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Investigating The Origins of Teachers' Beliefs of Students' Algebra Development

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Abstract

Elementary, Middle, and High School mathematics teachers (n=105) rank ordered a set of mathematics problems based on expectations of their relative problem-solving difficulty. The problems were designed to systematically compare the effects of arithmetic versus algebraic structure along one dimension, and symbolic versus verbal reasoning along a second dimension. Teachers also rated their level of agreement to a variety of statements on teaching and learning mathematics. Analyses of the difficulty ranking data and the attitude surveys suggest that teachers hold a textbook view of mathematical development, which leads them to believe that arithmetic reasoning strictly precedes algebraic reasoning, and that students' symbolic problem-solving skills develop before verbal reasoning abilities. Teachers accurately predicted a performance advantage for arithmetic problem solving over algebra, but incorrectly predicted that symbolic equation problems would be easier for students than matched word and story problems. Grade level differences among teachers were also found. High school teachers were most likely to hold the textbook view and made significantly poorer predictions of students' performances than did elementary and middle school teachers. Middle school teachers made the most accurate predictions of problem solving difficulty for students. Implications for mathematics instruction and teacher education are discussed.

Investigating the Origins of Teachers' Beliefs of Students' Algebra Development

The study of people engaged in cognitively demanding tasks must consider the relationship between people's judgements and actions, and the beliefs they hold. Several aspects of people's decision-making are well established. People do not strictly follow the laws of logic and probability when weighing information or following implications (Wason & Johnson-Laird, 1972; Cheng, Holyoak, Nisbett, & Oliver, 1986). In fact, most of the time, human decision making differs substantially from the normative logical process (Kahneman & Tversky, 1973; Rosch, 1973; Tversky & Kahneman, 1973; H.Simon, 1969). These characteristics, coupled with an appreciation of the inherent limitations of human attention, short-term memory and cognitive processing (e.g., Just & Carpenter, 1996; Baddely & Hitch, 1974; Miller, 1956) have led researchers of complex cognitive behavior to regard human decision making as "reasonable," rather than rational (Borko & Shavelson, 1990).

One area of complex cognitive behavior that is of particular interest is the study of teaching. The decisions and actions of teachers have been studied in order to uncover the links between teachers' practices and the knowledge and beliefs that are hypothesized to mediate them. For example, investigators have found that teachers' interpretations and implementations of school curricula are influenced greatly by their beliefs and knowledge of instruction and of student learning (Ball, 1988; Clark & Peterson, 1986; Ernest, 1988; Romberg & Carpenter, 1986; Thompson, 1984). Ball (1988) found that pre-service teachers' beliefs about mathematics instruction and learning were largely formed during their schooling years, and were heavily shaped by their own experience as mathematics students. Members of the Cognitively-Guided Instruction (CGI) program for elementary school mathematics (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter, Franke, and Carey, 1992) found that teachers who viewed students as active constructors of their own mathematical knowledge enacted different instructional practices than their peers, employing pedagogical strategies that involved listening more to their students' mathematical ideas and making more direct connections to students' prior knowledge and strategies.

Teachers' beliefs are crucial to the enactment of curricular goals. Yet Thompson (1992) cautions us to treat teachers' beliefs critically. "Teachers' often unexamined assumptions or beliefs about what children are capable or not capable of learning can render them impervious to matters of children's cognitions; we must find ways of helping teachers examine those assumptions" (pp. 142-3). For example, as Thompson (1992) reports, many teachers have the belief that mathematics instruction should emphasize student mastery of symbols and procedures. The instructional approaches that follow from this view of mathematics, in practice, tend to focus on the mechanics of symbol manipulation while the conceptual underpinnings of those symbols and procedures is rarely addressed.

Investigators in the CGI program have shown that teachers' instructional practices can be changed along the intended direction by providing them with well-organized information about children's thinking and strategy use during simple arithmetic story problem solving (Carpenter et al., 1989). Carpenter and his colleagues chose to focus on students' problem-solving behaviors and performances because of the perceived match with teachers' focus on content and instruction.

Programs designed to institute teacher change are unlikely to succeed unless they can be made compatible with teachers' existing belief systems (Eisenhart et al., 1988). Consequently, understanding teachers' beliefs about instruction and student learning is essential for instituting changes in teachers' practices, be it for teacher education, or for the implementation of reform-based curricula (Fenstermacher, 1979).

Teachers' Beliefs of Algebra Problem-Solving Difficulty

To understand teachers' beliefs about students' algebraic reasoning, Nathan and Koedinger (in press) analyzed teachers' expectations of high school students' problem-solving difficulties. The investigators asked high school mathematics teachers (n=67) to rank order the relative difficulty of mathematics problems that varied along two dimensions (see Table 1). Along the first dimension, the unknown value was positioned to be either the result of a problem (Table 1 problems P4, P5, and P6) as found in arithmetic, or at the beginning of the problem (Table 1 problems P1, P2, and P3) so that the unknown value was expressed in relation to other known quantities, as found in algebra level problems. Task analyses of these two problem types show that result-unknown problems are solvable by direct application of the arithmetic operators, while start-unknown problems extend beyond immediate arithmetic reasoning (e.g., Tabachneck, Koedinger, and Nathan, 1995).

