by doing a number of trials, measuring the precession angle for each trial, and plotting the precession angle versus its location in phase space. Monodromy is then shown in that fact that when the trials are picked such that they walk in a complete circle around a singularity in phase space, the value of the precession angle does not come back to the starting value.

However, in order to fully understand the goal of this experiment, it is important to understand something of the background behind monodromy and the swing spring system.

1.3 Swing Spring System

The swing spring system is a simple one that most people understand reasonably well after their first introductory physics course. The swing spring consists of a spring that is fixed at one end with a mass attached at the other, and, true to its name, it can either engage in pure swinging motion or pure springing motion. If the parameters of the system are tuned such that the frequency of the springing motion and the frequency of the swinging motion are in a 2:1 resonance, the motion of the swing spring will evolve from pure springing into swinging, and back again indefinitely. Not only that, but every time the swing spring goes into swinging motion, the plane that it swings in precesses. This precession angle between the swing planes is constant and is sensitively dependent on the initial conditions given to the system.

Monodromy in the swing spring is not found in the oscillating between swinging and spring motion or in the fact that the swing plane precesses. It is not even possible to observe monodromy in one single trial. Monodromy is found, rather, in what we observe of the motion of the swing spring in phase space over several different trials.

1.4 Monodromy Theory

All the possible motions of the pendulum can be described in phase space by points inside a curve that looks similar to the one found in Fig. 1, with the axes being angular momentum on the x axis and the cubic terms of the Hamiltonian on the y axis. For example, on this figure, the point at the origin corresponds to pure springing motion, and the far left point on the L axis corresponds to spherical pendulum motion. Therefore, for specific values of angular momentum and the Hamiltonian, using this figure the motion of the swing spring and the value of the precession angle can be predicted.

The key to monodromy is that the point at the origin is a singularity, a special point. If other points are chosen, specific values for the Hamiltonian and the angular momentum, such that they go in a circle around this point, with the predicted precession angle being determined at each point, it would be expected that upon arriving back at the starting point, the same value for the precession angle as the starting value would be obtained. However, the bizarre thing is that this does not hap-

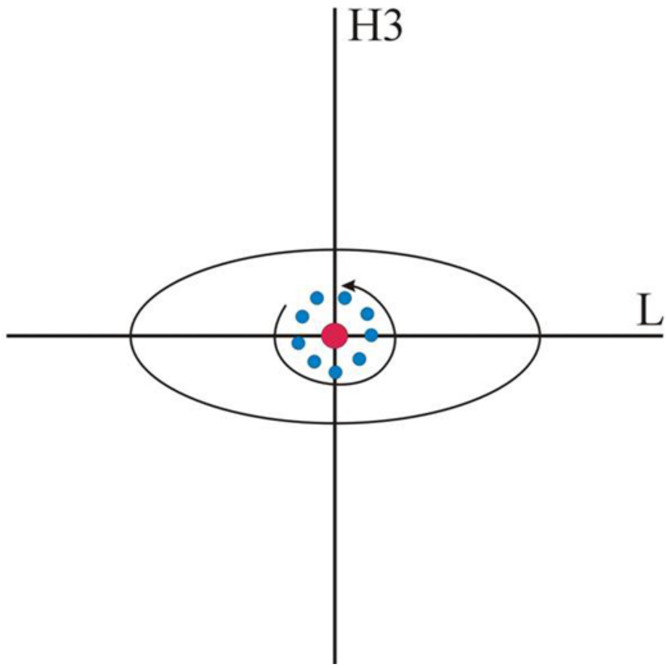

Figure 1: The motion of the pendulum can be described by points in phase space.

pen. Since the point at the origin is a special point, when traversing around it, one actually finds that upon coming back to the starting point, a different value for the precession angle is found. This is monodromy, a warping of phase space.

Monodromy cannot, therefore, be shown in the swing spring system in one single trial. It can only be shown by performing separate trials, with different initial conditions corresponding to separate points in phase space, and by measuring the precession angles at these separate points. In plotting the obtained values for the precession angle versus their location in phase space, monodromy is demonstrated in the fact that when coming back to the starting point, the value obtained for the precession angle does not come back to the same as the starting value.

