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# Difficulty Factors in Arithmetic and Algebra:

# The Disparity of Teachers' Beliefs and Students' Performances

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## Abstract

In Experiment 1, teachers' (n=68) predictions about the difficulties experienced by early algebra problem solvers is assessed using a problem ranking task. The rank order data match closely a proposed normative view of algebra development: (a) start-unknown problems are predicted to be much more difficult for students than result-unknown problems; and (b) verbally presented problems such as story problems and word equations are seen as more difficult than equation problems. While teachers are accurate in the first claim, student data from two prior experiments (n1=76, n2=171; Koedinger, Nathan, and Tabachneck, 1996) show systematic discrepancies. Teacher predictions are explained well by the normative model, but student performances deviate considerably from that model. In particular, students show a pronounced advantage for verbally presented problems in arithmetic and algebra, and are better fit by the verbal-precedence model. Analyses of students' solution protocols reveal a great likelihood for verbally presented problems to elicit alternative strategies that provide some built-in validity checks on their answers. Experiment 2 replicated the ranking pattern among teachers in a new region of the United States (n=105) and found that the ranking data was best predicted by teacher grade level (elementary, middle, high school) and teachers' responses to belief items from the normative view. Elementary teachers showed superior ability to predict students' areas of difficulty than high school teachers. It is suggested that the normative view plays a major role in shaping teachers' practices. Rather than building upon students' (taught) symbol manipulation skills, however, it is recommended that teachers should work within students' natural developmental trajectories and emphasize their verbal reasoning abilities as the foundation for healthy development of one's symbol processing skills. Implications for teacher cognition are presented.

> "...[T]here is a grave scarcity not only of models of the teaching of algebra but also of literature dealing with the beliefs and attitudes of algebra teachers." (Kieran, 1992)

Researchers and mathematics educators have made great strides in identifying the problem-solving processes and difficulties that young children experience as they develop their early mathematical reasoning abilities. Earlier work on students' arithmetic story problem solving showed, for example, the importance of distinguishing between classes of problems based on the solution approaches that they elicited from students rather than on an analysis of the underlying mathematical structure (Carpenter & Moser, 1982; Cummins, Kintsch, Reusser, & Weimer, 1988; Riley, Greeno, and Heller, 1983). Subsequent work showed that solution approaches were strongly influenced by the semantic characteristics of the problems. One may group problems such as Join, Separate, Compare and Combine together because of the common underlying mathematical expressions that describe them. However, the problems are quite different from a psychological perspective, and tend to elicit solution methods from students that reflect the different semantic structure of each (Fuson, 1990).

Researchers have also shown how students' solution approaches become more abstract and general over time, as direct modeling gives way to counting strategies and the use of numerical relationships (e.g., Carpenter et al., 1989; Case & Griffin's work, reported in Bruer, 1993; Resnick, 1987). We have also become aware of the importance of studying students' mathematics learning conjointly with mathematics teachers' beliefs and instructional practices (Carpenter & Fennema, 1988; Grouws, 1991). Studies of elementary school teachers' pedagogical content knowledge (Shulman, 1986) have identified the extent to which teachers are aware of the relative difficulty of these problem types, their influence upon students' problem-solving success and solution strategies, and their developmental accessibility (e.g., Carpenter, Fenema, Peterson, & Carey, 1988).

Our current investigation is of the development of early algebraic reasoning — reasoning that builds upon and extends young people's arithmetic thinking to include unknown quantities and general patterns. The body of work on arithmetic story problem solving (e.g., Carpenter et al., 1988; De Corte, Greer, & Verschaffel, in press); Shalin & Bee, 1985) provides a firm methodological and theoretical basis for the study of older students' algebraic reasoning, its development and impediments, and teachers' pedagogical content knowledge in this area.

Our inquiry focuses on the learning and instruction of early algebra which is typically encountered in middle school and beginning high school level instruction. It is rooted in the question "What kinds of activities qualify as algebra or as algebraic thinking and how does this thinking develop?" The series of studies presented here has several objectives. In Experiment 1, we set out to examine teachers' pedagogical content knowledge and beliefs about the factors that make mathematics problems difficult for early algebra students. The views of a sample of mathematics educational researchers were also studied. These views are examined with respect to a normative view of early algebra development (discussed in the next section), and a model underlying teachers' predictions is proposed. The accuracy of teachers' beliefs was examined by comparing their predictions to students' problem-solving performance data collected in another study (Koedinger, Nathan, & Tabachneck, 1996). Systematic matches and discrepancies between teachers' predictions and students' performances were identified. Detailed analyses of students' solution processes document the mathematical knowledge of early algebra students and lead to the formation of an alternative model of early algebra development that extends current views of algebraic reasoning. Differences between the models attributed to teachers and to students are considered as a major cause for the observed discrepancies.

Experiment 2 further examined aspects of teachers' pedagogical content knowledge. It asks how teachers' basic beliefs about mathematics instruction, student learning, and the domain of mathematics relates to teachers' expectations of students' problem difficulties. This relationship was examined for teachers from primary to secondary school levels, and important differences across grade level were identified. Some implications for early algebra learning and instruction are presented.

## Theoretical Framework

Before we turn to the study of teachers' beliefs, it is important to review some of the theoretical and empirical distinctions that have been made with regard to mathematics problem solving in arithmetic and early algebra. Previous work on the characterization of algebra-level problem-solving and its relation to arithmetic (e.g., Tabachneck, Koedinger, & Nathan, 1995) has suggested the following characterization: The unknown quantity in a problem may be placed either at the end of the problem or refer to the concluding amount to be determined, and is referred to as a <u>result unknown</u> problem (cf. Carpenter et al., 1980). Alternatively, the unknown quantity may refer to a value needed to specify the quantitative relationship that leads to a known result — a <u>start unknown</u> problem. A start-unknown problem (e.g., ? x 4 = 20, and 5 x ? = 20) describe a quantitative relationship in a way that impedes direct application of arithmetic procedures. It requires a student to compute the value of the unknown quantity through the application of a variety of other methods such as (a) those that may be associated with algebra level reasoning (i.e., manipulation of terms within an algebraic equation where symbols explicitly represent unknown quantities), (b) inverting the arithmetic operators given in the problem (e.g., changing the multiplication operation to division) and applying them in the reverse order, or (c) application of model-based methods such as guess-and-test and simulation techniques (cf. Hall et al., 1989; Herr & Johnson, 1993). Result unknown problems (e.g., 5 x 4=?) may be solved through direct application of mathematical techniques typically acquired earlier, such as (a) direct modeling, (b) counting methods, or (c) application of arithmetic procedures and number facts. Empirical findings show that start unknown problems are significantly more challenging for students in elementary grades (Carpenter et al., 1989) and at the college level (Koedinger & Tabachneck, 1995 ??).

The format in which a mathematical problem is presented also bears on problem difficulty. Story problems are typically distinguished from mathematically equivalent symbol equation problems because performance on story problems tends to be much lower than on equations (Carpenter et al., 1980; Cummins, Kintsch, Reusser, & Weimer, 1988; Mayer, 1982). The observed performance difference between start-unknown and result-unknown problems applies to symbolically presented problems as well as those presented linguistically (e.g., Tabachneck et al., 1995).

Symbolic problems are those shown in equation form. Symbolic arithmetic problems are typically described as number sentences. Story problems differ from symbolic problems in their verbal format, and because they contain contextual information about the problem situation which can be used by the solver as a source of problem elaboration, reframing, and solution constraints (Baranes et al.; CTGV story problems; Lave story problems; Nathan, Kintsch, & Young, 1992). One can also consider a presentation format that is intermediate to the story problem and the symbolic equation format. This word equation format is verbal in presentation form and provides a description of the relationship among pure quantities (both known and unknown) with no story context. Examples of these three different types of problems are presented in Table 1.