The second factor of problem difficulty compared problems presented in a verbal format with a context (a story problem, Table 1 problems P1 and P4), problems in a verbal format with no context (a word equation problem, Table 1, problems P2 and P5), or problems in a symbolic format (a symbol equation, Table 1 problems P3 and P6). Table 1 shows a sample problem of each type, organized by the two major factors of unknown position and presentation format.

Place Table 1 about here

Prior studies have found that teachers' expectations are only partially accurate in predicting students' problem-solving performance (Nathan & Koedinger, in press). Children and adults have much higher levels of performance with result-unknown than with start-unknown problems (e.g., Riley, Greeno, & Heller, 1983; DeCorte & Vershaffel 1996; Fuson 1988; Heffernan, & Koedinger, 1997; Koedinger & Nathan, 1999; Tabachneck, Koedinger, & Nathan, 1995). For example, Koedinger & Nathan (1999; see also Table 2, Column 5) found that high school students' (n=76) level of problem-solving performance on the start-unknown algebra problems is significantly below that of result-unknown problems, $\underline{F}(1,75) = 48.9$, $\underline{p} < .0001$. This general finding was replicated the following year (n=171). This performance pattern was accurately predicted by the majority of teachers in the sample (84%), though some high school teachers (6%) had the reverse expectation.

Place Table 2 about here

The high school students in Koedinger and Nathan's samples also showed a pronounced advantage for verbally presented problems (story and word equations) over symbolic equations, $F(2,74) = 12.6$, $p<.0001$ (Table 2). This finding was replicated the following year, and has been found among students in several populations, across 6th through 9th grades, in several regions of the U. S. (Koedinger, Tabachneck, & Nathan, 1995; Koedinger & Nathan, 1999; Verzoni & Koedinger, 1997). However, teachers' predictions on this matter are generally inaccurate. The majority of teachers (70%) ranked verbally presented problems as consistently more difficult than symbol equations within both levels of arithmetic and algebra (Nathan & Koedinger, in press).

Analyses of students' solution strategies and errors revealed widespread use of informal solution methods such as guess-and-test and unwinding (for more details of students' strategies see Koedinger & Nathan, 1999; Nathan & Koedinger, in press). The guess-and-test method models the problem situation, and uses an iterative approach to substitute a chosen value for the unknown (the guess) to see if a contradiction is reached with the quantitative constraints given in the problem (e.g., Hall et al., 1989). Unwinding belongs to a class of "working backwards" methods (e.g., Kieran, 1988; Polya, 1957). It strips away the given relations in the opposite order they are given, while at the same time inverting the mathematical operations given. Unwinding essentially takes a multi-step, start-unknown problem and turns it into a series of one-step, arithmetic calculations (Nathan & Koedinger, in press).

Verbal problems were far more likely to elicit informal strategies such as unwinding and guess-and-test than were symbol problems (cf. Mayer, 1982). Interestingly, these informal methods were far more likely than formally taught symbolic procedures to produce a correct answer — even though all of the high school students had gone through a traditional Algebra I curriculum. While formal symbol manipulation led to a correct answer on start-unknown problems 54% of the time, the informal unwinding method was correct 72% of the time, and the guess-and-test strategy was correct 68% of the time (Nathan & Koedinger, in press).

The teachers' problem difficulty ranking data suggests that teachers are largely unaware of the efficacy of students' informal solution strategies, and of the role verbal reasoning plays in the development of students' mathematical problem solving.

Because of the emphasis on verbal reasoning shown in students' solution methods, Nathan & Koedinger (in press) described students' developing algebra competence within a verbal precedence model which emphasized the central role of verbally based reasoning (such as constraint untangling). They contrasted this to the textbook view, where symbolic reasoning, as a form of "pure mathematics," is considered to be a pre-requisite for later verbal "applications." In a recent analysis of four widely used textbook series on pre-algebra and algebra, Nathan & Long, (1999) showed that textbooks tend to sequence symbolic problems before verbal problems in both arithmetic and algebraic lessons. As Greeno, Collins, & Resnick (1996) point out in their discussion of curricular design based on the behaviorist view,

> Typical sequences of instruction begin with training in a procedure, facts, or vocabulary in a simplified context, followed by presentation of the material in somewhat more complicated settings. Standard mathematics textbooks are examples, in that procedures for calculating are presented and practiced, followed by word problems (p. 33).

In a quantitative comparison between the textbook and verbal-precedence models of algebra development, Nathan and Koedinger (in press) showed that the verbal precedence model gave a better account of students' early algebra performance than the textbook view exhibited by most teachers. The verbal precedence model predicted the performances of 91% of students (88% of the students in the replication study), while the textbook model predicted only 62% of students (46% of the students in the replication study.

The Textbook Hypothesis

Nathan and Koedinger (in press) hypothesized that teachers operate with a textbook view of mathematical development that mirrors the view implicitly presented in popular algebra textbooks. Initial evidence for this hypothesis was obtained using Kendall's rank correlation test.

The test between teachers' ordering and the curricular sequence offered by several mathematics textbook series revealed highly significant agreement between the two, $\tau(12)=.867$, $p<.02$. One likely interpretation of this pattern is that, through reliance on and repeated exposure to mathematics textbooks, teachers internalize the textbook view as a basis for their predictions of problem difficulty for students. Alternatively, both teachers and textbook authors may base their views of algebra development on a common perspective, such as the component view of mathematics learning (cf. Greeno, Collins, & Resnick, 1996).

