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Teachers' Judgements About Algebra Problem Difficulty

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Problems presented in words and stories are generally perceived as the most difficult tasks facing algebra students. Textbooks typically save these problems for the end of chapters (Nathan and Long 1999); students find these problems least favorable; even comic strip folklore (Figure 1) presents story problems as the bane of formal education (Larson 1997).

Place Figure 1 about here

Yet however it is that mathematical difficulties are portrayed, it is the beliefs that teachers hold about students' mathematical abilities and learning processes that influence teachers' pedagogical decisions, planning activities, and instructional practices most (Borko and Shavelson 1990). Teachers consider student ability to be the characteristic that has the single greatest impact on instructional decisions. Because teachers' views of their students are so important, we investigated the relation between teachers' perceptions and students' actual performances on a set of mathematics tasks. Our goal is to present teachers with an accurate picture of students' abilities, and to dispel some myths about student performance at the algebra level. Providing a realistic picture of students' mathematical development is especially important as schools throughout the United States consider algebraic instruction for earlier grades.

The Problem Difficulty Ranking Task

Consider your own expectations about how students will perform on the small set of problems presented in the Problem Difficulty Ranking Task (Figure 2). A similar version of this task has been given to groups of high school mathematics teachers attending district-sponsored professional development workshops. The first of these was given to 67 teachers during a summer workshop (Nathan and Koedinger in press). The second was administered to 39 high school mathematics teachers in a different area of the country during the fall of the following school year.

We invite the reader to take a few minutes and perform the ranking task shown in Figure 2 before reading further. We will then present views commonly reported by teachers.

Place Figure 2 about here

Teachers' Predictions

Table 1 shows how the problems given in the ranking task can be organized according to the area of mathematics of concern and the presentation format of the problem. The rows of Table 1 show problems that are either arithmetic (with the result as the unknown) or algebraic (with a starting quantity as unknown). The columns of Table 1 differ by either verbal or symbolic format. The word equation problem (problems P2 and P5 of Table 1) is like the "pick a number" game (Usiskin 1997). It differs from a story problem because it verbally describes the relationship among pure quantities without a story context.

Place Table 1 about here

When high school teachers were asked to judge the relative difficulty of the problems, their pattern of responses was clear (Table 2). First, arithmetic problems (P4, P5, and P6) were generally considered to be easier than algebra problems (P1, P2, and P3) regardless of their presentation in word, story, or symbolic format. Second, verbal problems (word and story problems like P1, P2, P4, and P5) were considered by teachers to be more difficult for students than symbol problems for both arithmetic and algebra problems. Finally, algebra word and story problems were considered to be most difficult for students.

Place Table 2 about here

By now you have probably compared your own predictions with these findings. There is certainly some variability in the responses, and people may also differ on their reasons. However, we have found the pattern to be quite general with several groups around the U. S., including other high school teachers, and a group of mathematics educational researchers who focus on algebra learning and instruction (Nathan and Koedinger in press, Nathan Tabachneck and Koedinger 1996). Let us now see how teachers' predictions compare with student performance.

Student Performance and Teacher Expectations

One hundred and seventy-one high school students who had completed the Algebra I curriculum took a test made up of problems like those in the ranking task (Koedinger and Nathan 1999). This test was administered by the students' teachers. The test revealed that students tended to find arithmetic problems easier to solve than algebra problems — as predicted by teachers' responses (Table 2). They solved 70% of the arithmetic problems correctly, while algebra problems were solved correctly 46% of the time. Comparisons of students' verbal and symbolic problem-solving performance contradicted teachers' expectations, however. Students solved about 65% of the word and story problems, but only 43% of the symbolic arithmetic and algebra problems. Students scored high on story and word-equation problems regardless of the presence of a problem context. Students also showed a surprising result: The accuracy rate for arithmetic equation problems (56%) was basically the same as for algebra story problems (60%) — suggesting that algebra story problems may not deserve their infamous reputation.