### Normative view of mathematics

As Table 2 illustrates, the general structure of early algebra problems may be characterized in terms of problem presentation type (the columns contrasting story, word equation, and symbolic equations) and the position of the unknown (the rows contrasting result- and start-unknown). Inspection of textbooks and educational practices suggests a normative view in which the primary distinction is between arithmetic (i.e., result-unknown problems) and algebra (i.e., start-unknown problems; see the two rows of Table 2). A secondary distinction is made between symbolic and verbal problems. Examples of each problem type are presented in Figure 1, where the index numbers from Table 2 (#1-#6) point to the different problem types.

All other things being equal, the normative view holds that algebra (problems #1, #2, and #3, Figure 1) problems are inherently more difficult than arithmetic problems (problems #4, #5, and #6). It also views the solutions of verbally presented story problems and word equations problems (#1, #2, #4, #5) as mathematically more demanding than solutions of problems presented as symbolic equations (#3 and #6)[1].

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[1] Some conceptions of algebra (e.g., Usiskin, 1988) would consider both problems #3 and #6 of Figure 1 to be algebra problems, because of the presence of a literal symbol. We operate under a different conception where structural differences (as depicted in Table 2) distinguish algebra from arithmetic.

The normative view also makes certain predictions for the relative difficulty of the six problem types of Table 2. Easiest is problem type #6 (symbolic arithmetic) followed by problem types #4 (arithmetic story problems) and #5 (arithmetic word problems). At the next level of difficulty come problems of type #3 (symbolic algebra), with problem types #1 (algebra story) and #2 (algebra word) seen as most difficult.

The normative view prescribes the following developmental sequence for early algebra learning: First, teach problem-solving skills on arithmetic problems and then move to algebra level problems. Within each of these levels, first introduce the skills associated with processing formal mathematical equations before moving on to its application for reasoning about verbally presented problem situations. The pinnacle is achieved when a student is ready to tackle algebra story problems.

## Hypotheses

Experiment 1 investigated the extent to which the normative view is held by teachers and mathematics education researchers when they predicted areas of student difficulty in problem solving. Teachers' predictions were compared to students' performances on the six different problem types. It was hypothesized that the normative view played a central role in shaping teachers' predictions, dictating teachers' decisions even when the normative view deviated from students' actual performances.

# Experiment 1a: Teachers' Beliefs About Mathematical Difficulty

## Method

### Subjects

Participants (n=68) were recruited from a professional development workshop during the summer of 1994. They represent a self-selected sample of highly dedicated public school mathematics teachers of a major metropolitan region in the southeastern United States. Participants included seventh through twelfth grade math teachers from a wide range of settings and socio-economic communities, including predominantly minority-based inner city schools, rural communities, and suburban areas.

Procedure

A single, one-sided form was distributed to workshop participants with the instruction that the teachers do not discuss their views with others until the forms were collected. The subjects had 10 min. to perform this task, after which time the forms were collected. Teachers were asked to rank order problems of the six different types shown in Figure 1, from easiest to most difficult, based on the criterion, "How hard do I think these problems are for my students?" The problems can be mapped on to the factors presented in Table 2 using the number entries in the table (e.g., #1) as an index to the problems in the handout sheet of Figure 1. Teachers also provided information about the student grade level that the teacher considered in developing the ranking, and whether they assumed the student used a calculator[2].

## Results and Conclusions

The average rank ordering produced by the mathematics teachers in our sample is presented in the first column of Table 3. Teachers' rank ordering of the problem types was analyzed using a 2-way, repeated-measures ANOVA with position unknown (result-unknown v. start-unknown) and presentation format (story v. word-equation v. symbol) as within-subjects factors, and rank as the dependent measure.

Position unknown

Teachers tended to rank order result-unknown (arithmetic) problems as easiest for students, regardless of format. This resulted in a significant main effect for the position unknown factor, $F(1,134)=5.9$, $p<.02$. The histograms of Figure 2 provide a visual sense of the comparative distributions of rankings that teachers provided. Intermediate values were used to account for ties. Eighty-four percent of all participants (see Table 4) ranked result-unknown problems (#4, #5, and #6) as easier than start-unknown problems, with arithmetic equations (#6) favored over arithmetic word-equations (#5), and word equations favored slightly over arithmetic story problems (#4). Approximately 10% of the participants made no distinction between result- and start-unknown problems, while 6% viewed the start-unknown (algebra) problems as easier than result-unknown problems. Looking at specific problem types, 78% of teachers rated story result-unknown problems as consistently easier than story start-unknown problems

[2] The authors recommend that the reader take a few minutes to perform this rank ordering task as instructed in Figure 1 before reading further.

(Table 5); 37% of teachers rated word-equation result-unknown problems as consistently easier than word-equation start unknown problems (Table 6); and 42% of teachers rated equation result-unknown problems as strictly easier than equation start unknown problems (Table 7). The histograms depicting the relative ranking of these problems is shown in Figures 2a through 2f.

Presentation format

Symbol-equation problems were ranked by participants as significantly easier than verbally presented problems (story and word-equation), $F(2,133)=$ , $p<.05$. The data show 42% of teachers ranked equations (ignoring position unknown) as strictly more difficult than word-equations on average, and 49% ranked equations as strictly more difficult than story problems on average. Fewer than 30% ranked verbal (story and word) algebra problems as being as easy to solve as symbolically presented arithmetic problems.

Start-unknown problems ranked as most difficult by teachers based on a post hoc comparison, $p=.05$. Within this category, verbally presented problems are considered particularly difficulty. Over 76% of the teachers ranked story and word start-unknown problems as the more difficult than all other problem types. A post hoc comparison among all six problems revealed that teachers (70%) ranked start-unknown (i.e., algebra) word problems as the most difficult problem type given, $p=.05$. The ANOVA, however, revealed no significant interaction between how teachers rank presentation format (story v. word v. equation) and position unknown (start v. result).

Teachers showed a strong tendency to rank algebra problems as more difficult than matched arithmetic problems, regardless of the presentation format. In comparing formats, teachers tended to rank verbally presented problems (i.e., story and word) as more difficult for students than symbolic equation problems in algebra and arithmetic. Furthermore, the rank orderings of teachers placed algebra word problems as the singularly the most difficult problems of our set, $p=.05$.

Kendall's rank correlation revealed a highly significant relationship between teachers' rankings and the ranking predicted by the normative view, $\tau(6)=.867$, $p=.015$. Teachers' rank orderings essentially followed the predictions made by the normative view of mathematical difficulty (Table 3). Teachers ranked start-unknown (algebra) problems as significantly more difficult than result-unknown (arithmetic) problems. They also ranked verbally presented problems within each category as more difficult than the corresponding symbol-equation problem. On the basis of these rank order data, it seems apparent that the decisions regarding problem difficulty made by teachers in this sample are based on a view of mathematics problem solving similar to the normative view discussed earlier. In support of our experimental hypothesis, teachers appear to use the normative view as the basis for their predictions. We discuss these findings and their relation to student performance data after an examination of data on the same task made by math education researchers.

## Experiment 1b: Math Ed researchers' views of problem difficulty (AWG)

To gain further insight into what is known about the difficulty factors that early algebra students face, rankings from a number of educational researchers whose focus is on algebra learning and instruction were solicited.

### Method

Subjects

Participants were all members of an algebra discussion group that communicated electronically, using electronic mail over the Internet. Volunteers were solicited to take part in a survey that was posted to the group in the Fall of 1994. Thirty-five people volunteered to take part in this survey.

Design, and Procedure

Participants (n=35) responded to a request posted to an algebra research and instruction group to rank order the six problems of Figure 1. The instructions were the same as those used for in-service teachers in Experiment 1a, above. Participants were told that the collective results of their responses would be posted to the entire group, and anonymity of individuals would be maintained. Responses were made over electronic mail and tabulated by the experimenters.

### Results and Conclusions

The average ranking produced by the 35 math researchers is shown in column 2 of Table 3.