The goal of this experiment was, then, to walk around two points in phase space, one being the special point located at the origin and the other being located elsewhere in phase space. It was hoped to show that when doing separate trials traversing around the point located elsewhere in phase space, the precession angle comes back to the same value as the starting one, but for the point located at the origin, it does not.

2 Experimental setup

The first task in setting up the experiment was to build a swing spring. An appropriate spring constant had to be chosen such that the frequency of the oscillations was high enough that the data taking process was not long and tedious. It was also important that the spring constant be chosen such that the damping in the spring was minimal. If the damping was too high, the precession angles between swing planes were not constant, and accurate data was difficult to obtain. It was discovered through the investigation of the damping of several springs with different spring constants that a stiffer spring, corresponding to a higher spring constant, had less of a damping effect on the motion.

Once a spring of sufficient spring constant was obtained, a bronze ball of known mass was attached to one end of the spring with fishing line, and the other end of the spring was suspended with fishing line. The length of the fishing line was tuned until the frequencies of the springing and the swinging were in the 2:1 resonance. The final length, spring constant, and mass were approximately 1.3 meters, 6 N/m, and 220 grams respectively.

The second task in setting up the experiment was to determine a method for mapping out the motion of the swing spring and measuring the precession angle between each swing plane. This was accomplished through the use of a digital video camera located underneath the pendulum that recorded the motion of the swing spring. A Matlab program, utilizing image processing tools, analyzed the footage of the motion frame by frame, located the spherical mass in each frame, found the coordinates of the center of the mass, and plotted them. After finding lines of best fit for the plotted data, the precession angles were determined by finding the angles between these lines. Examples of typical Matlab plots of the motion of the swing spring can be seen in Fig. 3.

The third task in the setup for the experiment was to determine the initial conditions given to the swing spring system, an important step for determining the point in phase space that corresponded to the observed precession angle. This was also accomplished through the use of digital video cameras, one located underneath the pendulum to find the initial conditions in the x and y directions and the other located to the side of the pendulum to find the initial conditions in the z direction. (A diagram of the completed setup can be seen in Fig. 2.) Matlab programs were also used to analyze the footage from these cameras and to find the initial positions and velocities in the x, y, and z directions.

In order for the Matlab programs to run smoothly and find the location of the center of the mass accurately, the mass had to be very bright, and the background had to be dark and uniform. The mass was therefore painted white and illuminated with several lights. Painstaking measures were also taken to make the background as uniform as possible through the use of a black felt enclosure that extended from ceiling to floor.

3 Procedure for Analysis of the Data

Once the setup was completed, the first step in the data taking process was to create initial conditions and then record the resulting motion of the spring pendulum.

