Proceedings of the 7th International Indoor Air Quality, Ventilation and Energy Conservation in Buildings Conference (IAQVEC 2010), August 15-18, Syracuse, NY

Impact of Time-splitting Schemes on the Accuracy of FFD Simulations

Jianjun Hu, Wangda Zuo, Qingyan Chen*

National Air Transportation Center of Excellence for Research in the Intermodal Transport Environment (RITE), School of Mechanical Engineering, Purdue University, 585 Purdue Mall, West Lafayette, IN 47907-2088USA

Corresponding email: yanchen@purdue.edu

ABSTRACT

Recently, a Fast fluid dynamics (FFD) scheme was proposed for fast flow simulations in buildings. The FFD used a time-splitting scheme to solve the Navier-Stokes equations. The principle of the splitting method is to break a complex equation into several simple sub-equations which can be easily solved. By this means, one can reduce numerical efforts on solving the complex equation. However, errors may be introduced during the splitting process. The FFD applied the first order Godunov method to split unsteady Navier-Stokes equations. This study tries to improve the FFD by applying a second-order Strang method for splitting. Then, the FFD model with Godunov and Strang methods are compared by simulating different flows. The results show that the FFD with the Strang method has the same performance as that with the Godunov method. The reason is that other numerical schemes applied in the FFD are first-order in time, such as a first-order implicit scheme for diffusion equation. As a whole, the FFD scheme is still a first-order method in time even though the splitting scheme is second-order. Thus, to improve the accuracy of the FFD scheme, one should apply higher order schemes not only for splitting, but also for other schemes in the FFD model.

KEYWORDS

Time-splitting Methods, Fast Fluid Dynamics, Strang Method, Godunov Method

INTRODUCTION

Many applications demand a fast solution for airflow both inside and outside a building. For instance, building designers need to know airflow distribution for a sustainable building design and emergency management personnel rely on smoke and fire prediction to evacuate people. The most popular models for airflow prediction are multizone network models, zonal models and Computational Fluid Dynamics (CFD) models (Chen, 2009).

Based on homogeneous assumption of indoor air, the simulation time of the multizone network models can be as short as a few seconds (Axley, 2007). But the homogeneous assumption is not proper when the air is stratified. Zonal models can also predict indoor airflow in a short time (Megri and Haghighat, 2007). However, it requires prior knowledge of indoor air distributions. By solving Navier-Stokes equations and other conservation equations for mass, energy, and species, the CFD can provide the most detailed information of indoor air distributions. However, the CFD simulations need tremendous computing efforts. Therefore, the CFD models cannot be applied to the cases when fast solutions of airflow are required.

The fast solution of airflow can be achieved by applying alternative models, which should be reliable and easy to implement. The FFD model fits the need. It was firstly developed by Stam (1999) for computer games. By sacrificing some accuracy, the FFD model can significantly reduce the computing effort. It has been proved that the model is an intermediate approach between nodal models and CFD models (Zuo and Chen, 2009). Different from CFD models and nodal models, the numerical method adopted by FFD model is time-splitting method. It's a reliable method for solving partial differential equation (Crandall, 1980; LeVeque, 1982). The concept has been developed by Godunov (1959). Strang (1968) further developed a high order splitting method. Because of its advantage that it can reduce a large amount of computing work to advance the solution one time step, several kinds of time-spitting methods have been studied and applied (Liu and Pletcher, 2007; Chertock, 2009). The numerical scheme to solve flow governing equations is the core of FFD model. Therefore, the FFD models can be further improved by employing alternative numerical scheme. This paper focuses on our effort to improve the accuracy of FFD simulations by improving the time-splitting scheme.