Position unknown

The average rank order is consistent with the general view that start-unknown problems are more difficult than result-unknown problems across all presentation formats. About 66% of the respondents consistently ranked start-unknown problems as harder than result-unknown problems across the 3 presentation forms (i.e., problem #1 of Figure 1 was ranked harder than #4, # 2 was ranked harder than #5, and # 3 was ranked harder than #6). None of the respondents exhibited the opposite in their rank ordering. Thirty-four percent of the respondents ranked result-unknown (i.e., arithmetic) problems as more difficult in some but not all cases (the most frequent case was ranking problem #6, result-unknown equations, as more difficult than #3, start-unknown equation). This is the same general pattern found in the teacher data presented earlier.

Presentation format

The rank ordering data also indicates a belief that story problems are harder than equations for start-unknown (i.e., algebra) problems, and that all forms of algebra problems are harder than all arithmetic problems. About 31% of the respondents consistently ranked equations as easiest. Only 23% of the respondents consistently ranked equations as harder than word and story problems within each of the two levels of the position unknown factor (i.e., problem #3 of Figure 1 was ranked harder than problems 1 and 2, and problem #6 was ranked harder than problems 4 and 5).

Interaction

Over 46% of respondents showed an interaction of the two factors by ranking start-unknown (algebra) equations easier than start-unknown verbal problems, and result-unknown equations harder than result-unknown verbal problems (see Table 3). Only 17% of participants ranked arithmetic equations as being the same or harder than the two algebra verbal problems (i.e., problems 1 and 2), while 69% ranked arithmetic equations as being easier than both verbal start-unknown forms. The remaining 14% of participants were split.

Summary

The perceptions of the relative difficulty of start-unknown problems is generally agreed upon by both researchers and teachers in our samples. Both communities also generally view verbal presentation as contributing to problem difficulty, with an effect of verbal presentation that further penalizes algebra problems, but adds somewhat to the solvability of arithmetic problems. These views generally match the perceptions that follow from the normative view described earlier. One test of this revealed a high correlation between the average ranking of teachers and that expected from the normative view. The accuracy of these predictions is assessed in the next section when we examine how these views compare to students' actual problem-solving performances on these items.

# Discussion

## Student Performance and Teacher Expectations: A Model Comparison

Students in two studies (Tabachneck et al., 1995; Koedinger, Nathan, & Tabachneck, 1996 ) were asked to solve problems generated from the six problem types shown in Table 2 and Figure 1. Koedinger and his colleagues termed their investigation a difficulty factors analysis (DFA) because it sought to identify the factors that affected problem-solving difficulty and the nature of their interactions. In the initial study (DFA 1), 76 ninth grade students completed a quiz which contained eight problems designed with the same underlying structure as those shown in Table 2 and Figure 1.

Student performance results and conclusions

The student performance data (percent correct) as a function of problem type is presented in Table 8. The rank ordering of problem difficulty based on students' levels of performance, are summarized in Table 3 (column 3). Student performance data showed highly significant effects of position unknown and presentation format (see Figure 3a). As the in-service teachers predicted, students scored much lower on start-unknown (algebra) problems than on result-unknown problems , $F(1,75) = 48.9$, $p<.0001$. Contrary to teachers' expectations, however, students' difficulties due to presentation format were due to their poor performance in solving symbol equation problems relative to verbally presented problems, $F(2,75) = 12.6$, $p<.0001$.Also in contrast to the general view, students' performances do not show algebra story and word problems to be among the most difficult. That status was reserved for algebra symbol problems. Algebra equation problems were significantly less likely to be correctly solved than either story problems or word-equation problems ($p<.01$ in a post hoc test). Algebra story problems and algebra word-equation problems were actually found to be equal in difficulty to arithmetic symbol problems, a result predicted by only 4.5% of the teachers in the sample. We refer the interested reader to Koedinger, Nathan, and Tabachneck (1996) for the details of this study and further results.

This experiment was replicated the following year with students from the same environment. The second difficulty factors analysis (DFA 2) largely paralleled the original findings (see Figure 3b). The results showed a greater tendency for students to accurately solve result-unknown problems than to solve start-unknown problems, ($F(1,170)=138$, $p<.0001$). The impact of presentation format was largely replicated in this study as well, ($F(2,170)=38.4$, $p<.0001$). Symbolic equations were found to be significantly more difficult than either story problems or word equation problems, $p<.01$. Unlike the original study, however, the data in the second study found that result-unknown word-equation problems were significantly more difficult than result-unknown story problems, $p<.01$. The distance between the success rate on story problems and word equations was perceivably larger in this study (Figure 3b) than in the original study (Figure 3a), but this difference is smaller than the distance between the success rate of word equations and equations.

There are three generalizations that can be drawn from the student data: (a) Start-unknown problems are harder for these students than result-unknown ($p < .001$); (b) Symbolic equation problems are harder than both word equation problems and story problems ($p < .001$). The latter two verbal problem types are about equal in difficulty in the original study, while word equation problems were more difficult for story problems in the replication; and (c) The relative difference in difficulty attributed to symbolism (b, above) is as large as the difference due to dealing with start-unknown problems (a, above) These findings are summarized graphically in Figure 3.

The start-unknown, or algebra, effect is well-established in the elementary school word problem-solving literature with single operator problems[3] (Briars and Larkin, 1984; Carpenter et al., 1989; De Corte, Greer, & Verschaffel, in press). The symbolism disadvantage challenges the oft-cited view that story problems are harder than symbolic math (E.g., Cummins et al., 1988; Mayer, 1982). On the other hand, a number of studies, mostly at the lower grade levels, have shown positive effects of situational context (e.g., Baranes et al., 1989; CTGV, 1993). The relatively good performance on situation-less word equations in this study suggests that the general performance advantage associated with verbal problems involves more than an advantage for situational context. Students at this stage can solve the harder problems in a situation-less word-equation form far more readily (by some 17%) than they can in symbols. Of course, this finding is certainly developmentally rooted — as symbolic skill develops and as students attack harder problems more often, advantages of symbolism are likely to appear.

The third generalized result (c, above) suggests that under certain circumstances students can do as well on simple algebra problems as they do on arithmetic problems. This occurs when the algebra problems are presented verbally (with or without a situational context), and the arithmetic problems are presented symbolically.

Teachers generally predicted much of what makes problems difficult for students. A Kendall's Rank Correlation (Tau) yields a significant relationship between teachers' ratings and the average performance of students in the two studies, $\tau(6)=.61$, $p=.03$ (see Table 3). However, data on teachers' and researchers' expectations uncovered some systematic areas where students' performances differ. We explore what these differences may be due to in this section.

Students' solution strategies

In their analyses of students' solution approaches, Koedinger and his colleagues (1996; Tabachneck, Koedinger, & Nathan, 1994) identified "alternative" solution strategies from both written solutions and interviews with students. These strategies supported their problem-solving processes, and appeared to allow students to solve problems which would otherwise be beyond their reach. Five major

[3] We have not found previous studies that systematically compare start-unknown vs. result-unknown problems with more than 1 operator at the middle school or junior high level.

groups of solution strategies were used by high school students to solve the 6 classes of problems in Table 2: Arithmetic, algebra, diagrams, guess-and-test, and unwinding (for examples of these strategies, see Figure 4(a)-(e), respectively). In addition, students' may have provided no response (a blank solution), or provided insufficient information to allow coding a strategy (e.g., they did it in their heads, or there was no discernible method in use). Arithmetic is the manipulation of sums, products and powers of numbers, while algebra strategies refer to these manipulations when carried out on letters instead of numbers to denote the generality of arithmetic procedures (Usiskin, 1988; see Figure 4a and 4b, respectively). Diagrammatic methods were scored when students used pictures or other graphical depictions of the (relative) quantities specified in the problem (Figure 4c). This included situational depictions, drawn markers used for direct modeling, and depictions of part-whole relations. Guess-and-test (Figure 4d) refers to the class of model-based methods used for iterative analysis (e.g., Hall et al., 1989). Unwinding methods (Figure 4e) allow the solver to work backwards and "unwind" or undo the quantitative constraints imposed by the problem, in order to free up the unknown (cf. Polya, 1959). It often parallels the steps referred to in the scenario of a story problem, or the order of mathematical constraints provided. Unlike the standard algebra approach, the unwinding strategy makes no use of equations or symbolic place-holders for unknown quantities. Unwinding may be done verbally by the solver (Koedinger & Tabachneck, 1995), or through the solver's written work.

