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Teachers' and Researchers' Beliefs About Algebra Development

Mitchell J. Nathan
University of Colorado

Ken Koedinger
Carnegie Mellon

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Abstract

Mathematics teachers and mathematics educational researchers were asked to rank order arithmetic and algebra problems for their predicted problem-solving difficulty for students. It was discovered that these predictions deviated systematically from actual algebra students' performances in important ways, but closely matched a view presented implicitly by common mathematics textbooks. An analysis of students' problem-solving strategies suggests specific ways that students' algebraic reasoning differs from the views held by most teachers and researchers in the sample, and portrayed in common textbooks. The Textbook view of algebra development — where symbolic problem solving precedes verbal problem solving, and arithmetic skills strictly precede algebraic skills — was contrasted with the Verbal Precedence (VP) model of development. The VP model provided a better quantitative fit of students' performance data. Implications for student and teacher cognition and for algebra instruction are discussed in light of these findings.

Teachers' and Researchers' Beliefs of Algebra Development

Our current investigation explores the relationship between teachers' predictions of the development of algebra reasoning, and students' actual performances. Teachers' beliefs about student ability and learning greatly influence their instructional practices. The study of these beliefs has revealed that teachers generally report that information about students is the most important factor in their instructional planning (Borko & Shavelson, 1990), and teachers consider student ability to be the characteristic that has greatest impact on their planning decisions. As students shift from an arithmetic approach to problem solving to an algebraic approach, teachers must also shift their practices and their views of the learner. Any improvements in our understanding of teachers' views of the development of students' knowledge and problem-solving abilities strengthens our picture of the complexities of teaching, and may ultimately enhance programs for teacher preparation. This article extends work on previous models of teacher cognition, and of how professional practitioners' knowledge and beliefs shape their instructional practices (Borko & Livingston, 1989; Knuth, 1999; Nathan, Knuth, & Elliott, 1998; Schoenfeld, in press; Shulman, 1986; Thompson, 1992).

This study reports on empirical results concerning high school teachers' and mathematics educational researchers' beliefs about the factors that make algebra problems difficult for beginning algebra students. Algebra has several interleaving aspects. First, algebra can be seen as generalized arithmetic, including the use of literal symbols such as letters as references to unknown quantities, and the generalization of arithmetic operations as applied to letters (MacLane & Birkhoff, 1967; Kieran, 1992; Usiskin, 1988, 1997). Second, algebra refers to the use of formal mathematical structures to represent relations, and includes the procedures that operate on those structures (Kieran, 1992; Usiskin, 1988). Third, algebra can be defined as a formal means to describe the relationships among quantities (Usiskin, 1988, 1997). We acknowledge the extent of the domain of algebra. In the present study we limit our scope, and specifically address the aspects of algebra and arithmetic as they relate to equation and word problem solving.

In this study, comparisons were made between teachers' and researchers' predictions of the relative difficulty of a set of theoretically designed problems, and students' actual problem-solving performances (Koedinger & Nathan, 1998; Koedinger, Nathan, & Tabachneck, 1996). In the analyses that follow, we examined discrepancies between teachers' and researchers' predictions and students' performances. The teacher data were also compared to the sequencing of problems presented in some pre-algebra and algebra textbooks. Results from the analyses of students' problem-solving strategies lead us to suggest specific ways that students' algebraic reasoning differs from the views of mathematical development commonly held by teachers and researchers, and those presented in popular algebra textbooks.

Theoretical Framework: Factors Affecting Problem-Solving Difficulty

We first review research that considers the relative impact of certain factors on algebra and arithmetic problem-solving difficulty. We specifically consider two important factors: (a) the position of the unknown quantity in the problem, and (b) the linguistic presentation of the problem. The body of work on arithmetic story problem solving of younger children (e.g., Carpenter et al., 1994; De Corte, Greer, & Verschaffel, 1996; Riley and Greeno, 1988) provides firm methodological and theoretical bases for the study of high school students' algebraic reasoning, and its development and impediments.

Riley and Greeno (1988), for example, found that problem difficulty is strongly affected by the role (or position) of the unknown quantity within the problem statement. Consider the case of a <u>result-unknown problem</u>, where the unknown quantity is the result of the events or mathematical operations described in the problem. An example result-unknown problem is shown in problem 6 (P6) of Table 1, with the symbol <u>X</u> standing for the unknown quantity. To find the value of <u>X</u> in this example the problem solver applies the indicated arithmetic operations on the left-hand side of the equation, first subtracting, then dividing. The problems P5 and P4 (Table 1) are also result-unknown problems, presented in two different verbal formats. Because result-unknown problems can be solved through the direct application of arithmetic operations, these can be considered arithmetic level problems.

In start-unknown problems, the unknown value (such as the hourly wage in P1 of Table 1, or X in P3) refers to a quantity needed to specify a relationship (Carpenter et al., 1994; Riley & Greeno, 1988). Start-unknown problems tend to subvert simple modeling and direct calculation approaches of arithmetic problems, and often require algebraic methods, or more sophisticated modeling (Hall et al., 1989). Because they can be solved through the application of standard algebraic procedures, start-unknown problems can be considered algebra level problems.

Riley & Greeno (1988) examined the performance of first grade students solving simple one-step problems with whole numbers. They found that while students were 100% correct on result-unknown (arithmetic) problems, they were only 33% correct on start-unknown (algebra) problems. However, we need not suspect that these students are using algebraic methods to solve these problems. Number fact retrieval and counting strategies can explain the non-trivial performance on start-unknown and open sentence problems at this level (Briars and Larkin, 1984; Kieran, 1992). This general pattern of problem-solving performance favoring result-unknown over start-unknown problems has also been found in mathematical problem solving at the college level for multi-step problems with rational numbers (Koedinger & Tabachneck, 1995).

Place Table 1 about here

Investigators have also examined the performance differences between problems presented in symbolic (or computational) formats and those presented in linguistic or verbal formats such as word and story problems. Story problems (problems P1 and P4 of Table 1) are those presented in a verbal format with contextual information about the problem situation which can be used by the solver as a source of problem elaboration, reframing, and solution constraints (cf. Baranes et al., 1989). Symbolic arithmetic problems (Table 1, problem P6) are typically described as number sentences.