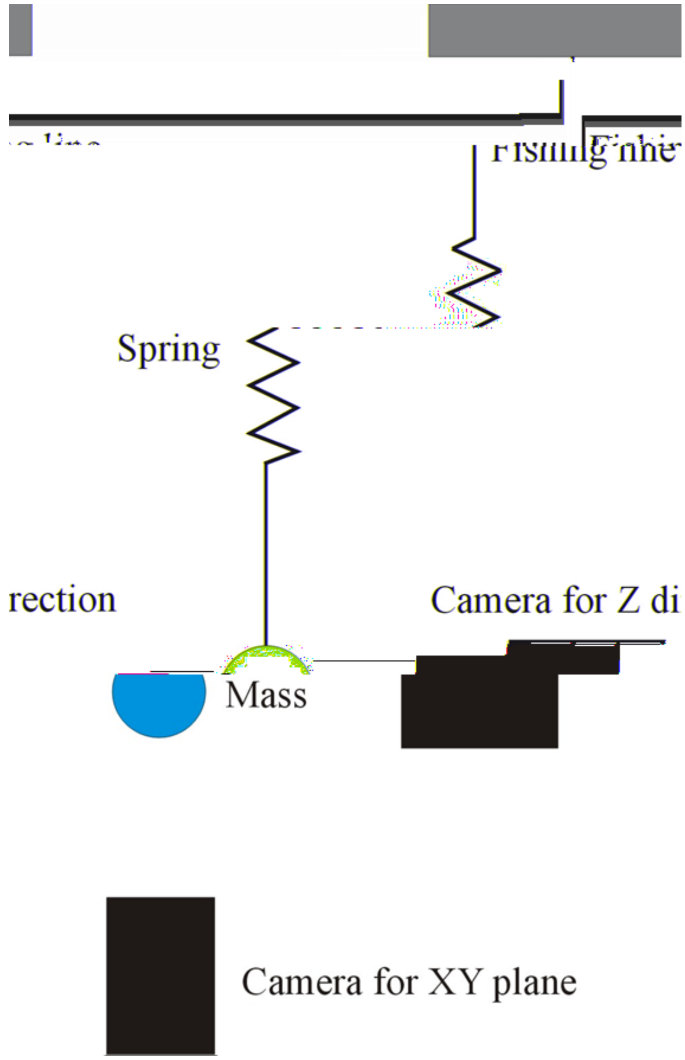

Figure 2: A diagram of the experimental setup.

At the beginning of this step before anything else was done with the spring pendulum, the two video cameras recorded a few seconds of the still ball to be used by the Matlab programs to set the xyz coordinates of the equilibrium point. To do this, the coordinates of the center of the spherical mass were found in each frame of the still ball, and the average of these coordinates was taken, the result being then set as the origin, or the equilibrium point.

When the recording of the still ball was finished, the next important concern in this step was how to create the necessary initial conditions. After much thought and deliberation about launching methods, it was determined that due to the sensitive nature of the swing spring, the best approach would be to create the initial conditions by hand. Through the use black clothing and gloves, the hand proved to work well and have no interference in the Matlab processing of the footage.

After the initial conditions were created by hand, the two video cameras were allowed to record the resulting motion of the swing spring for about three oscillations from springing into swinging and back again. The footage of the motion was then imported into Adobe Premiere Pro CS3, where it was split into clips. The clips required by the Matlab programs were individual clips of the still ball in the xy directions and the z direction, the initial position in the xy directions and the z direction, the initial velocities in the xy directions and the z direction, and the actual swinging and springing motion of the swing spring in the xy plane, broken down into different clips for each swing plane.

Once the splitting of the clips was complete, the second step in the data taking process was to determine the angle of precession. The clips of the individual swing planes were passed into a Matlab function entitled "getCirc4all2." The function found the lines of best fit for the swing planes and output the slopes of the lines. The precession angles were determined from this output by finding the arctangent of the slopes and adding or subtracting them, depending on the case, to find the angle between the lines of best fit.

The third step in the data taking process was to determine the initial conditions, which was accomplished through passing the appropriate clips into the corresponding Matlab functions, entitled "findInitXYPos," "findInitZPos," "findInitXYVel," or "findInitZVel."

The programs that find the initial positions, "findInitXYPos" and "findInitZPos," work first by finding the equilibrium point, as described earlier. Next, in the footage of the position of the ball just before release, the coordinates of the center of the ball in the last frame are found. The program outputs the difference between the coordinates of the equilibrium point and the coordinates of the position just before release, using the appropriate conversion factor between pixels and meters to express the answer. This difference is the displacement from the origin, which is taken to be the initial position of the ball.

The programs that output the initial velocities, "findInitXYVel" and "findInitZVel," work by finding rolling averages of the velocity over every three frames of the footage. The coordinates of the position of the ball in the first and the third frame are found, and the difference between them is taken to be how far the ball traveled. How long it took to travel that far is known since the camera records at 30 frames per second. The velocity can be therefore determined from this information, once again using the appropriate conversion factor between pixels and meters.