A detailed analysis of the strategies from original student data (DFA 1) was conducted (Koedinger et al., 1996). Arithmetic strategies, not surprisingly, were used overwhelmingly in solving result-unknown problems. When these were removed and only solution approaches to start-unknown problems were considered, an interesting picture emerged. As Table 9 shows, students tended to shift their strategy approaches to suit the problem presentation, even though the underlying mathematics and the location of the unknown quantity were controlled (cf. Carpenter et al., 1980). Story problems tended to elicit the unwinding strategy more than half of the time. Story problems seldom elicited the symbol manipulation methods associated with algebra (only 6% of the time). Situation-less word equation problems tended to elicit either a guess-and-test approach (31% of the time) or unwinding (26%). Symbolic equations resulted in no response 30% of the time, more than twice as often as the other problems. When students did respond, they tended to stay within the mathematical formalism and apply symbol manipulation methods (26%), or opted for the iterative guess-and-test method (20%).

Unwinding and guess-and-test methods showed a higher likelihood of success than use of the symbol manipulation approach (Table 13). This is because these methods rely on some built-in validity checks on their answers (Tabachneck, Koedinger, and Nathan, 1994). The guess-and-test algorithm includes in it a check on the consistency of the solution before deciding to terminate its execution or continue with a new guess. Unwinding operates in two modes. For story problems, the solver retains the situated nature of the values gleaned from the problem scenario (e.g., one maintains the units as one speaks about or writes the values), and thus minimizes the likelihood for producing absurd values that may be out of range. When used with problems with no context, the unwinding process essentially transforms the algebra problem into an arithmetic problem, that is then more readily solved (see example in Figure 4e). The findings reported above that equate student performance on arithmetic equation problems to that of algebra story and word problems provide empirical evidence for this.

We expect that teachers and researchers do not place as much stock on the power of these alternative solution strategies. Instead, it is our contention, based on findings from Experiment 1, that students' problem-solving abilities are largely thought of within the normative view of algebra development. Thus, teachers and researchers, examining a problem for its level of relative difficulty, make their decisions based on the question, "how far along the trajectory from symbolic arithmetic to algebra story problem solving is a student?" These data suggest that there are alternative trajectories for the development of one's algebra level reasoning ability. Two trajectories are considered as candidates for describing the student data in the next section.

Comparison of two developmental models

Koedinger, Nathan, & Tabachneck (1996) proposed two competing models of students' early algebra development (Figure 5). Figure 5a is the model that emerges from the normative view discussed earlier. As one moves from the top of this model to the bottom (with branching allowed), the level of development increases, and more types of problems are solvable because of the greater skill set. Initially it is believed that students cannot solve any of these problem. This is the "None" stage. All students are at least at this minimal stage in early algebra development. At their final stage of development, all problem types in our corpus can be solved, but by few students. The differences between the two models are how to specify the developmental trajectory from the initial to the final level of competency.

According to the normative view (Figure 5a), one first progresses from competence at none of the problems, to competency at solving symbolic arithmetic (result-unknown) problems (stage 2). This instantiates the notion that symbolic problem-solving skills are believed to develop prior to (and in fact serve as the foundation of) verbal problem-solving skills under the normative view. A student then moves to one of two competencies at stage 3: she may extend her symbolic problem-solving competency, adding algebra equation problem solving to her repertoire (see the left fork of stage 3 in Figure 5a), or she may extend her arithmetic reasoning ability to include verbally presented problems (right fork). These abilities come together in stage 4. Students at this developmental stage can solve all arithmetic problems (symbolic and verbal) as well as all equation problems (arithmetic and algebra), but fail to solve verbally presented algebra problems. In the final stage of the normative model of development, students demonstrate competence in solving verbal (story and word equation) algebra problems.

The model of Figure 5b presents an alternative developmental model that emerges from our empirical work. This model emphasizes the early development of students' verbal reasoning ability and its role in mathematics problem solving. It hypothesizes a progression from competence first in solving verbal arithmetic problems, through intermediate zones that add competence for either verbal algebra problems (left fork) or arithmetic equations (right fork). These abilities combine as a competency for all forms of verbal problems (arithmetic and algebra) and all arithmetic problems (verbal and symbolic) but poor performance on algebra equation problem solving. In the final stage, symbolic algebra problem solving is demonstrated.

Model comparison

Using the patterns of performance from each student in the original study (DFA 1), Koedinger, Nathan, and Tabachneck (1996) classified each student into these zones of increasing competence. A student's performance does not fit the model when it demonstrates one competency but lacks another that is presumed to be part of the of the predicted developmental sequence. Students classified as competent in either none or all of the problem categories fit both models trivially (i.e., they received either a zero score or a perfect score). If a student can do symbolic arithmetic but can't do verbal arithmetic, then that student fits the normative model but not the alternative model. This is because the normative model predicts that symbolic arithmetic problem solving (stage 2) precedes arithmetic verbal problem solving. On the other hand, if a student can do verbal arithmetic but not symbolic arithmetic, that student fits the alternative model but not the normative one. This is because the alternative model predicts that verbal behavior verbal arithmetic problem-solving strategies precede arithmetic equation problem solving. When a subject no longer follows the trajectory of zones or stages hypothesized by the model, that subject is rejected by the model. This is indicated in Figure 5 by an arrow that leads to no successive stage of development. For example, 17 subjects are rejected at stage 1 as the normative model tries to move from the "None" zone (where no problems are solved)—to the "symbolic-arithmetic" zone. Of the original 76 students in 1 data set, 53 remain in the model to be fit to the remaining developmental zones. Rejections of students indicate the limitations of the model to account for those students because their behavior diverges from the predicted developmental trajectory. The final percentage of students who fit any particular trajectory provides a measure of the power of the model to predict students' developmental process.

Considering only non-trivial fit (i.e., excluding those students who can solve none of the problems or all of the problems), only 31% of all subjects in the original study (DFA 1, n=76) fit the normative model while 83% of these subjects fit the alternative model. In fact, more than half of the subjects that fit the normative model do so at the intermediate "all arithmetic" zone (stage 3) which is the only zone that overlaps with the alternative model. This is the zone where subjects demonstrate competency in arithmetic problems (result-unknown problems) in both representations (symbolic and verbal), but are not able to solve algebra (start-unknown) level problems in either presentation format.

The findings from the teachers' estimations of problem difficulty reported in Experiment 1 show that teachers accurately predict certain factors— in particular, differences between result unknown and start unknown—that add to students' arithmetic and algebraic problem-solving difficulty. However, there appears to be important systematic differences in other areas between what math students find difficult in practice and what teachers predict will lead to their difficulty. Teachers' (and researchers') decisions appear to be strongly mediated by factors consistent with the normative view of early algebra development. The student data suggest that this view is far from optimal in predicting actual performance. As seen from the analyses of students' problem-solving strategies above, the normative view neglects the early development of students' verbal problem-solving ability. This may have important consequences for teachers' practices. In particular, teachers may erroneously withhold struggling students from verbal problems until they develop a certain level of symbolic skill performance. Teachers' predictions also underestimates the difficulty that most students have with symbolic problems. This may lead them to improperly design assessments of students level of math development, and may lead teachers to erroneously evaluate students' math abilities. The normative view, with its structualist emphasis (pitting start- vs. result-unknown) fails to take into account students' problem-solving processes when evaluating relative problem difficulty. This point is taken up more in the General Discussion.