To more fully understand the cognitive processes that underlie teachers' responses to the difficulty ranking task, the following study was conducted. Data on the relative difficulty ranking of problems were collected from a new sample of teachers, using the two dimensions of mathematical factors presented in Table 1. Teachers' views about several issues of mathematical performance, learning, and instruction were also examined using a belief instrument which included constructs that directly addressed teachers' support for the textbook view of algebra development. Additionally, participating teachers from a variety of grade levels (elementary, middle, and high school) were studied to examine the generality of these findings.

We set out to test two hypotheses. First, we hypothesize that teachers in the new sample will make predictions about problem difficulty in accordance with the textbook view of mathematical development, regardless of the grade level they teach, and will tend to rank order verbal problems as relatively more difficult to solve than matched symbolic equation problems. This would replicate previous findings, and extend their generality to a different region of the country, and across a greater range of grade level instruction. Second, we hypothesize that teachers' judgements about problem difficulty are correlated to their level of agreement with items consistent with the textbook view of mathematical development. This establishes a relationship between teachers' beliefs about learning and instruction in general, and their judgements concerning mathematics problem difficulty.

Method

Subjects

Participants of this experiment were 107 K-12 grade teachers who attended a school district-sponsored mathematics workshop during the fall. Of the 107 original participants, two produced forms insufficiently completed (less than half of the difficulty rank or belief statements had responses) and so those data were removed from the sample prior to analysis. This left a final count of 105 subjects. All participants were either mathematics teachers or elementary school teachers who taught all of the major subject areas, including mathematics. The teachers' reported grade levels of instruction ranged from 2nd to 12th grade. The students in this district are predominantly Caucasian (20% minority status) and live in predominantly suburban areas.

Design, Materials, and Procedure

Teachers received a sheet with the difficulty ranking activity (see Figure 1). The teachers were asked to rank order six mathematics problems, from easiest to most difficult. The specific problems given to the participants are shown in Figure 1 (see Table 1 for the underlying problem structure). These problems were chosen because they were representative of problems found in pre-algebra and algebra mathematics textbooks. Teachers were asked to "rank order the 6 problems shown starting with the ones you think are easiest for your students, and ending with the ones you think are hardest." Ties were allowed.

Place Figure 1 about here

In addition to the ranking task, the teachers in this sample were given a set of 47 statements. They were given 20 min to rate the degree to which they disagreed or agreed with each statement by selecting the appropriate number along a 6-point Lickert scale (with max. = 6.0). Larger numbers indicated greater disagreement. Participants received the 47 items in a randomized order, and were asked to "Circle the number to the right that corresponds most accurately with your beliefs about the accompanying statement." The six-item scale (1 through 6) was presented to the left margin of each statement, with associated anchors: "(1) strongly agree (2) agree (3) agree more than disagree (4) disagree more than agree (5) disagree (6) strongly disagree."

The forty-seven items comprised six constructs (item groups). Examples of each construct are presented in Table 3. Wherever possible, each construct presented items that were worded both positively (affirming the construct) or negatively (negating the construct). Many of the items were adapted from previously published work, including Witherspoon and Shelton (1991), Cobb (1990), and Peterson and colleagues (1989). These items were chosen because they broadly addressed basic issues of pedagogical practice, math learning, problem solving, and the role of algebra in the domain of mathematics. Included were statements on the textbook view of algebra instruction discussed earlier. The constructs Learning and Pedagogy provided the strongest declarations of student-centered and student-directed learning when stated in the affirmative. The remaining constructs (Algebra is Best, Textbook, Product over Process, and Solution Alternatives) voiced views that were procedure and curriculum-centered when stated in the affirmative.

Place Table 3 about here

Participants were told that the intent of the questionnaire was to learn about the views that teachers held regarding mathematics, learning and instruction, and that all information they provided would be kept confidential and only the collective results of the group would be shared. The belief instrument was administered immediately after completing the problem difficulty ranking task.

Results and Conclusions

Belief Instruments

Descriptive statistics and reliability analyses of the belief constructs were compiled. Five of the original 47 items were dropped as a result of the reliability analysis. Results are presented on the remaining 42 items grouped within the six original constructs.

The analyses of teachers' ratings show that reliability measures for the given items were generally high (Cronbach's alpha had a range of α =.65 to .84), indicating good agreement on items that were theoretically clustered together (see Table 4). The means of the items (max = 6.0) tended to cluster around the middle range of the scale (3.5), indicating that the 6-point scale given was generally sufficient for the teachers to express their level of agreement adequately. (Note that higher mean scores (M) indicate greater disagreement.) The one exception to this is the Product Over Process construct (M=5.1 out of 6.0 total points) which was skewed toward disagreement. This suggests that, on average, teachers in this district tended to reject this view, and might have disagreed to an even greater degree if a wider scale had been provided.