TIME-SPLITTING SCHEME

The purpose of this section is to elucidate the time-splitting scheme. Let's consider a differential equation:

$$ {}_t + \nabla \cdot \mathbf{f}(u) = \varepsilon \Delta u, \quad (\mathbf{x}, 0) = {}_0(\mathbf{x}) \tag{1}$$

Here,

$$\mathbf{x} = (x_1, \cdots, x_m),\ \mathbf{f}(u) = \big(f_1(u), \cdots, f_m(u)\big),\ \nabla = (\partial/\partial x, \cdots, \partial/\partial x_m),\ \Delta = \textstyle\sum_{i=1}^{m} \partial^2/\partial x_i^2 .$$

The principle of time-splitting scheme can be summarized as follows. We denote by S_H and S_P the solution operator of Eq. (2) and Eq. (3) respectively:

Hyperbolic equation

$$ {}_t + \nabla \cdot \mathbf{f}(u) = 0, \tag{2}$$

Parabolic equation

$$ {}_t = \varepsilon \Delta u \tag{3}$$

If the time step-size is Δt, then there are a fixed set of times $\Delta t_{1,h}, \cdots, \Delta t_{a,h}$ and $\Delta t_{1,p}, \cdots, \Delta t_{b,p}$, which satisfy

$$\Delta t_{1,h} + \Delta t_{2,h} + \cdots + \Delta t_{a,h} = \Delta t, \tag{4}$$

$$\Delta t_{1,p} + \Delta t_{2,p} + \cdots + \Delta t_{b,p} = \Delta t, \tag{5}$$

where either a=b or a=b+1. Let's say the initial value is u($\mathbf{x}$, t). Firstly, the Eq. (2) is solved on the time interval $(t, t + \Delta t_{1,h}]$, $u^*(\mathbf{x}) = S_H(\Delta t_{1,h})\, u(\mathbf{x}, t)$, then take the u^* as initial value to solve Eq. (3) on the time interval $(t, t + \Delta t_{1,p}]$, $u^{**}(\mathbf{x}) = S_P(\Delta t_{1,p})\, u^*(\mathbf{x})$. This state in turn is advanced using S_H on the time interval $(t + \Delta t_{1,h}, t + \Delta t_{1,h} + \Delta t_{2,h}]$. The procedure is continued until the final step of $\Delta t_{a,h}$ using S_H or $\Delta t_{b,p}$ using S_P is taken (depending on whether a=b+1 or a=b). This yields to an approximation of u($\mathbf{x}$, t+Δt). So we can get the final solution of Eq. (1) which evolves from u(**x**, t) to u(**x**, t+Δt) in a+b substeps. The procedure can be summarized by following:

$$u(\mathbf{x}, t+\Delta t) = S_H(\Delta t_{a,h}) \circ S_P(\Delta t_{b,p}) \circ \cdots \circ S_P(\Delta t_{1,p}) \circ S_H(\Delta t_{1,h}) u(\mathbf{x}, t), \text{ while a=b+1}$$

$$u(\mathbf{x}, t+\Delta t) = S_P(\Delta t_{b,p}) \circ S_H(\Delta t_{a,h}) \circ \cdots \circ S_P(\Delta t_{1,p}) \circ S_H(\Delta t_{1,h}) u(\mathbf{x}, t), \text{ while a=b}$$

The splitting accuracy of the approximation mostly depends on the choice of $\Delta t_{1,h}, \cdots, \Delta t_{a,h}$ and $\Delta t_{1,p}, \cdots, \Delta t_{b,p}$. The simplest example which involves only two steps is $\Delta t_{a,h} = \Delta t_{b,p} = \Delta t$. The splitting is first order accurate (the local error is o(Δt^2)) and it's called Godunov method (Godunov, 1959). Another widely spread method is Strang method (Strang, 1968), which involves three steps, $\Delta t_{1,h} = \Delta t/2$, $\Delta t_{1,p} = \Delta t$ and $\Delta t_{2,h} = \Delta t/2$. It has been proved that the method is second order accurate (LeVeque, 1982).