The strong role of the normative view in mediating teachers' reasoning is only speculative, however, with no direct indication that this was the basis of most teachers' views of problem difficulty. In order to more fully understand teachers' responses to the difficulty ranking task, a second study was conducted. Data on teachers' views about several issues of mathematical performance, learning, and instruction, including the normative view of algebra development, was examined in Experiment 2.

## Experiment 2: Teachers' difficulty rankings and their underlying beliefs

In this experiment, a new sample of teachers were tested. We set out to (a) replicate the original difficulty ranking findings across a wider range of grade levels and in a new region of the United States, and (b) test the hypothesis that teacher's beliefs regarding mathematics problem difficulty were mediated by the normative view of mathematics. Several other belief constructs concerning views of the nature of mathematics, math learning and instruction, and the status of algebra as a problem-solving method were also considered as possible factors influencing teachers' decisions.

## Method

### Subjects

Participants of this experiment were K-12 teachers who attended a district-wide mathematics in-service during the fall semester of 1995. Of the one hundred and seven participants, two produced forms insufficiently completed (less than half of the difficulty rank or belief statements had responses), leaving a final count of 105 subjects. All were either mathematics or elementary teachers in a western United States public school district. The students in this district are predominantly Caucasian and live in predominantly suburban areas.

### Design, materials and procedure

Teachers received the same difficulty ranking activity used in Experiment 1 (see Figure 1). In addition, the teachers in this sample were given a set of 47 statements. They were to respond on a 6-point Lickert scale the degree to which they disagreed or agreed with each statement. Participants were told that the intent of the questionnaire was to learn about the views that teachers held regarding mathematics, learning and instruction, and that all information they provided would be kept anonymous and only the collective results of the group would be shared. The belief instrument was administered immediately after completing the difficulty ranking task. Teachers were given 10 min. to complete this second activity before it was collected.

The forty-seven items comprised six constructs (item groups). Examples of each construct are presented in the Results section below. Participants received the items in a randomized order, and were asked to "Circle the number that corresponds most accurately with your beliefs about the accompanying statement." They were provided with a six-item scale (1 through 6) presented to the left go each statement, with associated anchors: "(1) strongly agree (2) agree (3) agree more than disagree (4) disagree more than agree (5) disagree (6) strongly disagree." Each construct presented items that were worded both positively (affirming the construct) or negatively (negating the construct). Many of the items were taken from previously published work, including Witherspoon (19XX), Cobb (1991). and Peterson et al., 1989) were chosen because they address basic issues of pedagogical practice, math learning, problem solving, and the role of algebra in the domain of mathematics.

## Results

### Belief instrument

Descriptive statistics and reliability analyses of the belief constructs were compiled. Five of the original 47 items were dropped as a result of the reliability analysis. Results are presented on the remaining 42 items.

The 6 constructs along with their reliabilities (Cronbach's alpha) and mean response values (max = 6.0) are presented below. Note that higher mean scores (M) indicate greater disagreement, while lower means indicate greater agreement.

- ***Algebra Is Best*** items (11 questions, $\alpha$=.76, M=4.3) present the view that algebraic procedures are the singularly most effective method for mathematical problem solving.

  Example items are:

  a. Using algebra for story problem solving is the most effective approach there is. (positive)

  b. There are many effective approaches to solving any algebra story problem, and manipulating symbols is only one method. (negative)

- ***Learning*** items (7 questions, $\alpha$=.70, M=2.5) present the view that students can learn and invent effective methods for problem solving that may differ from those taught.

  Example items are:

  a. Students enter the algebra classroom with intuitive methods for solving algebra story problems. (positive)

  b. Most students cannot figure out for themselves how to solve algebra story problems. (negative)

- ***Normative*** items (6 questions, α=.65, M=3.3) hold the view commonly expressed in math texts that arithmetic problems are easier and need to be presented before algebra. Also, within a math topic, math problems presented in words are most difficult and need to appear later in the curriculum.

  Example items are:

  a. Arithmetic story problems are easier for students to solve than algebra story problems. (positive)

  b. Solving math problems presented in words should be taught only after students master solving the same problems presented as equations. (positive)

- ***Pedagogy*** items (8 questions, α=.84, M=2.7) state that students may possess valid ways of reasoning as they enter the classroom, and may figure out for themselves effective problem-solving approaches.

  Example items are:

  a. Students should be encouraged to invent their own methods to solve mathematics problems. (positive)

  b. Rewarding right answers and correcting wrong answers is an important part of teaching. (negative)

- ***Product Over Process*** items (4 questions, α=.78, M=5.1) emphasize correct answers over a student's reasoning process.

  Example items are:

  a. Getting the correct answer is a better indicator of learning than is the ability to articulate a good solution approach. (positive)

b. Mathematical understanding is more clearly shown in a student's reasoning than in the final answer a student produces. (negative)

- ***Solution Alternatives*** items (6 questions, $\underline{\alpha}$=.68, $\underline{M}$=4.6) state that alternative methods such as arithmetic, guess-and-test, and other non-symbolic methods demonstrate gaps in the student's knowledge.

Example items are:

a. When a student uses an arithmetic approach to solve an algebra word problem, that indicates a weakness in that student's math abilities. (positive)

b. Use of a "guess and check" strategy to solve an algebra story problem shows an adaptive approach to problem solving. (negative)

The analyses of teachers' ratings indicate that reliability measures for the given items was generally quite high (range: $\underline{\alpha}$ =.65 to .84), indicating good agreement on items that were clustered together from a theoretical perspective. Also, the means of the items tended to cluster around the middle range of the scale, indicating that the 6-point scale given was generally sufficient for the teachers to express their level of agreement adequately. The one exception is the *Product Over Process* construct ($\underline{M}$=5.1 out of 6.0 total points) which was clearly skewed toward disagreement. This suggests that on average, teachers in this district tended to reject this view, and may have disagreed to a grater degree if they had been given a wider scale.

An analysis of covariance (ANCOVA) on the mean rating for each of the 6 constructs was performed using teacher instructional level (elementary, middle, high) as a factor, and years of teaching experience as a covariate. As summarized in Table 10, high school teachers are significantly more likely to agree (i.e., their mean responses are closer to 1.0) with the view that algebra is the uniquely best method for solving problems *(Algebra Is Best)* than are middle and elementary teachers, $F(2,104)=14.3$, $MS=3.99$, $p<.0000$. High school teachers are least likely to agree with the view that students can learn effective problem solving on their own *(Learning)*, $F(2,14)=9$, $MS=3.27$, $p<.0002$. Although teachers in general strongly disagree with this view, high school teachers are most likely to agree with the view

*(Product Over Process)* that the students' answer (product) is more important than the problem-solving process, $F(2,104)=9.9$, $MS=9.95$, $p<.0001$. High school teachers are most likely to agree with the *Normative* view that arithmetic is always easier than algebra, and symbol manipulation skills are a pre-requisite to verbal problem solving, $F(2,103)=5.5$, $MS=3.9$, $p<.005$ (with one missing value). High school teachers are least likely to agree that students should be encouraged to invent their own problem-solving methods and build on their prior knowledge rather than be explicitly taught methods from a teacher or textbook *(Pedagogy)*, $F(2,104)=14.4$, $MS=6.5$, $p<.0000$. High school teachers are most likely to agree with the view that alternative solution methods are indicators of weak skills or knowledge gaps *(Solution Alternatives)*, $F(2,104)=21.74$, $MS=6.5$, $p<.0000$.