There is also a presentation format intermediate to the story problem and symbolic equation format. This word equation format (Table 1, problems P2 and P5) verbally describes the relationship among pure quantities (both known and unknown) with no story context. A common example of this type of problem is the "pick a number" game (Usiskin, 1997).

While the advantages of result- over start-unknown problems are widely agreed upon, findings in the literature are inconsistent when examining the relative difficulty of verbal versus symbolic problems. Researchers have found circumstances where computational problems are solved more readily than word problems (e.g., Carpenter et al., 1980). Researchers have also found performance advantages for problem contexts over symbolic equations (e.g., Carraher, Carraher, and Schliemann, 1985). Brazilian children regularly engaged in street trade, for example, solved arithmetic problems more readily when the problems were presented in a practical context such as a story, an action sequence, or, preferably, as real life interactions of the street markets (Carraher et al., 1985; Guberman, 1987; Saxe, 1988). Specifically, contextualized problems presented either as typical word problems or as problems situated in a commercial transaction led to greater levels of performance than symbolically presented problems.

To add to the picture, Baranes, Perry, & Stigler (1989) found that the Brazilian children, but not the U. S. children, exhibited this advantage for problem context. Strategy selection appeared to be an important mediating variable in the U. S. data. Decontextualized problems tended to elicit less successful mental strategies in Brazilian children, with no corresponding differences among the sample of U. S. students. However, the U. S. students did show an advantage for problems when the context (e.g., money or time) more closely matched the numbers used (e.g., multiples of 25 cents or 15 mins.), suggesting that the activation of real-world knowledge facilitated problem-solving performance (cf. Nathan, Kintsch, & Young, 1992).

To further understand the relative impact of context as well as other factors on problem-solving difficulty, Koedinger and Nathan (1998; Koedinger, Nathan & Tabachneck, 1996) studied high school and college students in several large-scale, factorially designed assessments. They termed their investigation a difficulty factors analysis (DFA) because it sought to systematically examine the factors affecting students' problem-solving difficulties. The findings are reviewed in some detail here because of their significance in interpreting the current study of teachers' beliefs. For a more complete report on these findings, the reader is directed to Koedinger and Nathan (1998).

Students solved problems based on six problem types which were formed by varying problems along three levels of presentation format (verbal story with context, word equations with no context, and symbolic equations), and two levels of placement of the unknown quantity (result- or start-unknown problems).

In one large-scale assessment, seventy-six urban high school students completed quizzes administered in May by their teachers. Of these students, 58 were currently enrolled in one of 3 mainstream Algebra 1 classes, and 18 were 9th graders who took Algebra 1 the previous year and were currently enrolled in a 10th grade Geometry class. The courses followed standard curricula for algebra instruction. The student performance data (Table 2) showed highly significant effects of unknown position and presentation format. These results were replicated a year later with students from the same school population (n=171; Table 3). We review these student findings here, and later relate them to the judgment data provided by teachers and researchers as part of the current investigation.

Unknown Values

Students in the original study (n=76) exhibited much lower performance levels on start-unknown (algebra) problems than on result-unknown problems (see the rows of Table 2). Students correctly solved start-unknown problems 50% of the time, while result-unknown problems (arithmetic) were correctly solved 64% of the time, leading to a significant effect of unknown position on problem difficulty, F (1,75) = 48.9, $p<.0001$.

The following year, students in the replication study (n=171) showed a similar pattern of results (Table 3). They correctly solved start-unknown problems 46% of the time, while result-unknown problems were correctly solved 70% of the time. This produced a significant effect of unknown position on problem difficulty, F (1,170)=138, $p<.0001$.

Place Tables 2 and 3 about here

Presentation Formats

As discussed above, the format in which a mathematical problem is presented also bears on problem difficulty (Baranes et al., 1989; Carpenter et al., 1980; Carraher et al., 1987). Koedinger and his colleagues found that high school students in the original sample (n=76) experienced a nearly 20% drop in performance when solving symbol equation problems than with verbally presented problems with or without a context (Table 2). These differences resulted in a significant effect for presentation format, F (2,75) = 12.6, $p<.0001$. A post hoc test ($p<.01$) revealed that algebra equation problems were significantly less likely to be correctly solved than either story problems or word-equation problems, while algebra story and algebra word-equation problems were found to be equal in difficulty (Table 2).

Although no interaction between position-unknown and presentation format was evident, a post hoc analysis ($p<.01$) revealed that algebra story and algebra word problems were equal in difficulty to arithmetic symbol problems. This suggests two possible causes, which may also be operating simultaneously. First, there may be an inhibiting effect of the symbolic format that burdens the students' cognitive resources for arithmetic reasoning on the arithmetic equations. Second, there may also be a facilitating effect of verbal format (regardless of the presence of problem context) which lessens the demands of the start-unknown structure found in algebra story and word equation problems.

Students in the replication study ($n=171$) also showed the advantage of verbal presentation format on problem-solving performance. Symbolic equations were solved correctly about 25% less often than story problems and nearly 20% less often than word equation problems (see Table 3). These differences led to a significant effect for presentation format, $F\ (2,170)=38.4$, $p<.0001$. A post hoc test revealed that symbolic equations were significantly more difficult than either story problems or word equation problems, $p<.01$. However, the replication data differed from the results of the original study. A post hoc test ($p<.01$) revealed that result-unknown word-equation problems (P5) were significantly more difficult than result-unknown story problems (P4) for these students.

Three generalizations can be drawn from the high school student data: (a) Start-unknown (algebra) problems are harder for these students than result-unknown (arithmetic) problems ($p < .001$); (b) Symbolic equation problems are harder than both word equation problems and story problems ($p < .001$); and (c) verbal algebra problems are equal in difficulty to symbolic arithmetic problems.

Students' Solution Strategies

Koedinger and his colleagues observed four major types of solution strategies used by the high school students to solve the 6 classes of problems discussed earlier. The first two strategies — arithmetic and algebraic methods — are the standard school taught methods. The other two strategies —guess-and-test and unwinding — are informally adopted and invented strategies.