Table 2 shows the relative ranking given by teachers (column 1), and the order of problem-solving difficulty for students (column 2). Difficulty divisions (Easy, Medium, Hard) reflect statistically significant divisions ($p<.05$). Overall, teachers predicted much of what makes problems difficult for students, and they are more often right than not. However, the comparison also shows systematic discrepancies that appear to be based on misconceptions teachers have about students' mathematical reasoning. Most notably, teachers' expect story and word problems to be most difficult, while students actually find equations to be hardest. This misconception is pronounced and systematic, and may significantly influence teachers' curricular decisions.

A closer look at students' solution strategies

Part of the disparity between teachers' predictions and students' actual performance may be due to teachers' perceptions of the problem-solving process. Teacher education programs typically present algebra word problem solving in a two-step fashion (Mayer 1985). First, the verbal problem is translated into a set of algebra equations, and then the equation is solved using symbol manipulation to isolate the unknown. Since equation problem solving is only a part of the overall approach, it is logically expected to be easier than story problem solving.

Yet, analyses of students' solution methods reveal that students do not typically follow the two-step translate-and-manipulate approach. Students frequently use informal methods such as Guess-and-Test and Unwinding to solve algebra word problems when they are given the freedom (Kieran 1992, Nathan and Koedinger in press). Guess-and-Test uses arithmetic procedures in a forward manner to solve algebra word problems iteratively, once a value has been substituted for the unknown quantity (see Figure 3). The Unwinding method untangles the quantitative relations of an algebra problem by inverting the mathematical operations and the order of quantities (see Figure 4). Unwinding circumvents the need for symbol manipulation, and instead transforms the algebra story problem into a sequence of arithmetic tasks. These alternative solution approaches show how students make use of their prior knowledge about the world to facilitate their reasoning. For example, in the Unwinding solution shown in Figure 4, the student keeps the numbers situated in a money ($) context which helps the student to avoid mis-aligning the place values — a common error during subtraction. These informal methods are far more successful (about 70%) than the formal methods (about 30%) in producing a correct answer for algebra story problems. Their use increases students' performance on the problems that teachers expect to be most difficult.

Place Figures 3, and 4 about here

Where do these views come from?

If the views expressed by teachers in the ranking task are so widely held, where do they come from? And if these beliefs are inaccurate, why do they persist? One answer is that they fit into a view of learning that is deeply rooted historically, and promoted implicitly by many mathematics textbooks. Algebra and pre-algebra textbooks often present symbolic problems as easier than verbal problems. Symbolic problems are placed earlier in arithmetic and algebraic lessons, while verbal problems are typically presented at the end, as "challenge" or "application" problems (Nathan and Long 1999). This sequence is based on images of mathematics learning inherited from the behaviorist view of curricular design. The behaviorist view advocates first teaching procedures in the most simplified context to isolate the new skills, and then presenting the material in somewhat more complicated settings, such as word problems (Greeno Collins and Resnick 1996).

While teachers' expectations parallel the problem sequence found in many textbooks, students' problem-solving performances do not (Table 2). The textbook view — which places arithmetic before algebra and symbol equations before verbal problems — describes the performance of only 46% of the students studied (Nathan and Koedinger in press). In contrast, the verbal precedence view of algebra development places verbal problem-solving skills before symbolic reasoning, and is consistent with the performance of 88% of the high school students.

Conclusions

Students' methods for solving problems are diverse and can lead to very different patterns of performance than one might expect based on commonly held views of mathematical development. Most students do not develop symbolic skills before they are able to solve verbal problems of similar mathematical structure. Furthermore, verbal problems elicit solution strategies from students that are more successfully applied than the symbol manipulation methods typically promoted in Algebra I classes. Our tests show that students do not acquire symbol manipulation skills at the levels that high school teachers expect. This suggests that we need to look for new ways to teach algebra concepts and skills that build directly upon students' informal solution methods and their verbal skills. In future articles we will present some of the specific curricular approaches that we have used to extend students' algebraic reasoning. Here, we will discuss implications of this work for teaching algebra at the middle school and high school grades.

Students' informal solution strategies are very powerful, although often relatively inefficient, and are used extensively by students in elementary, middle and high school, as well as at the college level. These informal solution strategies can serve as excellent stepping stones — conceptual bridges — to formal symbol-manipulation procedures. The Guess-and-Test method highlights the structural aspects of algebraic equations (Kieran 1992). The interactive process relies heavily on students' number facts, and encourages students to explore how the relations among the quantities act as constraints. It also reifies the concept of the variable for the student.