Middle school teachers appear to largely parallel elementary school teachers. The major exception is that middle school teachers are significantly more likely to disagree with the *Pedagogy* view, that students should be encouraged to invent their own problem-solving methods rather than receive these from texts and teacher instruction. Middle school teachers' responses do generally agree with this view, however. Since many early concepts of algebra (such as generalized expressions, slope, etc.) are presented at that stage of education, it is worthwhile to summarize middle school teachers' responses independently of the analyses presented above. Middle school teachers in this district generally perceive students' prior mathematical knowledge as potentially very effective. They agree with the *Learning* and *Pedagogy* views, and disagree with the view expressed by the *Solution Alternatives* construct. Collectively, they are fairly neutral to the *Normative* view that verbal skills must be based on symbolic ones. They disagree that algebra is inherently best *(Algebra Is Best)* as a solution approach, and that answers are more important than the processes that lead to them (as stated by *Product Over Process)*. It indicates a basic confidence in many of the reform-based principles for mathematics learning (such as the NCTM standards, NCTM, 1989).

## Problem ranking by difficulty factors

### Start-unknown vs. result-unknown problems.

Teachers in our sample (n=105) tended to rank Start-Unknown (algebra) problems as consistently more difficult than Result-Unknown (arithmetic) problems (32% of teachers ranked it so), while they seldom ranked Start-Unknown problems as consistently easier (2.8%) (see Table 11). This matches student performances and the other teacher rankings, as presented in Experiment 1. This general pattern held across grade levels, and represents accurate and widely available pedagogical content knowledge.

Presentation format.

Students showed significant performance deficits with symbolic problems as compared to mathematically equivalent problems presented verbally, in contrast to the estimations of teachers in the original pool (Experiment 1). In the current experiment, the teachers on average were split. High school teachers were more likely to rank verbally presented story and word-equation problems as more difficult than symbol-equation problems (36% vs. 5%, Table 11). This is in accord with teachers in our earlier sample, but in contrast to our students. Elementary teachers (who were not represented in the sample of teachers in Experiment 1) were far more likely to rank symbol problems as more difficult than verbal problems (46%), and never ranked verbal problems as more difficult than symbol problems. Middle school teachers were more moderate, though they leaned toward elementary school teachers' views (see Table 11).

Relating beliefs to problem difficulty (DFA) ranking: A regression analysis.

All 6 belief constructs, along with school level (elementary, middle, high) were considered as factors in a regression analysis to evaluate which of these factors was the most effective predictor of teachers' difficulty (DFA) ranking. Years of teaching experience, a continuous variable, was used as a covariate, but proved to not be a reliable predictor in any of the analyses.

As the above analysis shows, school level (elementary, middle, high) proved to be an important in predicting teachers' difficulty rank orderings. In addition to this, the *Normative* construct was a significant factor as well. In predicating teachers' rank ordering of symbolic (equation) problems (#3 and #6, Figure 1), the factors School and Normative produced the most reliable model, $F(2,96)=37.7$, $MSe=45$, $p<.0001$. Table 12 shows the relative contribution of each factor in predicting the dependent variable. For predicting verbal problems, School and Normative again provided the best model, $F(2,96)=23.3$. $MSe=10.8$, $p<.0001$. Although all of the constructs were reliable, only the Normative was a reliable predictor for ranking, accounting for error above and beyond that explained by School. The number of years of teaching experience reported by teachers proved to be statistically unreliable as a factor in predicting teachers' difficulty rank ordering.

## Conclusions

Findings from the original teachers' rank orderings from Experiment 1 were largely replicated among the second set of mathematics instructors. School level proved the best predictor of teachers' difficulty ranking in Experiment 2. Teachers at different grade levels have significantly different expectations for their students, and differ greatly on their views of the contributors of mathematical problem-solving difficulty. Elementary teachers believe that arithmetic is easiest for their students, and that couching problems in a verbal format aids students in their problem solving. This should not be too surprising given the mathematical experiences of most elementary school students. With little training in the uses of formal procedures and notation, the accessibility of verbal problems, particularly word problems, with no confusing cover story, it is understandable that they would be rated the easiest to solve. High school teachers tend to carry the view that algebra is significantly more difficult than arithmetic under all circumstances, and linguistic presentations contribute to problem difficulty. High school students' struggles to apply symbol based solution procedures to algebra story problems support this view as well. Middle school teachers receive students midway, and their responses to these issues reflect that. In this light, it is not surprising to see that school level proved reliable in determining teachers' predictions.

The role of the normative view as a predictor may be less apparent. Teachers' views of the sequencing of various factors such as presentation format translated into decisions about the relative difficulty of problems. The teacher rankings from Experiment 1 were based on views of math teachers in a different region of the country. When the results of these two experiment are combined, there is considerable support for the hypothesis that this view is a major influence in teacher' assessment of problem difficulty.

# General Discussion

## Teacher Predictions and Student Performance

This series of studies reveals that students' problem-solving behaviors differ in some systematic ways from those predicted by a normative analysis, and from the expectations of teachers. Algebra-level problems, those problems that perform operations on an initial unknown quantity, are accurately predicted by teachers to be more difficult than arithmetic (result unknown) problems. However, two separate investigations of the beliefs held by teachers from two different areas of the country (the southeast and the west), along with a sample of math educational researchers, showed that the impact of verbal presentation formats on students' problem-solving processes is misunderstood. These formats tend to trigger problem-solving strategies that fall outside of the normative view. These strategies— unwinding and iterative guess-&-test—are general and fairly robust, providing, in many instances, ways for the solver to verify the accuracy of the solution produced. As is evident from the advantages gained from the contextless word-equation format, use of these alternative strategies is not limited to contextually rich problem situations, but appears to extend over a range of verbal presentation types. It is important that students be given an opportunity to explore and refine them.

While many teachers appear to be influenced by the normative view in their decisions concerning problem difficulty, the student performance data suggest that an alternative model of the development algebraic reasoning may be more accurate. This model places greater emphasis on students' ability to reason about verbally presented problems. These alternate forms of quantitative reasoning may developmentally precede symbol manipulation ability for many students. The use of these methods by college students who have been away from algebra for a while have also been observed (Tabachneck et al., 1994), suggesting they are relatively resistant to extinction — unlike algebraic symbol manipulation skills. By building on students' knowledge-based strategies, it may be possible to enhance the acquisition and retention of algebraic reasoning, provide a richer conceptual basis for symbol manipulation skills, and develop within the student, a stronger sense of his agency and a problem solver and mathematician.

## Implications for Research on Teacher Cognition

By relying on the normative view, teachers tend to make inaccurate predictions of how students will perform and leads to a poor developmental model. This is of great interest since these beliefs of problem difficulty affect teachers' instructional planning and the development of their assessments. The normative view is quite popular, and has held the attention of educational researchers, cognitive scientists, and educators for some time (e.g., Mayer, 1982, etc.). Beliefs do not easily change. If we want to ultimately bring teachers' views into closer alignment with the empirical findings, it is imperative that teachers are made aware that they hold these views, that the views are explicitly characterized for them, and that their strengths are acknowledged while the limitations of the views are exposed.

At this preliminary stage, our data suggest that, rather than building upon students' (taught) symbol manipulation skills, it is more fruitful to emphasize verbal reasoning abilities. These abilities may in fact serve as the necessary foundation for healthy development of one's symbolic skills.

It is both interesting and ironic that elementary teachers seem to be for more accurate at recognizing high school students' areas of difficulty than are high school math teachers. There are perhaps two reasons that suggest themselves. The first is that students may not develop their symbol manipulation skills as far as high school teachers hope and believe. Thus, these students still are weak in the skills that need the most attention in elementary school (i.e., symbol manipulation and calculation) despite years of intense practice. A more reasonable and optimistic view is that elementary teachers are, by their training and their interactions with students more generally acquainted with the many facets of their students' abilities. They teach reading, science, social studies/history, as well as math their students. They see students' verbal as well as their computational reasoning. High school teachers are generally more specialized — in two ways. First, as more expert mathematicians and secondary level math problem solvers they may be "further" from the difficulties of the novice. Second, they interact with their students in less diverse arenas and are less likely to see them use a variety of forms of reasoning.