The guess-and-test strategy refers to the class of model-based methods used for iterative analysis or "hand simulations" of the events of the problem (e.g., Hall et al., 1989; Kieran, 1992). An example of the application of this strategy is shown in the student work of Figure 1. Here, the student is solving a word-equation start-unknown (algebra) problem by guessing different possible numbers that could be added to the number 25 to yield a sum of 66.40. Since the student never achieves this goal, the search for the original number (that is, some number multiplied by 4) was apparently ignored. It can also happen that in 2-step problems such as this, steps get dropped because of the large demands that problem-solving activities place on one's limited working memory (Anderson, Reder, & LeBiere, 1996).

In contrast to the guess-and-test method, the unwinding method allows the student to "work backwards" from the givens of the problem and "unwind" or undo the quantitative constraints imposed, in order to isolate the unknown (cf. Graves & Zack, 1997; Kieran, 1992; Kieran & Chalouh, 1990; Polya, 1957). Unwinding often inverts the steps referred to in a story problem, or the order of mathematical constraints provided in a symbolic or word equation problem. Figure 2 shows a student's work while solving a start-unknown story problem using the unwinding method. Here, the student unwinds the waiter's tips from the total earnings of $90, and receives a value of 24. The student then unwinds the multiplication by 6 (for the 6 hours that Ted worked) using long division to obtain the initial quantity of $4.00 — the waiter's hourly wage.

Unlike traditional algebraic approaches such as transposing terms while maintaining a balanced equation, the unwinding strategy circumvents use of equations or symbolic placeholders for unknown quantities. It operates directly on the numbers in a computational way, rather than operating on the symbol structure of the equation (Kieran, 1992). Unwinding may be done verbally by the solver, or through the solver's written work (Koedinger & Tabachneck, 1995). By unwinding the mathematical relationships step by step, the solver systematically does two things. First, a multi-step problem is transformed into a set of simpler one-step problems, each of which is solved separately by unwinding a single operation. Second, each step in the unwinding procedure is arithmetic, using the inverse operations of those given in the original problem. This essentially circumvents the need to perform (or even know about) the rules of algebra symbol manipulation or equation balancing. Because the unwinding strategy essentially transforms multi-step algebra problems into a sequence of single-step arithmetic problems, it can be hypothesized to be a cause for the finding (reported above) that students' problem-solving performances on algebra story and word-equation problems are at the same level as when solving arithmetic equation problems.

An analysis of problem-solving strategy use from the initial student data (n=76) was conducted. In addition to the two strategies discussed, students' solution approaches were coded as "no response" when they left a problem blank, or as "unknown" when there was insufficient information to allow coding a strategy. Not surprisingly, the arithmetic strategy was used overwhelmingly in solving result-unknown problems. For the purposes of this paper, we focus on solution methods used to solve start-unknown problems (Table 4). For these problems, non-standard solution approaches were preferred, and were relatively more successful than the standard algebraic methods (cf. Baranes et al., 1989).

As Table 4 shows, algebra story problems tended to elicit the unwinding strategy from students over 50% of the time — more than all other response categories combined. Even though students in the sample all had extensive instruction in algebraic methods, story problems rarely elicited the symbol manipulation methods. Situation-less word equation problems tended to elicit either a guess-and-test approach or unwinding more than half the time. Symbolic equations resulted in no response from these algebra students an alarming 30% of the time, more than twice as often as the other problem types. When students did respond, they tended to stay within the algebraic formalism and apply symbol manipulation methods, or opt for the iterative guess-and-test method. Algebra symbol equation problems elicited alternative methods about half as often as algebra story and word equation problems.

Place Table 4 about here

As the data of Table 5 show, the informally acquired unwinding and guess-and-test methods had the highest likelihood of success, and led to the greatest number of correct solutions when they were applied to start-unknown problems. One reason offered for this performance advantage is that these methods employ some built-in validity checks on the answers (Tabachneck et al., 1994). Guess-and-test, for instance, includes a check on each cycle for the consistency of the solution before deciding to terminate (upon reaching the correct answer), or continue with a new guess. Similarly, when unwinding a story problem, the solver uses the situated nature of the values gleaned from the problem scenario (e.g., one maintains the units as one writes the values). This minimizes the likelihood of producing absurd values and helps align numbers for computation, among other things. When used with verbal problems with no context, unwinding essentially transforms the algebra word problem into a result-unknown (arithmetic) problem, that is then more readily solved.

Place Table 5 about here

A variety of students' solution methods have been noted in the literature on algebra level problem solving (e.g., Hall et al., 1989; Kieran, 1998, 1992; Polya, 1959; Petito, 1979; Tabachneck et al., 1994). However, as Rachlin (1989) notes, "there is a need for research on the learning and teaching of the curriculum at two levels — that of the students and that of the teachers" (p. 259). Because teachers' beliefs about student ability and learning greatly influence their instructional practices, we set out to observe the impact of unknown position and presentation format on high school teachers' judgements of student problem-solving difficulty.

Teachers are not the only people whose judgements influence algebra instruction. Mathematics educational researchers also play a vital role today in shaping classroom practices and designing curricula. If teachers have prevalent misconceptions about algebra problem solving, it is valuable to know if they are also held by researchers in the field. If these views are only held by teachers, than it is likely that they would relinquish these views in light of the findings available to researchers. However, if these views are also held by researchers who are informed by the current research, we may expect that these views are firmly rooted and may be more resistant to change. In this case it would be necessary to alert researchers to these widely held misconceptions, and consider new approaches for improving teachers' understanding of students' thinking. For these reasons we include in our investigation the judgements of algebra researchers as well as those of teachers.

Method

Subjects

Participants were U. S. mathematics teachers (n=67) and mathematics educational researchers (n=35). The teachers were from the same state in the southeastern United States, and taught in a wide range of settings and socio-economic communities, including predominantly minority-based inner city schools, rural communities, and middle income suburban areas. They were recruited from a teacher workshop during the summer, and their current teaching responsibilities included 7th through 12th grades. The researchers were dispersed throughout the U. S., and were recruited via an Internet discussion group specifically focused on issues of algebraic thinking and instruction.