The Unwinding method exposes students' intuitions about manipulating quantities. Even young students will naturally "undo" the relations presented in a story problem, and carry out the unwinding procedure using separate arithmetic calculations. Unwinding provides a natural entrée to inverse operations and equation balancing — crucial aspects of symbolic skill development.

Students' ideas about algebraic relations may need to be honed, but they are largely effective. By identifying these intuitive solution methods, it is possible to show students that they already have much of the conceptual foundation for algebraic reasoning in place. As early as the sixth grade, students can discuss these approaches in depth with their peers, and explain them to others (Nathan et al. 1998). This, in turn, supports students' ownership of mathematical ideas. However, the teacher must be careful in the selection of problems: If they are too easy (e.g., they use very common number facts) students may apply these methods automatically, without really noticing their solution approach. Choosing numbers that require students to write things down and present all their steps helps students to see their own reasoning process. Teachers may also introduce the notion of a "problem solvers' toolkit"; that is, a collection of solution methods that students may list in their notebooks or on a classroom bulletin board. Students can then discuss their strategy decisions explicitly during problem-solving activities, and reflect on the factors that influenced their choices. Explicit discussion of the trade-offs of different strategies (e.g., efficiency, computational difficulty) raises the level of classroom discourse to one of solution methods and problem demands, rather than talking only about computations and answers.

Students possess and use a variety of effective solution methods, many of which signal profound understandings of quantitative relationships. By building on these powerful intuitions, it is possible for teachers to ground important abstract ideas of mathematics to a conceptual base that will support mathematical development, and enable students to see the range of mathematical thinking that is within their grasp.

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Figures and Tables

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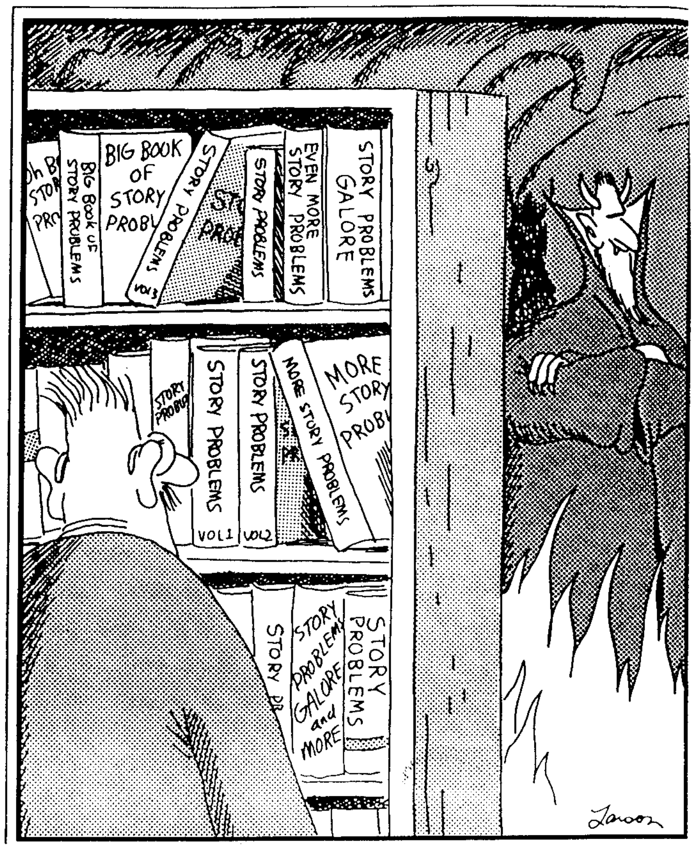
Figure 3. The Guess-and-Test strategy.

Figure 4. Figure 4. A middle school student using the Unwind strategy to solve an algebra story problem. Note how the situated nature of the numbers in this strategy helps the student to catch a common place value alignment error.



Hell's library

Figure 1. Story problems are commonly presented as the most undesirable of tasks.