Place Table 4 about here

Across grade levels, teachers in our sample (n=105) reflected recent mathematical reform views, such as those presented by NCTM (1989, 1991, 1999). These data are summarized in Table 4. Teachers agreed with the reform-based views expressed in the Learning (M= 2.5) and Pedagogy constructs (M= 2.7) that students can develop effective solution methods on their own, and that this is a valuable approach toward teaching mathematics. Teachers tended to disagree with those views that challenged reform-based views, such as Algebra Is Best (M= 4.3), Product over Process (M= 5.1), and the Solution Alternatives construct (M= 4.6). These are constructs which emphasized correct answers over students' reasoning and solution methods, and minimized the use of alternative solution methods. Teachers were, on average, largely neutral on the Textbook view (M= 3.3) of problem sequencing.

An analysis of covariance (ANCOVA) on the mean rating for each of the 6 constructs was performed using teacher instructional level (elementary, middle, high) as a factor, and years of teaching experience as a covariate. From this analysis, all six constructs showed significant differences in the ratings of elementary and high school teachers.

Grade Level Differences

As summarized in Table 4, high school teachers were least likely to agree with reform views expressed in the survey. They were less likely than their colleagues to agree with the view that students can learn effective problem solving on their own (Learning), $F(2,104)=9$, MS=3.27, $p<.0002$. And, though teachers in general strongly disagreed with the Product Over Process view, high school teachers were less likely than elementary school teachers to disagree with the view that the students' answers (i.e., the product) were more important than their problem-solving processes, $F(2,104)=9.9$, MS=9.95, $p<.0001$.

High school teachers also did not give students' invented strategies as much credit as their colleagues in middle and elementary school. Despite the efficacy of students' informal methods, the survey data showed that high school teachers were less optimistic of the successes of students' inventions than were elementary and middle school teachers. Thirty-one percent of high school teachers disagreed or strongly disagreed with statements from the Pedagogy construct that students' invented methods were valid and signaled an understanding of mathematics. This compared to 17% of middle school teachers and 2.6% of elementary teachers. This produced a significant difference between high school teachers' responses on the Pedagogy construct and that of their colleagues, $F(2,104)=14.4$, MS=6.5, $p<.0001$.

High school teachers were less likely than their elementary and middle school colleagues to disagree with the view that alternative solution methods are indicators of weak skills or knowledge gaps (Solution Alternatives), $F(2,104)=21.74$, MS=6.5, $p<.0001$. Hardly any middle school (3%) and elementary school (0%) teachers agreed with this view held by 15% of the high school teachers in our sample. In a similar manner, high school teachers were significantly less likely than their peers to disagree with the view that algebra is the best method for solving problems involving unknowns (Algebra Is Best), $F(2,104)=14.3$, $MS=3.99$, $p<.0001$. Twenty-three percent of high school teachers agreed with this construct, while 17% of middle and 2.6% of elementary school teachers supported this view.

High school mathematics teachers were also more likely to agree with the Textbook view that arithmetic is always easier than algebra, and symbol manipulation skills were a pre-requisite to verbal problem solving, $F(2,103)=5.5$, $MS=3.9$, $p<.005$ (with one missing value). Thirty-one percent of high school teachers agreed or strongly agreed with the Textbook view, as compared to 10% of middle school and 8% of elementary school teachers.

It is worthwhile to summarize the thirty middle school teachers' responses as well, since many early algebra concepts (such as generalized expressions, slope, etc.) are presented at that stage of education. Middle school mathematics teachers generally perceived students' prior mathematical knowledge as potentially very effective. They agreed with the Learning and Pedagogy views, and disagreed with the view expressed by the Solution Alternatives construct. They disagreed that algebra is inherently best as a solution approach (Algebra Is Best), and that answers were more important than the processes that led to them (Product Over Process). Collectively, they were fairly neutral to the Textbook view that verbal skills must be based on symbolic ones. The views of middle school teachers in our sample largely paralleled those expressed by elementary school teachers. The major exception was that middle school teachers were significantly less likely than elementary school teachers to agree with the Pedagogy view, though they generally agreed with this view.

Teachers' Problem Difficulty Ranking

Teachers views on the belief instrument largely reflect current reform ideas expressed by the mathematics education community, and indicate a basic confidence in many of the reform-based principles for mathematics learning and instruction (NCTM, 1989, 1991, 1999). Elementary teachers expressed the strongest agreement to principles of student-centered and student-directed learning, while high school teachers, though generally in agreement with these views, supported these views least, with middle school teachers falling midway on every construct.

Teachers were also asked to rank order the relative difficulty of problems that implicitly compared arithmetic to algebra problems along one dimension, and verbal problems to symbol equations along another dimension. With this foundation of teachers' views established among our sample, along with the variations, we now examine how teachers evaluate the difficulty of certain mathematical tasks for their students.

Start-unknown versus result-unknown problems.

Teachers in our sample (n=105) received start-unknown problems in three different presentation formats: as a story problem with context, as a verbal description of quantitative relations with no story context, and as a symbolic equation. A consistent ranking means that teachers were consistent in their comparisons across all three levels of problem presentation. Most teachers (65%) were not consistent in their ranking of the effect of unknown position. When teachers were consistent, they tended to rank Start-Unknown (algebra) problems as more difficult than Result-Unknown (arithmetic) problems (32% of teachers ranked it so), while teachers seldom ranked Start-Unknown problems as consistently easier (2.8%). Table 2 shows that this general pattern held across teacher grade levels.