Procedure

The participants were asked to rank order twelve mathematics problems, from easiest to most difficult, without discussion among them. The twelve problems were composed of two problems for each of the six types discussed above (see Table 1). These problems were chosen because they were representative of problems used in the mathematics textbooks used in the teachers' school districts.

Specifically, the teachers were given the following instructions.

> Below are 12 problems that are representative of a broader set of problems that are typically given to public school mathematics students. My colleagues and I would like you to help us by answering this brief (10 min) survey. We are happy to share the results we obtain with you this spring.

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What we would like you to do:
Rank these 12 problems starting with the ones
you think are easiest for your students to the
ones you think are hardest. You can have ties
if you like.
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Unlike the teachers, those researchers who participated in this study were self-selected on the basis of their own interest. All were members of an on-line discussion group on algebra learning and instruction. Researchers received the same problems as the teachers. They were told, "Below are problems that are representative of a broader set of problems that have been given to urban high school students at the end of an Algebra 1 course -- they are mostly 9th grade students." Participant researchers were then asked to "rank these problems starting with the ones you think were more difficult for these students to the ones you think were easier." The materials were distributed and collected electronically over e-mail.

The problems from the survey were generated using the three presentation formats — story, word equation, and symbolic equation — and the two unknown positions — result and start. Note that the underlying mathematical relationships are the same across the six problem types given to participants. This underlying structure, and the problem variations, were not discussed with the participants, but were crucial in our subsequent analyses of participants' predictions and students' performances.

Results

The average rank ordering produced by the mathematics teachers and researchers in our sample are presented, respectively, in the first and second columns of Table 6. Each of the rank orderings were analyzed using a 2-way, repeated-measures ANOVA with position of the unknown (result v. start) and presentation format (story v. word-equation v.

symbol) as within-subjects factors, and difficulty rank as the dependent measure. The divisions reflect statistically significant differences ($p<.05$) among the rank ordered data.

Place Table 6 about here

Unknown Values

Teacher data. Across all problem types, 84% of the teachers (n=67) ranked result-unknown (arithmetic) problems (problems like P4, P5, and P6 of Table 1) as significantly easier than start-unknown problems. Arithmetic symbol equations (P6) were predicted by teachers to be significantly easier than arithmetic word-equations (P5), while word equations were expected to be slightly easier than arithmetic story problems (P4). These ranking data showed a significant main effect for the unknown value factor, $F(1,134)=5.9$, $p<.02$. Six percent of the teachers in this sample viewed start-unknown problems as easier than result-unknown problems, and 10% of the teachers were inconsistent in their ranking.

Researcher data. The mathematics researchers (n=35) showed strong agreement with teachers. About 66% of the respondents ranked start-unknown problems as consistently harder than result-unknown problems across the 3 presentation forms, while 34% ranked result-unknown problems as more difficult in some but not all cases.

Relation to student performance. The relative rank of problem difficulty for students in both samples is shown in Table 6, columns 3 and 4. As predicted by teachers and researchers, students exhibited lower performance levels on start-unknown (algebra) problems than on result-unknown problems. As Tables 2 and 3 show, the high school students in our two samples scored between 14% and 24% higher on result-unknown problems.

Presentation Format

Teacher data. The data show that, after collapsing across the unknown value factor, 42% of teachers ranked symbol equation (P3, P6) problems as consistently easier than word-equations on average, and 49% ranked equations as consistently easier than story problems (P1, P4) on average. Fewer than 30% ranked verbal algebra problems (story and word equation combined) as being as easy to solve as symbolically presented arithmetic problems. Only 8 teachers (12%) consistently ranked equations as harder than verbal problems in both the arithmetic and algebra problems.

Verbally presented start-unknown problems (P2, P3) were considered particularly difficult. Over 76% of the teachers ranked story and word-equation start-unknown problems as more difficult than all other problem types. A post hoc comparison ($p=.05$) among all six problem types revealed that most teachers (70%) ranked start-unknown (i.e., algebra) word-equation problems as the most difficult form of problem given in the survey.

Researcher data. About 31% of the researchers consistently ranked equations as easiest for arithmetic and algebra level problems. Only 23% of the researchers consistently ranked equations as harder than word and story problems within each of the two levels of the unknown position factor.

Relation to student performance. Contrary to teachers' expectations, however, students' experienced greater difficulties when solving symbol equation problems than verbally presented problems (Table 6, columns 3 and 4). Also, in contrast to the view proffered by teachers or by researchers, students do not find algebra story and word problems to be most difficult. Rather, algebra equation problems were significantly less likely to be correctly solved than either story problems or word-equation problems.

Interactions

Over 46% of the mathematics education researchers in our sample showed an interaction of the two factors by ranking start-unknown (algebra) equations easier than start-unknown verbal problems, and result-unknown equations harder than result-unknown verbal problems (see column 2 of Table 6). Only 17% of researchers ranked arithmetic equations as being the same or harder than the two algebra verbal problems (i.e., problems P1 and P2 of Table 1), while 69% ranked arithmetic equations as being easier than both verbal start-unknown forms. The remaining 14% of the participants were split.

These data indicate that many mathematics education researchers view story problems as harder than equations for start-unknown (i.e., algebra) problems, and see all forms of start-unknown problems as harder than all result-unknown problems.

Summary

Teachers and researchers generally ranked algebra problems as more difficult than matched arithmetic problems, regardless of the presentation format. Teachers and researchers also tended to rank verbally presented problems (i.e., story and word-equations) as more difficult for students than symbolic equation problems in algebra and arithmetic. Both professional communities in our samples generally agreed upon the relative difficulty of algebra story and word equation problems.

Discussion

In the balance, teachers predicted much of what makes problems difficult for students. A Kendall's Rank Correlation (Tau) shows a significant relationship between teachers' ratings and students' performances, $\tau(12)=.61$, $p=.03$ (see Table 6). However, the data also show systematic discrepancies with students' performances — discrepancies which could have significant implications for teachers' instructional practices. The most salient discrepancy is teachers' predictions that verbal problems will be most difficult, while students find symbolically presented problems significantly more difficult. We now explore possible sources of this discrepancy by examining the structure of mathematics textbooks, and their role in teachers' decision-making. We then consider two competing models of algebraic development based on the student and teacher data sets.