However, the accuracy of time-splitting method is not only affected by the splitting method, the numerical error of each sub-equation is also important for the truncation error of time-splitting scheme. Randall J. LeVeque (1982) concluded that the truncation error of time-splitting scheme was the sum of splitting error and numerical error of each sub-equation.

$$TE_{TS} = E_{Split} + \sum TE_{SE} \tag{6}$$

Here, TE_{TS} is truncation error of time-splitting method, E_{Split} is the splitting error, $\sum TE_{SE}$ is the numerical error of each sub-equation.

FAST FLUID DYNAMICS

The FFD scheme was developed by Stam (1999) for flow simulation in computer games. It solves the Navier-Stokes equations as the CFD does.

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} + \upsilon \frac{\partial^2 u_i}{\partial x_j^2} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{f_i}{\rho} \tag{7}$$

Where u_i and u_j are fluid velocity components in x_i and x_j directions, respectively; υ is kinematic viscosity; ρ is fluid density; p is pressure; t is time; and f_i are forces, such as buoyancy force and other external forces.

By using the time-splitting scheme, the FFD splits the Navier-Stokes Eq. (7) into the following four equations and solve them in order.

Source:

$$\frac{\partial u_i}{\partial t} = \frac{f_i}{\rho} \tag{8}$$

Diffusion:

$$\frac{\partial u_i}{\partial t} = \upsilon \frac{\partial^2 u_i}{\partial x_j^2} \tag{9}$$

Advection:

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} \tag{10}$$

Projection:

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{11}$$

Stam (1999) solved Eqs. (8) and (9) with first order implicit scheme. Then, it applies a first order semi-Lagrangian method (Courant et al. 1952) to solve the advection Eq. (10). Finally, it ensures mass conservation by solving pressure equation (11) and continuity equation (12) together with a pressure-correction projection method (Chorin, 1967).

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{12}$$

In the previous research, the Godunov method has been applied in the FFD solver (Zuo and Chen, 2009). It is possible to further improve the accuracy of the FFD model by using a higher order Strang splitting method. In addition, it is worthy to know whether the sequence on solving those sub-equations will affect on the final results. Thus, this study proposed and evaluated three time-splitting schemes as follows:

Type-1:

$$u_i^{(0)} \xrightarrow[source]{} u_i^{(1)} \xrightarrow[diffusion]{} u_i^{(2)} \xrightarrow[advection]{} u_i^{(3)} \xrightarrow[projection]{} u_i^{(4)} \tag{13}$$

Type-2:

$$u_i^{(0)} \xrightarrow[source]{} u_i^{(1)} \xrightarrow[advection]{} u_i^{(2)} \xrightarrow[diffusion]{} u_i^{(3)} \xrightarrow[projection]{} u_i^{(4)} \tag{14}$$

Type-3:

$$u_i^{(0)} \xrightarrow[source]{} u_i^{(1)} \xrightarrow[advection(\Delta t/2)]{} u_i^{(2)} \xrightarrow[diffusion]{} u_i^{(3)} \xrightarrow[advection(\Delta t/2)]{} u_i^{(4)} \xrightarrow[projection]{} u_i^{(5)} \tag{15}$$

RESULTS

To evaluate the impacts of the proposed three splitting schemes on the FFD model, this study simulated three different flows by using the FFD with those schemes. The studied flows are ideal flow described by the Burgers' equation, lid-driven cavity flow and forced convection flow.

Burgers' equation

Burgers' equation is a nonlinear partial differential equation. The Burgers' equation is a typical unsteady advection-diffusion equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \upsilon\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = -\tanh x \tag{16}$$

The exact solution of this equation was given by the Hopf formula (Chertock and Kurganov). With υ=0.1, the three time-splitting schemes can be also applied to solve the equation. Obviously, the projection step is unnecessary in this case. Due to the lack of source term, it has only two sub-equations that need to be solved. It employs the semi-Lagrangian method whose truncation error is first order in space and second order in time to solve advection equation. The diffusion equation is solved by the second order central differential scheme. Type-1 and Type-2 time-splitting scheme applied Godunov splitting method and Type-3 scheme employed Strang method. So the splitting error of the three schemes are o(Δt^2) for Type-1 and Type-2, o(Δt^3) for Type-3. According to the Eq. (6), truncation error of the three time-splitting schemes will be:

Type-1 & Type-2:

$$TE_{1\&2} = E_{split} + TE_{Adv} + TE_{Diff} = \left[o\left(\Delta t^2\right)\right] + \left[o\left(\Delta x\right) + o\left(\Delta t^2\right)\right] + \left[o\left(\Delta t^2\right) + o\left(\Delta t \cdot \Delta x^2\right)\right]$$
$$= o\left(\Delta x\right) + o\left(\Delta t^2\right) \quad (17)$$

Type-3:

$$TE_3 = E_{split} + 2 \cdot TE_{Adv} + TE_{Diff} = \left[o\left(\Delta t^3\right)\right] + 2 \cdot \left[o\left(\Delta x\right) + o\left(\Delta t^2/4\right)\right] + \left[o\left(\Delta t^2\right) + o\left(\Delta t \cdot \Delta x^2\right)\right]$$
$$= o\left(\Delta x\right) + o\left(\Delta t^2\right) \quad (18)$$

Although the type-3 scheme has applied higher order splitting method, the truncation error of the three schemes are the same. It means that the third scheme has no remarkable advantage over other two schemes. We can also find the same conclusion from Fig.1.

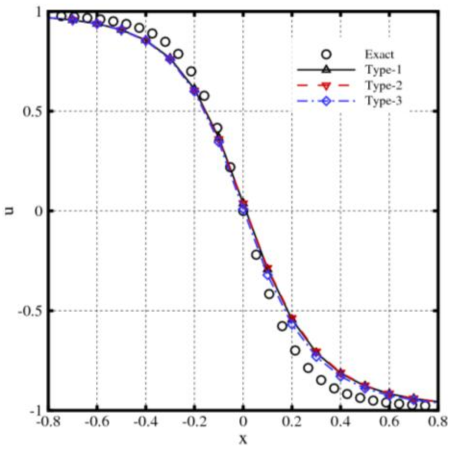


Fig.1 Comparison of Numerical Solution of Burgers' equation, t=2

Lid-Driven cavity flow

The flow in a square lid-driven cavity is like the circulated flow in a room (Fig.2). Assuming the density and dynamic viscosity of the fluid are 1.0kg/m^3 and 0.01kg/m.s, based on the cavity height (1 m) and lid velocity (1 m/s), the Reynolds number of the studied flow is 100. The flow under this Reynolds number is laminar. We have employed four different meshes (17×17, 33×33, 65×65, 129×129) and the results showed a mesh with 65×65 grids was fine enough. High quality CFD data from Ghia et al. (1982) was used as reference. The calculation results which employed the three time-splitting schemes have been shown in the Fig.3. The profiles indicate that all of these three schemes can obtain fine enough result. The first and second scheme predicted exactly same results. The third scheme which employed higher order splitting method didn't make remarkable improvement.