This prediction is consistent with the student performance data and indicates that teaches are somewhat sensitive to the arithmetic-algebra distinction, and familiar with its effects on students' problem-solving performance. This finding is also consistent with previous research examining elementary and high school educators' beliefs concerning problem difficulty (Peterson et al., 1989; Nathan & Koedinger, in press).

Story, word, and equation problems.

Teachers, on average, ranked word-equation problems as easier to solve than story problems (Table 2), a finding that is at odds with previous studies of high school teachers (Nathan & Koedinger, in press). As we will see, however, these data are primarily driven by the split among teachers along grade-level divisions.

High school teachers (n=39) tended to rank symbolic equations as easier than verbal problems. Thirty-six percent of the high school teachers consistently ranked verbally presented story and word-equation problems as more difficult than symbol-equation problems, while only 5% consistently favored the reverse order. The high school teachers, on average, placed arithmetic equations (P6, Table 1) in the easiest rank along with arithmetic word equations (P5), and ranked algebra equations (P3) with middle level difficulty, significantly ahead of all of the verbal algebra problems, and even ahead of arithmetic story problems. These judgements are consistent with judgement findings reported elsewhere (Nathan & Koedinger, in press). However, they are at odds with the student performance data (Table 2, column 5) which showed that verbal problems were solved by students far more readily than symbol equations.

The rank orderings produced by elementary teachers generally paralleled the ranking produced by high school teachers, though elementary teachers were more likely to rank problems on the basis of algebraic and arithmetic structure than on presentation format. Forty-six percent of elementary school teachers consistently ranked verbally presented algebra and arithmetic problems as easier than symbolic problems, while none of these teachers offered the reverse rank ordering. Arithmetic word equations were considered the easiest, as one elementary teacher mentioned, because "it tells you exactly what to do." They were closely followed by arithmetic symbol equations which presented the problems in "pure math." Like the high school teachers, the elementary level teachers regarded algebra equations as easier than verbal algebra problems, and ranked algebra word and story problems as most difficult, a judgement that directly contradicted the actual performance of students.

Middle school teachers stood out as a group. They showed their sensitivity to the presentation formats of the problems, and were far more likely than their elementary and high school colleagues to place verbal problems in the easiest ranks. Twenty-three percent of middle school teachers consistently ranked verbally presented algebra and arithmetic problems as easier than symbolic problems, while only 10% offered the reverse rank ordering. As Table 2 shows, arithmetic word equations were considered easiest by middle school teachers, while arithmetic and algebra story problems were ranked at mid-level difficulty. This rank ordering was consistent with student performance which placed success with verbal problems above that of symbolic problems, particularly algebra equations.

In fact, middle school teachers gave the closest match to the order of problem difficulty actually attained by students, as measured by the Kendall's rank correlation non-parametric statistic, τ (6)=.733, p=.034. The ranking of elementary teachers was marginally predictive of student performances, τ (6)=.67, p=.06, while the ranking provided by high school teachers was not significantly related to performance difficulty, τ (6)=0.

From these data, two general conclusions may be drawn. First, we found that teachers regardless of instructional grade level, accurately predict that arithmetic problems will be easier for students to solve than matched algebra problems. Second, we found that middle school teachers are most accurate in their predictions of the effect of presentation format, while high school teachers least accurately judged students' areas of problem-solving difficulty.

Relating Teachers' Beliefs to Difficulty Ranking: A Regression Analysis

Teachers' grade level and response data from the six belief constructs were considered as factors in a regression analysis used to evaluate which factors were reliable predictors of teachers' problem-solving difficulty ranking responses. The dependent variable was teachers' relative rank order of a given problem type. Composite scores from the belief instrument ratings were constructed and served as predictors, along with years of teaching experience, and grade level taught by the teacher. The average number of years of teaching experience ranged from 0 to 34 years, with a mean of 14.3 years. However, years of teaching experience proved to not be a reliable factor in any of the analyses, and so it was removed from the analyses. Grade level (elementary, middle, high) proved to be important in predicting teachers' difficulty ranking.

Place Table 5 about here

Table 5 shows the relative contribution of each factor in predicting the rank order for verbal and symbol problems. Although all of the belief constructs were found to be reliable (see Table 4), only Textbook was a significant predictor for teachers' problem difficulty ranking. In predicting teachers' rank ordering of symbolic (equation) problems (P3 and P6, Figure 1), the factors Grade and Textbook produced the most reliable model, $F(2,96)=37.7$, $MSe=45$, $p<.0001$, accounting for 52% of the variance. For predicting the rank ordering of verbal problems, Grade and Textbook again provided the best model, $F(2,96)=23.3$, $MSe=10.8$, $p<.0001$, accounting for 39% of the variance.

Summary

Middle school teachers were most accurate in predicting students' problem-solving performance, bucking the view held by many high school teachers, that symbolically presented problems (i.e., arithmetic and algebraic equations) were easier to solve than verbally presented problems (i.e., story and word equation problems). Teachers' levels of agreement with items consistent with the so-called textbook view of algebra development proved to be a reliable predictor of teachers' judgements regarding difficulty for both verbal and symbolic problems. This relation lends support to the hypothesis that the Textbook view significantly influences teachers' judgments regarding students' mathematical development. Middle school teachers tended to be neutral on the Textbook construct. In the following discussion we will speculate on how teachers' beliefs affect their instructional practices, and how these beliefs may shape the learning experiences of children. We conclude with some broad reflections on the study of complex cognitive behavior and the role such studies may play in fostering teacher change through professional development programs and reform efforts.

