RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

August, 2020

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

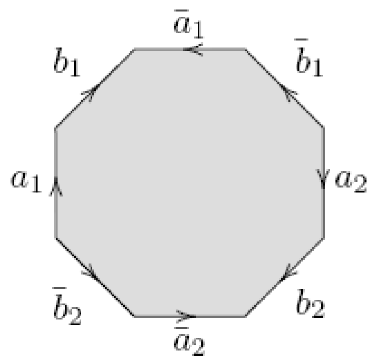

Figure 1: $O \subseteq \mathbb{R}^2$ with gluing data for Σ

Topology

Problem 1.

Let $f\colon X \to Y$ be a function between topological spaces.

(a) Prove that if f is continuous, then whenever a sequence (x_n) converges to x in X, then $(f(x_n))$ converges to $f(x)$ in Y.

(b) Prove that the converse of (a) holds if X is first countable.

Problem 2.

Consider the closed octagonal disk O as in the figure, topologized as a subspace of $\mathbb{R}^2$. Let Σ be the quotient space obtained by identifying the directed edge a_1 with $\bar{a}_1$, a_2 with $\bar{a}_2$, b_1 with $\bar{b}_1$ and b_2 with $\bar{b}_2$. Use the Seifert Van-Kampen Theorem to determine the fundamental group of $\pi_1(\Sigma, y)$ in terms of generators and relations for any base point $y \in \Sigma$.

Problem 3.

Let $\mathbb{R}P^n$ be the quotient of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ by the anatipodal map $a\colon S^n \to S^n$, $a(x) = -x$. You may assume that $\mathbb{R}P^n$ is path connected, locally path connected and semilocally simply-connected. You may assume that the product of path connected spaces is path connected and that the product of semilocally simply-connected spaces is semi-locally simply-connected.

(a) Prove that the product $X_1 \times X_2$ of locally path connected spaces X_1 and X_2 is locally path connected.

(b) Determine the number of equivalence classes of path connected covering spaces over $\mathbb{R}P^n \times \mathbb{R}P^n$. Give a representative for each equivalence class. Prove any claim you make about the the fundamental groups of $\mathbb{R}P^n$ and $\mathbb{R}P^n \times \mathbb{R}P^n$.

Geometry

Problem 4.

(a) Give a careful definition of the tangent bundle of a manifold M. Define the manifold structure of TM.

(b) Show that the tangent bundle of the sphere S^2 is not bundle isomorphic to $S^2 \times \mathbb{R}^2$.

Problem 5.
Consider the following two vector fields on $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \text{and} \quad Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial z} \ .$$

Show that there is no nonempty smooth surface $S \subset \mathbb{R}^3$ that is tangent to both vector fields at each of its points.

Problem 6.

(a) Let $a : S^n \to S^n$ be the antipodal map. Suppose that ω is a smooth form on S^n such that $a^*\omega = \omega$. Prove that if ω is exact, then there is a smooth form η with $\omega = d\eta$ and $a^*\eta = \eta$.

(b) Use (6a) to deduce that on the projective space $\mathbb{RP}^n$ every closed k-form with $0 < k < n$ is exact.

Hint: You may use that for $0 < k < n$ every closed k-form on S^n is exact.