Textbook View

In the course of the investigation of possible sources of teachers' and researchers' predictions, two commonly used mathematics textbook series adopted by the teachers' school district were analyzed for their treatment of arithmetic and algebra concepts (Nathan, 1998). Textbooks have been identified as a primary resource — and often the only source — of the content planning performed by expert and novice high school mathematics teachers (Borko & Livingston, 1989; Borko & Shavelson, 1990; Cooney, 1985). Both textbook series (Harcourt, Brace, and Jovanovich, and Glencoe) comprised a a pre-algebra text and an Algebra I text. The series first present arithmetic computations in symbol form, followed by the application of these procedures to stories and scenarios. The algebraic formalism was introduced next, along with rules of symbol manipulation and worked-out examples showing how the rules can be applied to problems. Story problems were then introduced as applications of the formalism. The textbooks showed solutions to story problems by first translating the verbal problems to a symbolic format, and then applying symbol manipulation procedures. The chapter organizations of the textbooks followed the general sequence shown in the "textbook" column of Table 6.

Kendall's rank correlation revealed a highly significant agreement between teachers' rankings (Table 6, column 1) and the curricular sequence offered by the mathematics textbook series, $\tau(12)=.867$, $p=.015$. Teachers ranked start-unknown (algebra) problems as significantly more difficult to solve than result-unknown (arithmetic) problems. They also ranked symbolic equation problems within each category as easier than the corresponding verbal problems.

One likely interpretation of this pattern is that, through reliance and repeated exposure to textbooks, teachers internalize the textbook view as a basis for their predictions of problem difficulty for students. Alternatively, both teachers and textbook authors may base their views of algebra development on a common perspective, such as the component view of mathematics learning (cf. Greeno, Collins, & Resnick, 1996).

Two Competing Developmental Models

The ranking data strongly suggest that teachers and researchers tend to view students' algebra development in the following light. First, students develop their symbolic skills in arithmetic, with instruction and practice limited to result-unknown problems. Then, students learn to apply and extend these skills to verbally presented arithmetic problems. Next, students learn to take on more general families of problems where the unknown quantity is no longer the result, but can occur at the beginning of the problem. Symbolic forms of these problems are initially presented as the simplest versions. For these problems, new procedures, concepts, and laws are introduced that support conventional symbol manipulation to isolate the unknown quantity as a means toward problem solving. Finally, these procedures are applied to the verbal format using translation and modeling rules so that the algebra word equation and story problems are reduced to algebra equations solvable via the symbol manipulation procedures acquired during the previous phase of instruction. teachers and researchers appear to be think of students' problem-solving development largely within the textbook view. Thus, teachers and researchers, examining a problem for its level of relative difficulty, make their decisions based on the question, "how far along the developmental trajectory from symbolic arithmetic to algebra story problem solving has a student progressed?"

However, the analyses of students' problem-solving process data suggest an alternative trajectory for the development of algebra reasoning ability that circumvents many of the difficulties of symbolic algebra. In this alternative view, verbal competence, and the associated reliance on guess-and-test and unwinding strategies, is hypothesized to precede symbol manipulation skill for both arithmetic (result-unknown) and algebraic (start-unknown) problems.

With these views in mind, one can frame students' performances and teachers' expectations in the form of two competing models of algebra development. One model, suggested by the analyses of textbooks and the teacher and researcher data, hypothesizes that students' symbolic reasoning skills develop first, with word problem solving developing later. This we refer to as the textbook model of algebra development. The other, suggested by the student data, hypothesizes that verbal reasoning precedes symbolic reasoning. This model we term the verbal precedence model of students' algebra development

The textbook and verbal precedence models of students' algebra development are compared in Figure 3. Each square in the figure depicts the level of problem-solving competency that the student can achieve unaided (cf. the zone of proximal development construct of Vygotsky, 1978). All sixteen possible levels are shown in Figure 3 with a descriptive label. The number in each level indicates the total number of students from the initial study (n=76) that have progressed to that level but not beyond, as determined by their problem-solving performance. These numbers are used to compare the predictive power of the two models based on student data.

Figure 3 illustrates two different pathways through this space of all possible competency levels. One path is consistent with the Textbook model (dashed lines) of development and the other (heavy lines) corresponds to the Verbal Precedence (VP) model of development. The model that follows from the Textbook view (right of center) favors development of arithmetic before algebra, and symbolic problem solving abilities over verbal reasoning.

Note that three states of competency are held in common by these two competing models. Both models account for "All" problems solved to occur at the end of development, while no problems solved ("None") occurs at the beginning. The two models also posit that "All Arithmetic" competency occurs midway.

Both models require that a student pass through the level of competency where the student can solve any and all result-unknown problems regardless of its presentation format. However, the two models differ in their predictions of the pre-requisite competencies. In the VP model, verbal arithmetic is hypothesized to precede result-unknown, while the textbook model hypothesizes that competency in symbolic arithmetic occurs first.

Quantitative Model Comparison

Using the patterns of student performance in the original study (n=76), it is possible to classify each student into a competency level showing the students' development. The domain can be characterized by 16 states of competency. Competing developmental models all work within these 16 states, but hypothesize a different ordering of these states that lead to skilled performance (the "All" state). States of competency may also be omitted by a model of development.

Students classified as competent in either none or all of the problem categories fit both models trivially (i.e., they received either a zero score or a perfect score on the assessment). A student's performance does not fit a model when the student occupies a state of the model that demonstrates one competency but lacks another that is hypothesized to precede it in the developmental sequence. For example, if a student's competency reaches symbolic arithmetic but no further, and the student cannot also solve verbal arithmetic problems, then that student fits the Textbook model but not the VP model. This is because the textbook model hypothesizes that competency in symbolic arithmetic problem solving proceeds competency in arithmetic verbal problem solving. On the other hand, if a student's competency level reaches verbal arithmetic problem solving but no further, and includes no symbolic arithmetic problem solving, she fits the VP model but not the Textbook model.