Discussion

No pedagogical theory is complete enough to stipulate in advance all of the instructional decisions that face teachers. Systems of beliefs about instruction and student learning are used by teachers to fill in the gaps and to organize the complex, dynamic, and uncertain demands of classroom planning and instruction (Eisenhart et al., 1988). Teachers' beliefs about students' abilities play a central role in shaping teachers' judgements and instructional practices (Borko & Putnam, 1996).

Some Factors That Influence Teachers' Judgements

Efforts to shape teachers' beliefs are made through educational programs and professional standards. The current study shows that the teachers in our sample generally hold standards-based views of mathematical learning and instruction. Even so, teachers tend to simultaneously hold a symbol-precedence view of mathematical development that is at odds with student performance. This is indicated most strongly by the relationship between the survey data and the problem ranking data. Teachers' views of students' learning and performance cluster reliably around all six of the belief constructs in our survey, but only one of these, the Textbook construct, proved to be a reliable predictor of teachers' judgements of problem difficulty.

The Textbook view of algebra development presents material in accordance with a task analysis that delineates the component sub-tasks and learning hierarchies that can then inform instructional sequences (e.g., Gagne, 1968; Mayer, 1985). This also breaks down the instructional units into manageable sizes for lesson planning. However, task analyses can produce misleading results if studied in isolation from the task context, including the prior knowledge of the student and the larger problem-solving situation itself. The analysis can mischaracterize the task by neglecting the dynamics of the problem-solving process. For example, analyses of students' behaviors have shown that successful informal methods tend to be elicited by verbal problems more often than symbolic problems (Nathan & Koedinger, in press). Such item effects change the demand characteristics of the task and alter subsequent problem-solving performance. Because of the prevalence of item effects, individual differences, and task demands, Glaser (1976) advocates performing detailed analyses of performance as well as of the task itself. Task analyses that do not acknowledge the dynamics of the problem-solving process will look very different than those that do, and will lead to very different predictions of future problem-solving behavior.

Textbooks provide instructors with a kind of organizational life raft in a sea of under-specified instructional decisions. However, Borko & Shavelson (1990) caution that textbooks and teacher manuals may actually interfere with teachers' personal decision-making processes, and serve as a kind of pedagogical "crutch" for novice teachers. Textbooks may also introduce dissonance when the prescriptions they offer poorly match a teacher's own beliefs about learning and instruction (McCutcheon, 1980). Yet textbooks and teachers' manuals serve as the major sources of content and instructional activities for many teachers (e.g., Clark, 1978-79). It is reasonable to surmise that the use of textbooks in structuring daily classroom lessons, weekly assignments, and year-long curriculum sequencing, leads teachers to internalize the images of mathematics they convey.

Implications for Teacher Decision Making and Instructional Practice

It is important to examine how the symbol-precedence view implicitly advocated by many algebra and transition textbooks can have the character of a self-fulfilling prophesy. Consider the case of a student who succeeds in solving a variety of symbol equation problems at the arithmetic or algebra level. We would expect, in accordance with the student data, that there is a high likelihood for success on comparable story problems. A teacher may interpret this success as support for her view that symbolic reasoning skills readily transfer to verbal problems. As one teacher put it in her written comments, "Teaching equation solving first provides the student with all the pieces in pure mathematics that he can later apply to word problems." However, our analyses show that this kind of transfer is very unlikely to occur. The better-fitting Verbal Precedence model of algebra competence (Nathan & Koedinger, in press) shows that a student who can solve symbolic problems within a level of arithmetic or algebra is further developed mathematically than the one who can only solve story problems. Since symbolic reasoning seems to lag behind verbal reasoning, it is more likely that we are not challenging the student to extend his or her mathematical reasoning when comparable verbal problems are held out as "challenge" or "application" problems to a student who routinely solves symbol equations.

If, however, a student fails to solve a symbol equation problem, a teacher with the symbol precedence view will likely withhold story problems from the student, until the student can demonstrate a certain level of symbolic skill performance. By making this decision, the teacher may never get to see how the student performs on verbal problems, and thus, never have these assumptions on mathematical development challenged.

Teachers' predictions of students' performances significantly underestimate the difficulty that most students have with symbolic reasoning. In so doing, they appear to pass over some of the conceptual and structural underpinnings of symbolic forms of representation (e.g., Kieran, 1992; Sfard, 1991). Instructional decisions from this perspective may not adequately address gaps in students' understanding of the representational structures and the mathematical procedures that manipulate them.

Data from the belief instruments also indicated that high school teachers in our sample — those most centrally charged with algebra instruction — were least aware of the efficacy of students' invented algebra solution strategies. High school teachers were also most likely to agree with statements suggesting that invented methods indicate deficits in students' mathematical knowledge, and that algebraic symbol manipulation is the best method for solving problems that deal with unknowns. Teachers who tend to hold formal strategies in such high regard also tend to discount children's mathematical ideas (cf. Carpenter et al., 1989; Fennema et al., 1992). Children in these classes will tend to be directed to learn and use abstract, and seemingly arbitrary, solution methods without bridging them to their own conceptualizations. This can alienate children from their own mathematical intuitions. And because students are actually less effective at generating correct answers with these formal methods, they will experience greater amounts of failure which can lead many students away from mathematics and science (cf. Dweck & Licht, 1980).