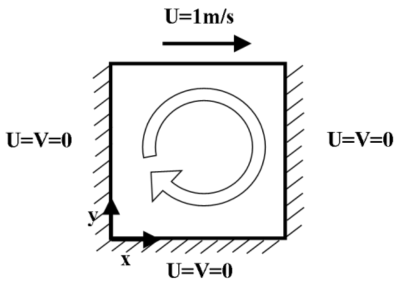
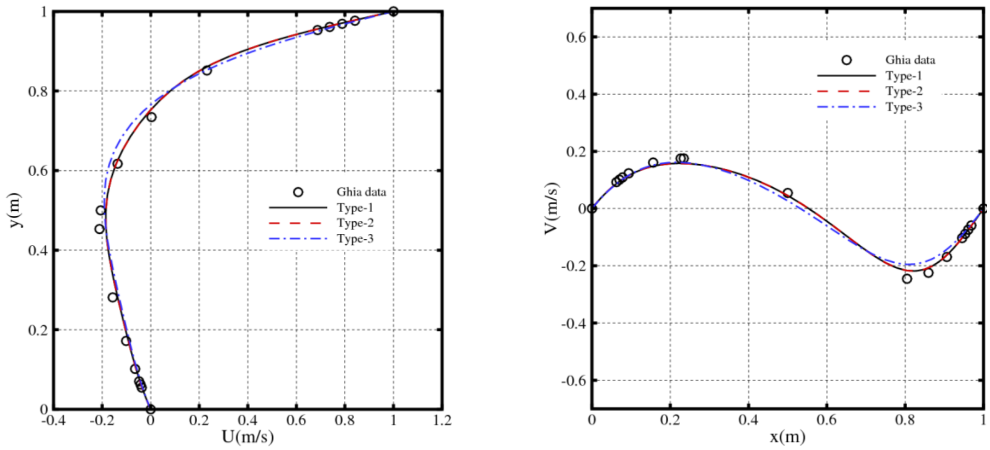


Fig.2 Schematic of Lid-Driven cavity flow



(a) U at x = 0.5m (b) V at y = 0.5m

Fig.3 Comparison of horizontal and vertical velocities in a lid-driven cavity flow predicted by the FFD with different time-splitting methods. The reference data is high quality CFD data by Ghia et al. (1982).

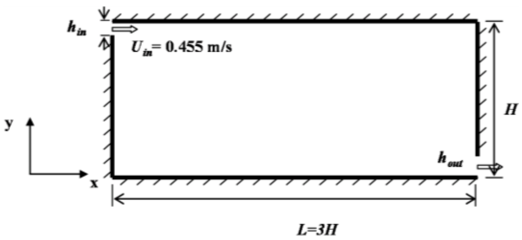
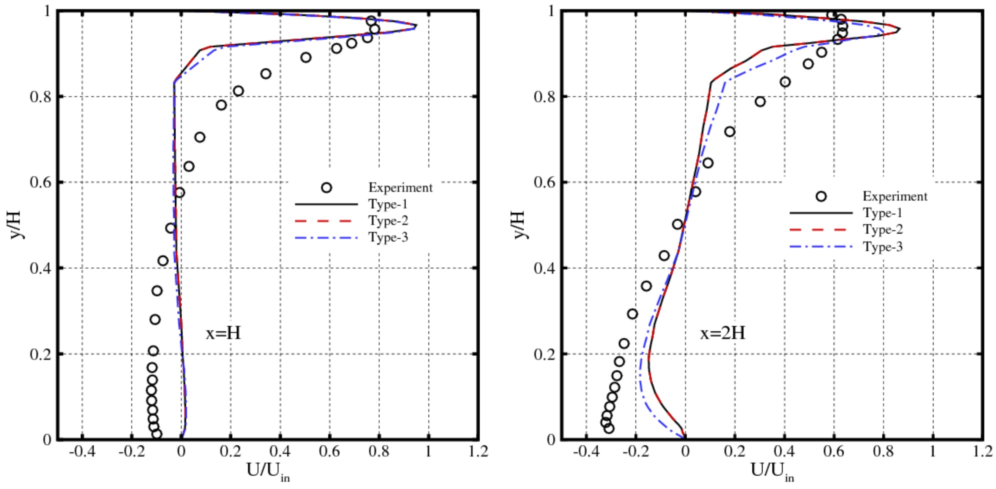


Fig.4 Schematic view of forced convection flow case

Forced convection flow

The forced convection in an empty room is based on Nielsen's experiment (Nielsen, 1990). The height of the room is 3m, which is defined by H. The length of the room is 3H (9m). The air horizontally goes into the room at a speed of 0.455m/s through a 0.168m wide opening at the upper-left corner. The opening for outflow is located at the lower-right corner with 0.48m wide. Based on the inlet velocity and the inlet opening width, the Reynolds number is 5000, so it's the fully developed turbulent flow. The flow pattern was two-dimensional. The results presented are based on a mesh with 60×20 non-uniform grids for all the simulations. Fig.5 indicates that we can capture the major airflow pattern with the FFD method. However, due to the lack of turbulence model for this case, the results calculated by the three time-splitting methods are not good enough when compared with measured data. The higher order time-splitting method which is the third method didn't improve the results. Further study should not just base on the time-splitting method.