Alternatively, a student's competence may follow a trajectory different than that hypothesized by either developmental model. For example, a lone subject was found to have competency at solving verbal arithmetic and symbolic algebra, but no other problems. Both models hypothesize that symbolic arithmetic will precede symbolic algebra. Consequently, performance of this subject is outside of both models, reducing the predictive power of each.

As a quantitative measure of the predictive power of each of the two models, the percentage of students who follow each of the hypothesized trajectories is compared. Of the 76 participants in the original study, 69 (or about 91%) of the students fit the VP model, while 47 (62%) fit the Textbook model. Most of those who fit the Textbook model are the 42 (55%) students in the three levels common to both models (the central column). Only 5 students (7%) uniquely fit the Textbook model, while 27 (36%) uniquely fit the VP model. Two students (3%) remain outside both. Based on the original student data, the Verbal Precedence model provides a better quantitative fit than the Textbook model.

As a further test, the VP and Textbook models are also applied to the data from the 171 students in the replication study. Here, 151 students (88%) fit the VP model. In contrast, only 79 students (46%) fit the Textbook model. As before, most of those students who fall along the developmental trajectory of the textbook model are the 65 students (38%) who are in the three competency levels common to both models. Only 10 students (6%) uniquely fit the Textbook model, while 86 students (50%) uniquely fit the VP model, and 7 students (4%) remain outside of either model. This provides added support to the hypothesis that the Verbal Precedence model of algebra development better reflects the problem-solving performance of students in the two samples than does the Textbook model. Furthermore, the VP model accounts for a high percentage of the students on an absolute scale, suggesting it captures something basic to students' algebra development.

Conclusions

While teachers accurately predict the differential performance of students on result- and start-unknown problems, we need to acknowledge that students' problem-solving behaviors differ in systematic ways from those predicted by teachers and researchers. These differences have significant impact on how teachers' perceive students' reasoning and learning. In this final section, we explore some of these issues and consider their implications for research on student thinking and for classroom instruction.

Implications for Research on Student Cognition and Development

The disadvantages of symbol problems apparent in our student data challenge the oft-cited view that story problems are inherently harder than symbolic ones (e.g., Cummins et al., 1988; Mayer, 1982). A number of studies, mostly at the elementary and middle grade levels, have shown positive effects of situational context (e.g., Baranes et al., 1989; Carraher et al., 1987; Cognition and Technology group at Vanderbilt, 1993). While the student data reported here shows a similar pattern, it must be noted that there was a general advantage of verbal problems overall. Although a context advantage may exist as well, as the student replication study suggests, studies comparing story and symbol problem solving have ignored the intermediate case of word-equation problems as a verbal problem format without a context, and so could not make the comparisons reported here. Students in secondary education have developed their verbal reasoning skills for a longer period of time than their skills with manipulating and reading mathematical symbol structures. The accessibility and use of students' alternative solution strategies suggest a view of mathematical development that matches the development of students' verbal reasoning abilities.

The student data also show that students can solve simple algebra problems as well as arithmetic problems, when the arithmetic problems are presented symbolically while the algebra problems are presented verbally (with or without a context) and thus elicit from students powerful alternative strategies such as the guess-and-test and unwinding methods. These alternative strategies are general and fairly robust, providing, in many instances, ways for the solver to reduce error opportunities and to verify solution accuracy. The use of these alternative methods by college students who have been away from algebra for a while has also been observed (Kieran, 1992; Tabachneck et al., 1994), suggesting they are relatively accessible and resistant to extinction — unlike algebraic symbol manipulation skills, which were relatively poorly executed by the algebra students in our samples.

Implications for Research on Teacher Cognition

Understanding teachers' views of problem difficulty is of great interest since these beliefs of problem difficulty are likely to affect teachers' instructional planning and the design of their assessments (cf. Borko & Shavelson, 1990; Carpenter et al., 1980). If teachers misperceive the relative difficulty of symbolically presented problems, they may choose to withhold verbally presented problems from a struggling student, with the rationale that verbal problems are simply out of reach for the student. Our results suggest that the development of students' algebraic reasoning and problem solving must be examined more closely before making curricular decisions. This is especially important as school districts throughout the U. S. are exploring ways to teach students algebra in the primary grades.

Mathematics educators need to be made aware of the efficacy and flexibility of students' alternative reasoning strategies. Findings from this study suggests, however, that any attempts to address this, will need to explicitly address the beliefs held by teachers. We must remember that deep-seated beliefs do not easily change. If we want to ultimately bring teachers' views into closer alignment with empirical findings, it is imperative that teachers are made aware that they hold these views. These views must be explicitly characterized for teachers, and the strengths must be acknowledged while the limitations are exposed.

Implications for Mathematics Instruction

The finding that algebra learning is formidable for students is not news. However, insights into how students cope with their attempts to solve new problems, learn formal representational systems, and think in new ways is of great interest. The informal strategies employed by students demonstrate an intuitive understanding of quantitative relations that may prove advantageous to later instruction.

Several studies have reported the advantages of explicitly addressing these informal methods during instruction, and building upon them as a means to teach formal algebraic methods. Research has shown problem-solving performance advantages when using informal and formal methods in combination (Koedinger & Tabachneck, 1994; Petito, 1979). The advantage of multiple strategies often are viewed in a compensatory manner, where the weaknesses of one are made up for by the strengths of another (Koedinger & Tabachneck, 1994). Guess-and-test and other substitution methods have proven to be beneficial for students developing their understanding of a balanced equation (Kieran, 1988). These methods apparently helped the students to see equations structurally, a view that ultimately facilitated their acquisition of formal methods that involved performing equivalent operations on both sides of the equation.

An alternative approach of algebra instruction has recently been used which builds on sixth graders' facility with informal methods to teach the concepts and procedures of the formal approach such as the symbolization of situations, the manipulation of symbolic expressions, solving systems of linear equations, and identifying and representing pattern generalizations (French, 1999; Knuth, 1999; Koedinger & Alibali, 1999; Nathan, 1999). In this approach, the unwinding and guess-and-test methods used spontaneously by students served as grounding representations for the new, more abstract objects and procedures. Key to this instructional approach is the role of diagnostic pre-tests that identify students' methods and inform bridging instruction aimed at connecting the new concepts to the learner's prior knowledge. Previous work using animations of situations (Nathan et al., 1992) and concrete arithmetic instances (Koedinger & Anderson, 1997) has shown that instruction which bridges formal algebra instruction to previously grounded representations helps students learn processes such as algebraic modeling of verbally presented relations. The two studies differed in the type of grounded representations used, yet yielded similar results, suggesting that a crucial feature of success was the role of grounded intermediate representations in students' learning.