Once the initial position and velocity in the x, y, and z directions were found, the correlating point in phase space was located through the transformations between xyz and chi (scaled value of the Hamiltonian) and lambda (scaled value of the angular momentum) provided by Holger Dullin. A Matlab programoi31(ogram)-rdi-463(oOeSwtinSsprin"e)-260utdi-oOe(ogr260(Dulli'ss)-22(transformations)3423tto)-301(tak
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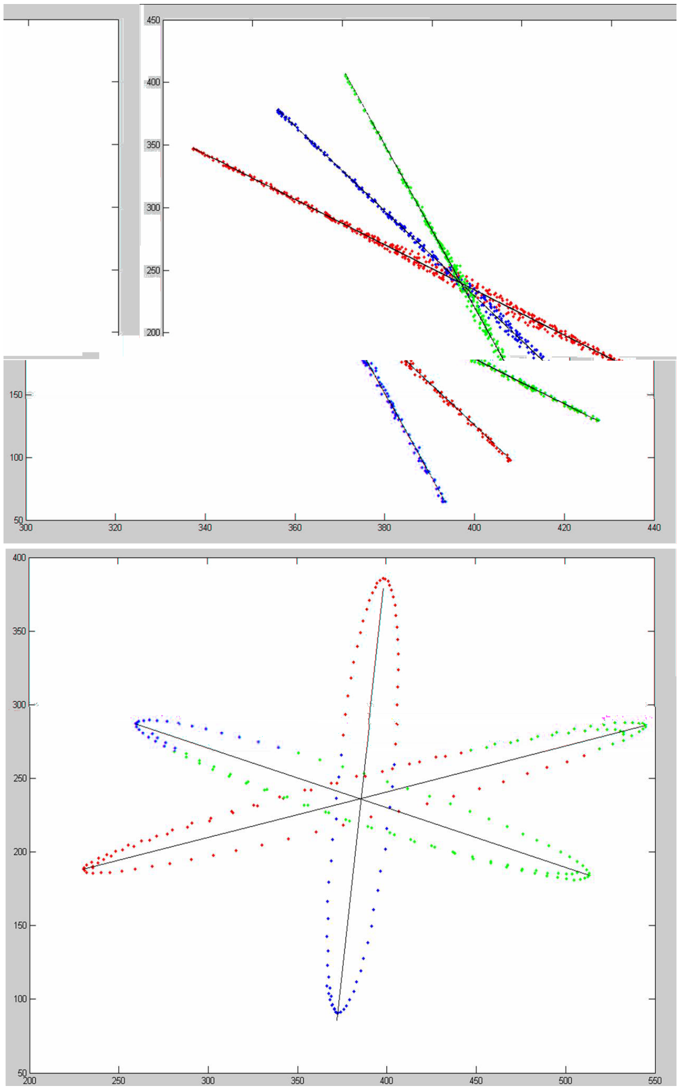

Figure 3: Two examples of typical data sets.

gle trial. Each point in both figures is the location of the center of the ball in a given frame, and each color corresponds to a different swing plane. The axes in the figures are in units of pixels. The black lines are the lines of best fit, and the angle between these lines is the angle of precession.

This experiment was successful in that the measured precession angles and the theoretical precession angles were very much in agreement with each other. In most cases, the measured precession angle was within about a degree of the theoretical angle. Fig. 3 contains two such examples. In the top figure, the measured precession angle was 7.6^{o}, while the theoretical precession angle was 6.6^{o}. In the bottom figure, the measured precession angle was 67.5^{o}, while the theoretical precession angle was 66.6^{o}.

5 Further Work

Unfortunately, the experiment has not progressed to the point where a circle around a point in phase space has been completed. The setup, however, is finished and working smoothly. The techniques are providing measurements for the precession angle that agree with the theoretically predicted values. What is left to be done, then, is just some final data taking. Enough data must be taken until points are found that walk all the way around these two aforementioned points in phase space. The measured precession angles must then be plotted against their location in phase space to show that for the point located elsewhere in phase space, the precession angle comes back to the value it started with, while for the point located at the origin, it does not.

This experiment has been centered on an investigation of classical monodromy; however, monodromy also has quantum manifestations which are planned to be investigated as well. Finally, the demonstration of monodromy in this experiment will, hopefully, be written up and submitted as a journal article, confirming something that has never been demonstrated before.