(a) x=H (b) x=2H

Fig.5 Compares of horizontal velocities of forced convection flow calculated by FFD with different time-splitting methods

CONCLUSIONS

Our study focused on improving the accuracy of FFD simulations. By changing the splitting method and the calculation sequential order, three types of time-splitting scheme have been tested. The study led to the following conclusion:

(1) Although higher order splitting method can be applied to the time-splitting scheme, the final truncation error of the time-splitting scheme could be the same as the one with lower order splitting method.
(2) The calculation sequential order has no great impact on the final accuracy.
(3) By applying higher order schemes on both splitting method and time advanced schemes for sub-equations, the improvement of the accuracy of FFD scheme can be achieved.

ACKNOWLEDGEMENTS

This study was funded by the US Federal Aviation Administration (FAA) Office of Aerospace Medicine through the National Air Transportation Center of Excellence for Research in the Intermodal Transport Environment under Cooperative Agreement 07-CRITE-PU.

REFERENCES

Axley J. 2007. Multizone airflow modeling in buildings: History and theory. *HVAC&R Research*, **13**, 907-928.

Chen Q. 2009. Ventilation performance prediction for buildings: A method overview and recent applications. *Building and Environment*, **44**, 848-858.

Chertock A., Kurganov A., and Petrova G. 2009. Fact explicit operator splitting method for convection-diffusion equations. *International Journal for Numerical Methods in Fluid*, **59**, 309-332.

Chertock A. and Kurganov A. On splitting-based numerical methods for convection-diffusion equations. *Quaderni di Matematica*, accepted.

Chorin A.J. 1967. A numerical method for solving incompressible viscous flow problems. *Journal of Computational Physics,* **2**, 12-26.

Courant R., Isaacson E., and Rees M. 1952. On the Solution of Nonlinear Hyperbolic Differential Equations by Finite Differences. *Communication on Pure and Applied Mathematics* vol. **5**, 243–255.

Crandall M. and Majda A. 1980. The Method of Fractional Steps for Conservation Laws. *Numerische Mathematik*, **34**, 285-314.

Ghia U., Ghia K.N., and Shin C.T. 1982. High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method. *Journal of Computational Physics,* **48**, 387-411.

Godunov S.K. 1959. Finitedifference methods for numerical computation of discontinuous solutions of the equations of fluid dynamics. *Mathematics of the USSR Sbornik*, **47**, 271–295.

Leveque R.J. 1982. Time-Split Methods for Partial Differential Equations. *Ph.D. Thesis*, Stanford University.

Liu K. and Pletcher R.H. 2007. A Fractional Step Method for Solving the Compressible Navier-Stokes Equations. *Journal of Computational Physics,* **226**, 1930-1951.

Megri A.C. and Haghighat F. 2007. Zonal modeling for simulating indoor environment of buildings: Review, recent developments, and applications. *HVAC&R Research,* **13**, 887-905.

Nielsen P.V. 2004. Computational fluid dynamics and room air movement. *Indoor Air,* **14**, 134-143.

Nielsen P.V. 1990. *Specification of a two-dimensional test case*, Aalborg, Denmark, Aalborg University.

Stam J. 1999. Stable fluids. In: *Proceedings of 26th International Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'99)*, Los Angeles, pp. 121-128.

Strang G. 1968. On the construction and comparison of difference schemes. *SIAM Journal on Numerical Analysis*, **5**, 506–517.

Zuo W. and Chen Q. 2009. Real-time or faster-than-real-time simulation of airflow in buildings. *Indoor Air*, **19**, 33-44.