Approaches such as those briefly reviewed address curricular goals that encourage middle grade students to "develop and apply a variety of strategies to solve problems" (NCTM, 1989, p. 75) and solve equations "using concrete, informal, and formal methods" (p. 102) so that they will "develop technical facility" with the later concepts of algebra (p. 150). However, the interpretation of the use of informal methods must be cautionary as well as optimistic. Informal methods may facilitate performance when "technical facility" is lacking. But informal strategies can themselves be limited in their efficacy. For example, the unwinding strategy breaks down when an unknown quantity has multiple occurrences. Likewise, the guess-and-test strategy can be inefficient, highly demanding of cognitive resources, and limited to finding numerical answers that are likely to be guessed (e.g., whole numbers and common fractions; Tabachneck, Koedinger, & Nathan, 1995).

Informal strategies also show their limitations as problem complexity increases. Verzoni & Koedinger (1997) found that middle school students (grades 6 through 8) performed best on easy (one-operator) problems when they were presented in a grounded story problem format, rather than as an abstract number sentence, because the story problems elicited more successful informal strategies. However, when more complex problems were given that involved two operators and, additionally, the use of negative numbers, performance was higher on formal number sentence problems, because the greater complexity interfered with the execution of the informal solution methods.

Informal methods have also been shown to inhibit the acquisition for formal algebraic solution strategies in an instructional setting. Although guess-and-test users in Kieran's (1988) study were more apt to learn how to isolate unknown terms by maintaining a balanced equation, the students who preferred to use working-backwards methods like unwinding had greater learning difficulties. Learning suffered because working-backwards methods seem to reinforce a procedural view of algebra equations, rather than supporting a structural view.

With these cautions in mind, the role of informal reasoning to support formal algebra instruction must be approached with caution as well as with interest. As research into mathematical learning and instruction continues, we expect to provide teachers and members of the research community with a greater understanding of students' mathematical conceptions and development. And as studies of teachers' knowledge and beliefs continue, we can look to enhanced programs of teacher preparation and the development of theoretically and empirically rooted approaches to classroom instruction.

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Table 1: Sample problems used to elicit difficulty ranking judgements from high school teachers and mathematics education researchers.

P1) When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?

P2) Starting with some number, if I multiply it by 6 and then add 66 I get 81.9. What did I start with?

P3) Solve for *X*: $X * 6 + 66 = 81.90$

P4) When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?

P5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?

P6) Solve for *X*: $(81.90 - 66) / 6 = X$

Table 2. Student performance (n=76) on an assessment of problems from the six problem types of Table 1. Data are taken from Koedinger & Nathan (1998).

		Verbal		Symbolic	Total
	Value Unknown ↓	Story	Word-Equation	Symbolic-Equation	
Student Performance (n=76)	Result unknown (arithmetic)	73%	67%	53%	64%
	Start unknown (algebra)	59%	54%	37%	50%
	Total	66%	61%	45%	57%

Table 3. Replication of student performance (n=171) on an assessment of problems from the six problem types of Table 1. Data are taken from Koedinger and Nathan (1998).

		Verbal		Symbolic	Total
	Value Unknown ↓	Story	Word-Equation	Symbolic-Equation	
Study 2 (n=171)	Result unknown (arithmetic)	80%	74%	56%	70%
	Start unknown (algebra)	60%	48%	29%	46%
	Total	70%	61%	43%	58%

Table 4. Frequency of solution strategies (%) used by solvers (n=76) as a function of problem presentation for start-unknown (algebra) problems.

	Solution strategies					
Problem Presentation	Unwind	Guess & Test	Symbol Manipulation	No Response	Unknown strategy	Total
Story Problems	56	10	6	14	14	100%
Word-Equation	26	31	8	13	22	100%
Symbol-Equation	13	20	26	30	11	100%

Table 5. Likelihood that solution strategies used by students (n=76) on start-unknown (algebra) problems led to a correct answer.

Strategies	Number of correct problem solutions	Likelihood strategy leads to correct answer
Unwinding	83	72%
Guess-&-Test	37	68.5%
Symbol Manipulation	19	54%
Unknown Strategies	20	49%
Total	159	61%

Table 6: Difficulty rank given by teachers' and researchers and student performances.

	Teachers mean rank (n=67)	Researchers mean rank (n=35)	Student Performance (n=76) [and % correct]	Student Performance (n=171) [and % correct]	Textbook view
Easy*	Arithmetic-Eqn (P6) Arithmetic-Word (P5) Arithmetic-Story (P4)	Arithmetic-Word (P5) Arithmetic-Story (P4) Arithmetic- Eqn (P6)	Arithmetic-story (#4) [73%] Arithmetic-Word (#5) [67%]	Arithmetic-Story (#4) [80%] Arithmetic-Word (#5) [74%]	P6 P5 P4
Medium-easy*	Algebra- Eqn (P3)	Algebra- Eqn (P3)	Algebra-Story (#1) [59%] Algebra-Word (#2) [54%] Arithmetic-Eqn (#6) [53%]	Algebra-Story (#1) [60%] Arithmetic-Eqn (#6) [56%] Algebra-Word (#2) [48%]	P3
Medium-hard*	Algebra-Story (P1)				
Hard*	Algebra-Word (P2)	Algebra-Word (P2) Algebra-Story (P1)	Algebra-Eqn (#3) [37%]	Algebra-Eqn (#3) [29%]	P2 P1