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THE PROBLEM DIFFICULTY RANKING TASK

Below are 6 problems that are representative of a broader set of problems that are typically given to public school students during middle school and high school mathematics courses.

What we would like you to do:

Take about 5 minutes to complete this task. Rank order the problems below based on your expectations of their difficulty for students in your class, starting with the ones you think will be easiest for these students to the ones you think will be harder. You can have ties if you like. For example, if you think the fourth problem (#4) was the easiest, the 3rd was the most difficult, and the rest were about the same, you would write:

4 (easiest)

2 1 5 6

3 (hardest)

Problems:

1) When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?

2) Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?

3) Solve for x: x * 6 + 66 = 81.90

4) When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?

5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?

6) Solve for x: (81.90 - 66) / 6 = x

Figure 2. The problem difficulty ranking task given to teachers.

Problem. Katrina's allowance is $2.75 more than Sarah's allowance. Combined the allowances of the two girls is $14.75 each week. What are their allowances?

Student written work	Student speech
Step 1 K - S = $2.75 K + S = $14.75	S: Katrina, K is two seventy-five more than Sarah. S. K and S is fourteen dollars and seventy-five.
Step 2 K = 7 S = 7.75 7 - 7.75 =	S: So, seven and seven point seven five is 14 point seven five. But…too small [referring to the difference.]
Step 3 6 - 8.75 = 8.75 - 6 = 2.75	S: Go up here [writes 6]… and down one here [writes 8.75] … Oh! Switch them, so eight point seven five minus six … is … Yea!

Figure 3. The Guess-and-Test strategy

Problem. Minerva bought 3 pairs of the jeans she loves. In the next store she bought a pair of shoes for $64. She spent a total of $125.50. How much did each pair of jeans cost?

Student written work	Student speech

Step 1 (with error): Undo the price of shoes.
 	
`  $125.50`
`–      64`
`  $124.86`
	S: One hundred and twenty-five dollars and fifty cents, minus, MINUS sixty-four dollars for the shoes.
	So she has a remainder of one hundred and twenty-four dollars and eighty-six cents.

Step 1 (without error): Undo the price of shoes.
 	
`  $125.50`
`–$ 64.00`
`  $ 61.50`
	S: Oh! [S catches her error and performs a new subtraction.] She has a remainder of sixty-one dollars and fifty cents.

Step 2: Undo the price of each pair of jeans.
 	
`   $ 20.50`
`3)   61.50`
`    -6`
`     01`
`    -00`
`      15`
`     -15`
`      00`
	S: Now, divide that by three.
	They each are twenty bucks. Twenty dollars and fifty cents. [Uses calculator to check the division calculation.]

Figure 4. A middle school student using the Unwind strategy to solve an algebra story problem. Note how the situated nature of the numbers in this strategy helps the student to catch a common place value alignment error in step 1.

Table 1. The underlying structure of the problems given in the difficulty ranking task.

Presentation type →	Verbal problems		Symbol Problems
Area of Mathematics ↓	Story	Word Equation	Symbol Equation
Arithmetic	P4. When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?	P5. Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?	P6. Solve for X: (81.90 - 66) / 6 = X
Algebra	P1. When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?	P2. Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?	P3. Solve for X: X * 6 + 66 = 81.90

Table 2. The average difficulty rank ordering of problems given by teachers, along with the order of problem solving difficulty for students (with % correct), and the order these problem as presented in common textbooks.

	Teachers Rank ordering (n=67)	Student Performance (n=171) [and % correct]	Textbook View
Easy	(P6) Arithmetic-Eqn (P5) Arithmetic-Word (P4) Arithmetic-Story	(P4) Arithmetic-Story [80%] (P5) Arithmetic-Word [74%]	(P6) Arithmetic-Eqn (P5) Arithmetic-Word (P4) Arithmetic-Story
Medium	(P3) Algebra- Eqn	(P1) Algebra-Story [60%] (P6) Arithmetic-Eqn [56%] (P2) Algebra-Word [48%]	(P3) Algebra- Eqn
Hard	(P1) Algebra-Story (P2) Algebra-Word	(P3) Algebra-Eqn [29%]	(P2) Algebra-Word (P1) Algebra-Story

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