Perhaps there is, lurking in our investigation, some support for holistic instruction, or at least a recommendation that educators need to be knowledgeable about more facets of the student than those covered in the math curriculum. There is the suggestion from our data that the level of specialization currently practiced in most secondary math programs may work against teachers. It allows these teachers to develop very skewed senses of how students reason, and where their strengths and weaknesses lie. These limitations to their pedagogical content knowledge may greatly impede teachers' ability to properly assess and instruct students. If this is so, then a greater emphasis on teacher exchanges and the sharing of research findings about the varieties of students' reasoning is essential to properly inform teachers. Math educators need to be made aware of the power and flexibility of students' alternative mathematical problem solving strategies. Findings from this study suggests, however, that any attempts to address the discrepancies between teachers' difficulty assessments and students' actual performances, will need to first address the Normative view held by many teachers, principally those at the high school levels of instruction.

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## Tables

Table 1. Presentation format and context as underlying factors of early algebra problems.

| | Context? No | Context? Yes |
|---|---|---|
| **Symbolic presentation** | Equation<br>$C + 4 = 7$ | N/A |
| **Verbal presentation** | Word equation<br>*Four added to some number is seven.* | Story problem<br>*After receiving four additional cookies, Aaron now has seven.* |

Table 2. Matrix of mathematics problems by presentation types.

Numbers refer to problems shown in Figure 1.

| | Verbal | | Symbolic |
|---|---|---|---|
| Value Unknown ↓ | Story | Word-Equation | Symbolic-Equation |
| Result unknown (arithmetic) | #4 | #5 | #6 |
| Start unknown (algebra ) | #1 | #2 | #3 |

Table 3. Summary of mean rank ordering of teachers' (column 1) and mathematics researchers from Experiment 1 on the six problem types of Figure 1, and students' performances on these problems (columns 3 and 4, reprinted from Koedinger et al., 1996 Tech report).

| Rank ↓ | Teachers' from Expt 1a (n=68) | Math Researchers' (n=35) Mean rank ordering | Student Performance (n=76) [and percent correct] (DFA 1) | Student Performance (n=171) [and percent correct] (DFA 2) |
|---|---|---|---|---|
| Easiest* | Arithmetic-Eqn (#6)<br>Arithmetic-Word (#5)<br>Arithmetic-Story (#4) | Arithmetic-Word (#5)<br>Arithmetic-Story (#4)<br>Arithmetic- Eqn (#6) | Arithmetic-story (#4) [73%]<br>Arithmetic-Word (#5) [67%] | Arithmetic-Story (#4) [80%]<br>Arithmetic-Word (#5) [74%] |
| Medium* | Algebra- Eqn (#3)<br>Algebra-Story (#1) | Algebra- Eqn (#3) | Algebra-Story (#1) [59%]<br>Algebra-Word (#2) [54%]<br>Arithmetic-Eqn (#6) [53%] | Algebra-Story (#1) [60%]<br>Arithmetic-Eqn (#6) [56%]<br>Algebra-Word (#2) [48%] |
| Hardest* | Algebra-Word (#2) | Algebra-Word (#2)<br>Algebra-Story (#1) | Algebra-Eqn (#3) [37%] | Algebra-Eqn (#3) [29%] |

* Difficulty divisions in Table 3 (Easy vs. Medium Vs. Hard) reflect significant differences in the percentages of the respondents who provided that rank ordering (columns 1 and 2) or in levels of performance (columns 3 and 4).

Table 4. Raw count and percentage of Experiment 1 teachers ranking Result-unknown relative to Start-Unknown problem types.

| Relative rank, all problems | Number of Teachers | Percentage of Teachers |
|---|---|---|
| Result Easier than Start | 56 | 84% |
| Start Easier than Result | 4 | 6% |
| Result = Start | 7 | 10% |
| **Total** | **67** | **100%** |

Table 5. Raw count and percentage of Experiment 1 teachers ranking Story Result-unknown relative to Start-Unknown problems.

| **Relative rank, Story** | **Number of Teachers** | **Percentage of Teachers** |
|---|---|---|
| Result Easier than Start | 52 | 78% |
| Start Easier than Result | 8 | 12% |
| Result = Start | 7 | 10% |
| **Total** | **67** | **100%** |

Table 6. Raw count and percentage of Experiment 1 teachers ranking Word Result-unknown relative to Start-Unknown problems.

| **Relative rank, Word** | **Number of Teachers** | **Percentage of Teachers** |
|---|---|---|
| Result Easier than Start | 25 | 37% |
| Start Easier than Result | 36 | 54% |
| Result = Start | 6 | 9% |
| **Total** | **67** | **100%** |

Table 7. Raw count and percentage of teachers in Experiment 1 ranking Equation Result-unknown relative to Start-Unknown problems.

| **Relative rank, Equation** | **Number of Teachers** | **Percentage of Teachers** |
|---|---|---|
| Result Easier than Start | 28 | 42% |
| Start Easier than Result | 32 | 48% |
| Result = Start | 7 | 10% |
| **Total** | **67** | **100%** |

Table 8. Student performance (percent correct) for the original study (n=76) and the replication (n=171) on the six problem types of Table 2. Data are taken from Tabachneck, Koedinger, & Nathan, 1995

| | | Verbal | | Symbolic | **Total** |
|---|---|---|---|---|---|
| | Value Unknown ↓ | Story | Word-Equation | Symbolic-Equation | |
| Study 1 (n=76) | Result unknown (arithmetic) | 73% | 67% | 53% | 64% |
| | Start unknown (algebra ) | 59% | 54% | 37% | 50% |
| | **Total** | **66%** | **61%** | **45%** | **57%** |
| Study 2 (n=171) | Result unknown (arithmetic) | 80% | 74% | 56% | **70%** |
| | Start unknown (algebra ) | 60% | 48% | 29% | **46%** |
| | **Total** | **70%** | **61%** | **43%** | **58%** |

Table 9. Solution strategies (%'s) employed by solvers (DFA 1, n=76) as a function of problem presentation for start unknown (algebra level) problems.

| Problem Presentation | Unwind | Guess & Test | Symbol Manipulation | No Response | Unknown | Total |
|---|---|---|---|---|---|---|
| Story Problems | **56** | 10 | 6 | 14 | 14 | 100 |
| Word-Equation | **26** | **31** | 8 | 13 | 22 | 100 |
| Symbol-Equation | 13 | 20 | **26** | **30** | 11 | 100 |

Table 10. Experiment 2 teachers' mean responses to various belief constructs by grade level. (1=Strongly agree, 6=strongly disagree).

| Construct | Elementary (n=36) | Middle School (n=30) | High School (n=39) | Total (n=105) |
|---|---|---|---|---|
| Algebra Is Best | 4.61 | 4.41 | 3.98* | 4.3 |
| Learning | 2.25 | 2.43 | 2.82* | 2.5 |
| Normative | 3.55 | 3.42 | 2.94* | 3.3 |
| Pedagogy | 2.31 | 2.73* | 3.13* | 2.7 |
| Product-Process | 5.38 | 5.13 | 4.67* | 5.1 |
| Solution Alternatives | 4.95 | 4.74 | 4.16* | 4.6 |