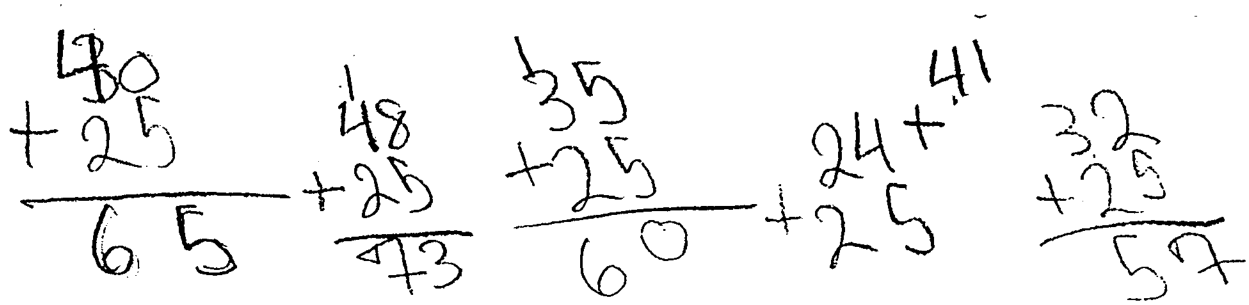
*Difficulty divisions (Easy, Medium, Hard) show significant differences ($p<.05$) in mean ranking by teachers and researchers, or in student performance levels.

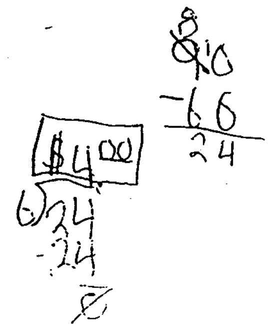
Figure Captions

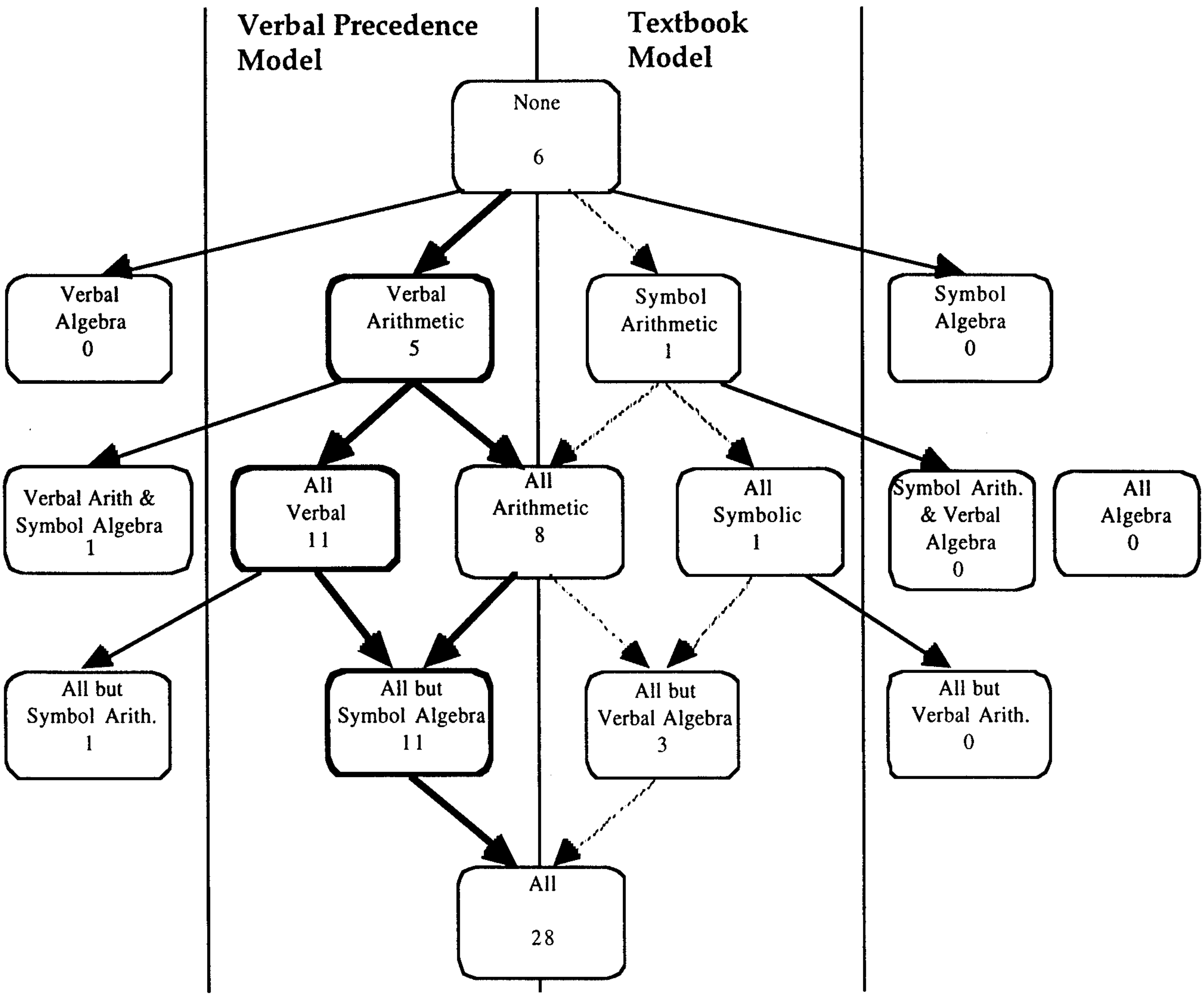
Figure 1. Guess-and-test strategy used by Subject #103.

Figure 2. Unwind strategy used by Subject #99.

Figure 3. Two models of algebra development (Verbal Precedence model with heavy lines, and Textbook model with dotted lines), and the quantitative fit with the original student data (n=76).






Verbal Precedence Model
Textbook Model
None
6
Verbal Algebra
0
Verbal Arithmetic
5
Symbol Arithmetic
1
Symbol Algebra
0
Verbal Arith & Symbol Algebra
1
All Verbal
11
All Arithmetic
8
All Symbolic
1
Symbol Arith. & Verbal Algebra
0
All Algebra
0
All but Symbol Arith.
1
All but Symbol Algebra
11
All but Verbal Algebra
3
All but Verbal Arith.
0
All
28