RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Geometry/Topology

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. All problems are weighted equally for grading purposes.
2. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.
3. Label each answer sheet with the problem number.
4. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

(1) Let T and T' be topologies on a space X, with $T \subset T'$. Prove or disprove each of the following statements:

(a) If X is compact in the topology T, then X is compact in the topology T'.

(b) If X is compact in the topology T', then X is compact in the topology T.

(2) (a) State carefully the definitions of:

- Hausdorff topological space;
- Regular topological space.

(b) Prove the following theorem: *Every compact Hausdorff space is regular.* (In proving this, you *may not* use, without proof, the theorem stating that every compact Hausdorff space is normal.)

(3) Let M be a compact connected oriented manifold of dimension n with boundary. Assume that the boundary of M has two connected components M_0 and M_1, and let $i_k \colon M_k \to M$ be the inclusion maps. Let α be a form of degree p on M and β be a form of degree $(n-p-1)$ on M. Assume that $i_0^*\alpha = 0$ and $i_1^*\beta = 0$. Prove that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

(4) Let $g = y^\lambda \,(dx^2 + dy^2)$, $\lambda \in \mathbb{R}$, be a Riemannian metric on $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. For this metric compute Levi-Civita connection and write (but do not solve) the differential equation for geodesics.

(5) Recall that $\mathbb{R}P^n$ – the real projective space of dimension n – is defined as the quotient of S^n by the equivalence relation $x \sim -x$, $x \in S^n$. Here S^n is the unit sphere in $\mathbb{R}^{n+1}$. Let $n \geq 2$.

(a) Compute the fundamental group of $\mathbb{R}P^n$.

(b) Show that every continuous map $\mathbb{R}P^n \to S^1$ is homotopic to a constant one.

(6) For a matrix U in the ring $M_{n\times n}(\mathbb{R})$ of $n \times n$ matrices with real entries, we define the *exponential* e^U of U by

$$e^U = I + U + \frac{U^2}{2!} + \frac{U^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{U^k}{k!}$$

(I denotes the identity matrix).

(a) Show that, for any $U \in M_{n\times n}(\mathbb{R})$,

$$U = \frac{d}{dt}e^{tU}\big|_{t=0}.$$

(Of course, tU denotes the product of the scalar t and the matrix U. Also, the differentiation on the right-hand side is element-wise; that is, the derivative of a matrix is the matrix of the derivatives.) You may assume that the derivative on the right-hand side exists, and that differentiation works for infinite sums here just as it would for finite sums.

(b) For a subgroup N of the group $GL(n,\mathbb{R})$ of invertible matrices in $M_{n\times n}(\mathbb{R})$, we define the *Lie algebra* $\mathrm{Lie}(N)$ of N by

$$\mathrm{Lie}(N) = \{U \in M_{n\times n}(\mathbb{R})\colon e^{tU} \in N \text{ for all } t \in \mathbb{R}\}.$$

Show that, if $n = 3$ and N is the *Heisenberg group*:

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subset GL(3,\mathbb{R}),$$

then

$$\mathrm{Lie}(N) = \left\{ \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Hint: part (a) above may be of use.

(c) Any $U \in \mathrm{Lie}(N)$ defines a differential operator on the algebra $C^\infty(N)$ of infinitely differentiable functions on N by the formula

$$Uf(n) = \frac{d}{dt}f\big(n \cdot e^{tU}\big)\big|_{t=0} \quad (f \in C^\infty(N), n \in N).$$

Let $Z \in \mathrm{Lie}(N)$ be the matrix with a one in the second row, third column, and zeroes elsewhere. Show that, as a differential operator on $C^\infty(N)$,

$$Zf = \frac{\partial f}{\partial b} + a\frac{\partial f}{\partial c}.$$

(Here, we are thinking of $f \in C^\infty(N)$ as a function $f(a, b, c)$ of the given coordinates of N.)