RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

August, 2022

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Show that a topological space X is Hausdorff if and only if $\Delta := \{(x,x) \mid x \in X\} \subset X \times X$ is closed.

Q.2 A topological group is a topological space G with a group structure such that

- group multiplication $G \times G \to G\colon (g,h) \mapsto gh$ is continuous
- inversion $G \to G\colon g \mapsto g^{-1}$ is continous.

Assume that G is a compact Hausdorff topological group and $g \in G$. Let $A = \{g^n \mid n = 0,1,2,\ldots\}$. Show that $\bar{A}$ – the closure of A – is a subgroup of G. Is this true without compactness of G?

Q.3 Recall that $\mathbb{R}P^n$ – the real projective space of dimension n – is defined as the quotient of S^n by the equivalence relation $x \sim -x$, $x \in S^n$. Here S^n is the unit sphere in $\mathbb{R}^{n+1}$.

a) Compute the fundamental group of $\mathbb{R}P^n$ when $n \geq 2$.

b) Show that every continuous map $\mathbb{R}P^n \to S^1 \times S^1$ is homotopic to a constant one when $n \geq 2$.

c) Is the previous statement true for $n = 1$?

Q.4 Let $M = \mathbb{R}^2$, and consider the vector fields

$$X = 2x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}, \qquad Y = x\,\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial y}$$

on M.

(a) Show that there exist local coordinates (s,t) in some neighborhood U of the point $(1,1)$ such that the restrictions of X and Y to U are given by $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

(b) Find such coordinates explicitly, and verify directly that they satisfy the conditions $X = \frac{\partial}{\partial s}$ and $Y = \frac{\partial}{\partial t}$.

Q.5 Let $a, b \in \mathbb{R}$, and consider the subset S of $\mathbb{R}^3$ defined by the equations

$$x^2 + y^2 - z^2 = a, \qquad x + 2y = b.$$

(a) Show that if $b^2 \neq 5a$, then S is a regular submanifold of $\mathbb{R}^3$.

(b) Describe the set S when $a = b = 0$. Is it a regular submanifold of $\mathbb{R}^3$?

Q.6 Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the quotient map. Let (x,y) be the standard coordinates on $\mathbb{R}^2$, and consider the 2-form

$$\omega = dx \wedge dy$$

on $\mathbb{R}^2$.

(a) Show that ω is closed and exact on $\mathbb{R}^2$.

(b) Show that there exists a 2-form η on $\mathbb{T}^2$ such that $\pi^*\eta = \omega$. (Hint: it suffices to show that for any transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$f(x, y) = (x + n, y + m)$$

for some integers m, n, we have $f^*\omega = \omega$.)

(c) Show that η is closed, but *not* exact, on $\mathbb{T}^2$. (Hint: Compute the integral of η over $\mathbb{T}^2$.)