It is interesting that the middle school teachers in our sample were the most accurate at recognizing students' areas of difficulty because high school mathematics teachers tend to specialize more, and generally receive the most extensive content area training. We offer three reasons why this might be the case.

First, students may not develop their symbol manipulation skills as far as high school teachers hope and believe. Even though there is a great deal of time and attention paid to symbol processing students are still weak in these areas. Second, middle school teachers are typically better acquainted with their students, and interact with them in a wider range of settings. Elementary school teachers see their students in even more diverse settings and also gave accurate predictions. However, the match of middle school mathematics may better inform teachers' views of mathematical development at this transition phase from arithmetic to algebraic reasoning. Many of the middle school teachers at the time of this study participated in the team approach, with common planning periods, and multiple opportunities to connect with students throughout the day in multiple classroom settings. Third, because high school teachers tend to have greater expertise in their content areas, they are personally more distant from the difficulties of their novice students. This may make high school teachers more susceptible to a kind of "expert blindspot" that prevents them from being aware of certain things because their pattern-matching skills are so highly tuned (Ericsson & Smith, 1991; Koedinger & Nathan, 1999).

Perhaps there is, lurking in our data, some support for a holistic view of instruction where students develop a variety of quantitative reasoning methods each with its own strengths and demands, and use them in a variety of educational settings. As a minimum, these data suggest that mathematics educators need to be more aware of the range, efficacy, and flexibility of students' alternative mathematical problem solving strategies, and the difficulties students have with symbolic reasoning. In support of enhancing teachers' pedagogical content knowledge in this area, it is natural to recommend that the research community find ways to disseminate its findings on students' reasoning methods. But a more concerted effort to share findings about the varieties of students' reasoning is not enough. Students' methods may be manifold; and their variability may be extensive. Ultimately, we will need to look to teachers to draw out students' alternative reasoning methods, and share them within their professional community.

Implications for Teacher Education

Beliefs and knowledge play a central role in complex cognitive behavior (Schoenfeld, 1983 Newell, 1989). So, the intuition goes, these beliefs must also play a key role in eliciting change in complex behavior. For this reason, belief elicitation is a central part of many programs of conceptual change (e.g., Posner & Strike) and reform based professional development (e.g., Richardson, 1994; Fenstermacher, 1994; Nathan & Elliott, 1996; Nathan, Elliott, Knuth, and French, 1997). At the same time, we must acknowledge that the structure provided by beliefs is a major source of any resistance to change. People are generally reluctant to give up their beliefs about important aspects of their lives or their professional practices because of the cognitive disorder that would ensue (Eisenhart et al., 1988). Thus, plans to effect change must provide a teacher with new beliefs that are accessible, and more valid than the old ones. One way to address the accessibility issue is to identify those aspects of education that teachers already tend to focus on, and those they find most favorable. In planning their instruction, teachers focus mainly on the subject matter content, and on instructional activities (Borko & Shavelson, 1990). Student teachers have also reported their most positive feelings toward instructional and in-class activities, especially those which allow teachers to take responsibility, exercise their own control, and create environments that make student progress salient (Eisenhart, 1988).

Establishing the validity of new instructional prescriptions to replace inadequate beliefs poses a greater challenge. Learning theories, no matter how elaborate, are not theories of instruction (Cobb, 1988; Goldman, 1991; M. Simon, 1995), and cannot specify all of the aspects of a complex learning setting. Previous efforts at curriculum and instructional reform have fallen short partly because reformers failed to account for the decision-making processes of the teachers implementing the programs (Fennema et al., 1992). Implementation of instructional and curricular goals will nearly always rest on decisions that lie outside of the learning theory, decisions made by curricular designers and instructors based on their own beliefs about student development and instruction (Clark & Peterson, 1986; Nathan, 1998). As we move closer to a scientific foundation for classroom instruction and teacher education, we must heed these limitations, and acknowledge the significant role of teaching professionals in translating theories of learning into practice and in specifying the myriad details that are necessary to actually teach. The ways in which these details are ultimately addressed is influenced by many factors — the teacher's prior learning experiences, her grade level and professional education, the available resources, and her beliefs about how students learn. A richer picture is sure to emerge of teachers' instructional decision making and practices as we continue to study the beliefs teachers hold and their origins.

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Table 1. The problems given to students and teachers can be organized by the presentation type (3 levels) and the position of the unknown value (2 levels).

Presentation type →	Verbal problems		
Unknown value ↓	Story	Word	Symbol
Result-unknown (Arithmetic)	P4. When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?	P5. Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	P6. Solve for <u>X</u>: (81.90 - 66) / 6 = <u>X</u>
Start-unknown (Algebra)	P1. When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?	P2. Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	P3. Solve for <u>X</u>: <u>X</u> * 6 + 66 = 81.90

Table 2: Summary of teachers' rank ordering of problem types and students' performances

Rank ↓	Teachers (n=105)	Elementary Teachers (n=36)	Middle School T (n=30)	High School Teachers (n=39)	Student Performance (n=76) [and percent correct]
Easiest*	Arith-Word-Eqn (#5)	Arith-Word-Eqn (#5)	Arith-Word-Eqn (#5)	Arith-Word-Eqn (#5)	Arithmetic-story (#4) [73%]
		Arithmetic-Eqn (#6)		Arithmetic-Eqn (#6)	Arithmetic-Word (#5) [67%]
Middle*	Arithmetic-Story (#4)	Arith-Story (#4)	Arith-Story (#4)	Algebra-Eqn (#3)	Algebra-Story (#1) [59%]
	Arithmetic-Eqn (#6)	Algebra-Eqn (#3)	Algebra-Story (#1)		Algebra-Word (**#2**) [54%]
					Arithmetic-Eqn (#6) [53%]
Hardest*	Algebra-Story (#1)	Algebra-Story (#1)	Alg-Word-Eqn (#2)	Arithmetic-Story (#4)	Algebra-Eqn (#3) [37%]
	Algebra-Eqn (#3)	Alg-Word-Eqn (#2)	Algebra-Eqn (#3)	Algebra-Word-Eqn (#2)	
	Alg-Word-Eqn (#2)		Arith-Eqn (#6)	Algebra-Story (#1)	

* Difficulty divisions indicate significant differences in teacher ranking or student performance, $p<.05$.

Table 3: Summaries and example items from the six different constructs used for the survey of teachers' views of mathematics, math instruction, and student learning, along with sample items presented positively and negatively.

Algebra Is Best items (11 items) present the view that algebraic procedures are the singularly most effective method for mathematical problem solving.

a. Using algebra for story problem solving is the most effective approach there is. (positive)

b. There are many effective approaches to solving any algebra story problem, and manipulating symbols is only one method. (negative)

Learning items (7 items) present the view that students can learn and invent effective methods for problem solving that may differ from those taught.

a. Students enter the algebra classroom with intuitive methods for solving algebra story problems. (positive)

b. Most students cannot figure out for themselves how to solve algebra story problems. (negative)

Textbook view items (6 items) hold the view commonly expressed in math textbooks that arithmetic problems are easier and need to be presented before algebra. Also, within a mathematics topic, math problems presented in words are most difficult and need to appear later in the curriculum.

a. Arithmetic story problems are easier for students to solve than algebra story problems. (positive)

b. Solving math problems presented in words should be taught only after students master solving the same problems presented as equations. (positive)

Table 3, continued.

Pedagogy items (8 items) state that students may possess valid ways of reasoning as they enter the classroom, and may figure out for themselves effective problem-solving approaches.

a. Students should be encouraged to invent their own methods to solve mathematics problems. (positive)

b. Rewarding right answers and correcting wrong answers is an important part of teaching. (negative)

Product Over Process items (4 items) emphasize correct answers over a student's reasoning process.

a. Getting the correct answer is a better indicator of learning than is the ability to articulate a good solution approach. (positive)

b. Mathematical understanding is more clearly shown in a student's reasoning than in the final answer a student produces. (negative)

Solution Alternatives items (6 items) state that alternative (unschooled) methods such as arithmetic, guess-and-test, and other non-symbolic methods demonstrate gaps in the student's knowledge.

a. When a student uses an arithmetic approach to solve an algebra word problem, that indicates a weakness in that student's math abilities. (positive)

b. Use of a "guess and check" strategy to solve an algebra story problem shows an adaptive approach to problem solving. (negative)

Table 4. Teachers' mean responses to various belief constructs, and by grade level.

(1=Strongly agree, 6=strongly disagree).

Construct	Cronbach's α	Elementary (n=36)	Middle School (n=30)	High School (n=39)	All Teachers (n=105)
Algebra Is Best	.76	4.61	4.41	3.98*	4.3
Learning	.70	2.25	2.43	2.82*	2.5
Textbook	.65	3.55	3.42	2.94*	3.3
Pedagogy	.84	2.31	2.73*	3.13*	2.7
Product-Process	.78	5.38	5.13	4.67*	5.1
Solution Alternatives	.68	4.95	4.74	4.16*	4.6

* $p < .005$

Table 5. The correlations of the reliable factors with teachers' problem-solving difficulty rank orderings.

	Factors	
Dependent Variable	Grade Level Adjusted-R^2	Textbook Adjusted-R^2
Rank of Verbal-Problems	.26	.13
Rank of Symbol-Problems	.40	.12

Figures

Figure 1. The difficulty ranking task given to teachers

A SURVEY

Below are 6 problems that are representative of a broader set of problems that are typically given to public school students at the end of an Algebra 1 course -- usually 9th grade students. My colleagues and I would like you to help us by answering this brief (5 min) survey. We are happy to share the results we obtain with your class this spring.

What we would like you to do:

Rank these problems starting with the ones you think were easiest for these students to the ones you think were harder. You can have ties if you like. For example, if you think the fourth problem (#4) was the easiest, the 3rd was the most difficult, and the rest were about the same, you would write:

4 (easiest)

2 1 5 6

3 (hardest)

(Feel free to include an explanation of any assumptions you made in the space below.)

Problems:

1) When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?
2) Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?
3) Solve for x: x * 6 + 66 = 81.90
4) When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?
5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?
6) Solve for x: (81.90 - 66) / 6 = x