* $p < .005$

Table 11. Summary of Experiment 2 Teachers' Rank Ordering of Problem Types and Students' Performances

| Rank ↓ | Teachers (n=105) | Elementary Teachers (n=36) | Middle School T (n=30) | High School Teachers (n=39) | Students' Performance (Avg. of studies 1 & 2) |
|---|---|---|---|---|---|
| Easiest | Arith-Word-Eqn (#5) | Arith-Word-Eqn (#5)<br>Arithmetic-Eqn (#6) | Arith-Word-Eqn (#5) | Arith-Word-Eqn (#5)<br>Arithmetic-Eqn (#6) | Arithmetic-Story (#4)<br>Arith-Word-Eqn (#5) |
| Middle | Arithmetic-Story (#4)<br>Arithmetic-Eqn (#6) | Arith-Story (#4)<br>Algebra-Eqn (#3) | Arith-Story (#4)<br>Algebra-Story (#1) | Algebra-Eqn (#3) | Algebra-Story (#1)<br>Arithmetic-Eqn (#6)<br>Algebra-Word-Eqn (#2) |
| Hardest | Algebra-Story (#1)<br>Algebra-Eqn (#3)<br>Alg-Word-Eqn (#2) | Algebra-Story (#1)<br>Alg-Word-Eqn (#2) | Alg-Word-Eqn (#2)<br>Algebra-Eqn (#3)<br>Arith-Eqn (#6) | Arithmetic-Story (#4)<br>Algebra-Word-Eqn (#2)<br>Algebra-Story (#1) | Algebra-Eqn (#3) |

Table 12. Regression model analyses yields to reliable factors for predicting the teachers' difficulty rank orderings obtained in Experiment 2.

| | Factors | |
|---|---|---|
| Dependent Variable | School Adjusted-R2 | Normative Adjusted-R2 |
| Rank of Verbal-Problems | .26 | .13 |
| Rank of Symbol-Problems | .40 | .12 |

Table 13. Likelihood that strategies used by students on start-unknown problems in first study (n=76) leads to a correct answer.

| Strategies | No. of correct problems | Likelihood strategy leads to correct answer |
|---|---|---|
| Unwinding | 83 | 72% |
| Guess-&-Test | 37 | 68.5% |
| Symbol Manipulation | 19 | 54% |
| Unknown Strategies | 20 | 49% |
| Total | 159 | 61% |

# Figures

**Figure 1.** The difficulty ranking task given to teachers and math education researchers in Experiments 1a, 1b (presented in electronic form over the Internet), and 2.

**A SURVEY**

Below are 6 problems that are representative of a broader set of problems that are typically given to public school students at the end of an Algebra 1 course -- usually 9th grade students. My colleagues and I would like you to help us by answering this brief (5 min) survey. We are happy to share the results we obtain with your class this spring.

**What we would like you to do:**

Rank these problems starting with the ones you think were easiest for these students to the ones you think were harder. You can have ties if you like. For example, if you think the fourth problem (#4) was the easiest, the 3rd was the most difficult, and the rest were about the same, you would write:

4 (easiest)

2 1 5 6

3 (hardest)

*(Feel free to include an explanation of any assumptions you made in the space below.)*

**Problems:**

1) When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour does Ted make?

2) Starting with some number, if I multiply it by 6 and then add 66, I get 81.9. What number did I start with?

3) Solve for x: x * 6 + 66 = 81.90

4) When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?

5) Starting with 81.9, if I subtract 66 and then divide by 6, I get a number. What is it?

6) Solve for x: (81.90 - 66) / 6 = x

## Figure 2. Histograms of ranking by problem type

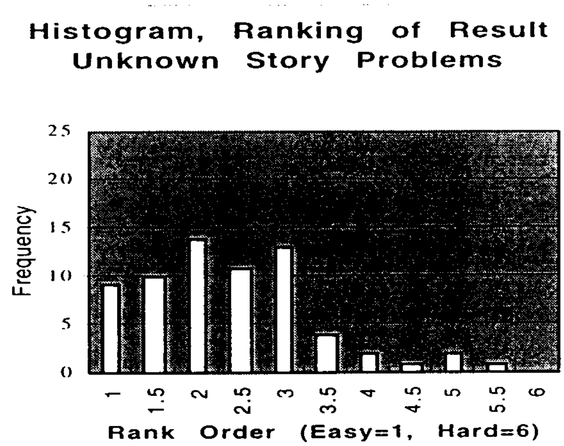



Figure 2a.

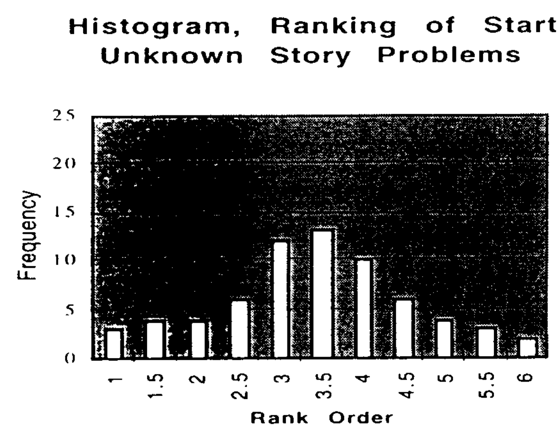



Figure 2b.

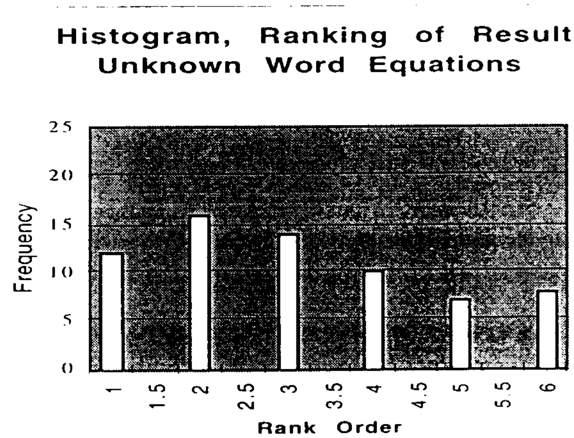



Figure 2c.

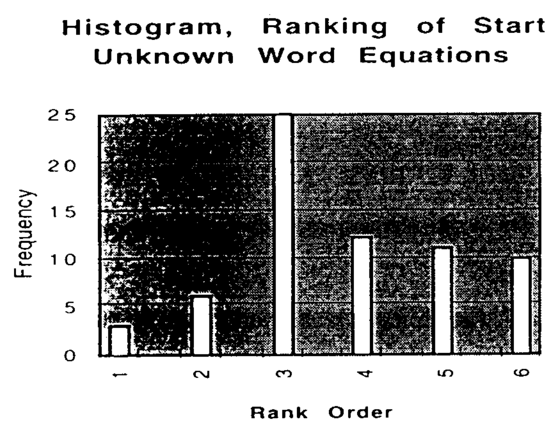



Figure 2d.

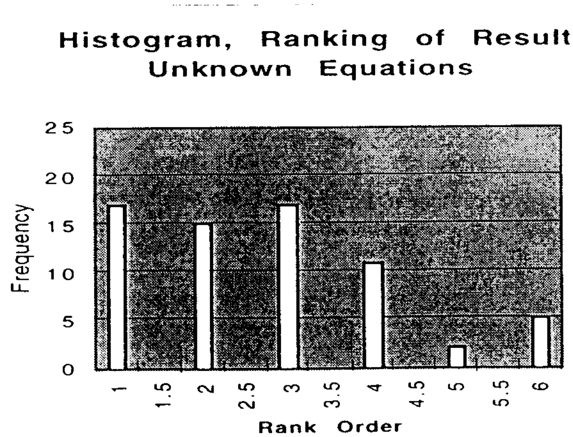



Figure 2e.

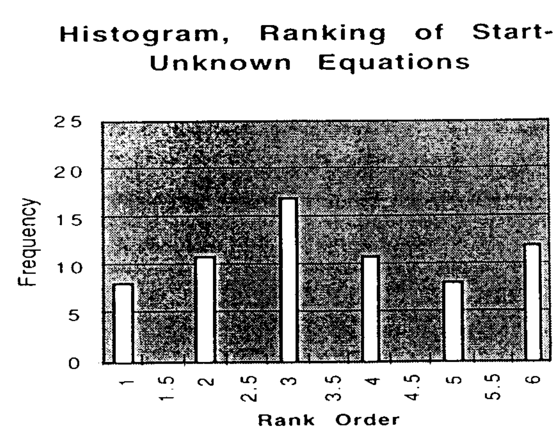
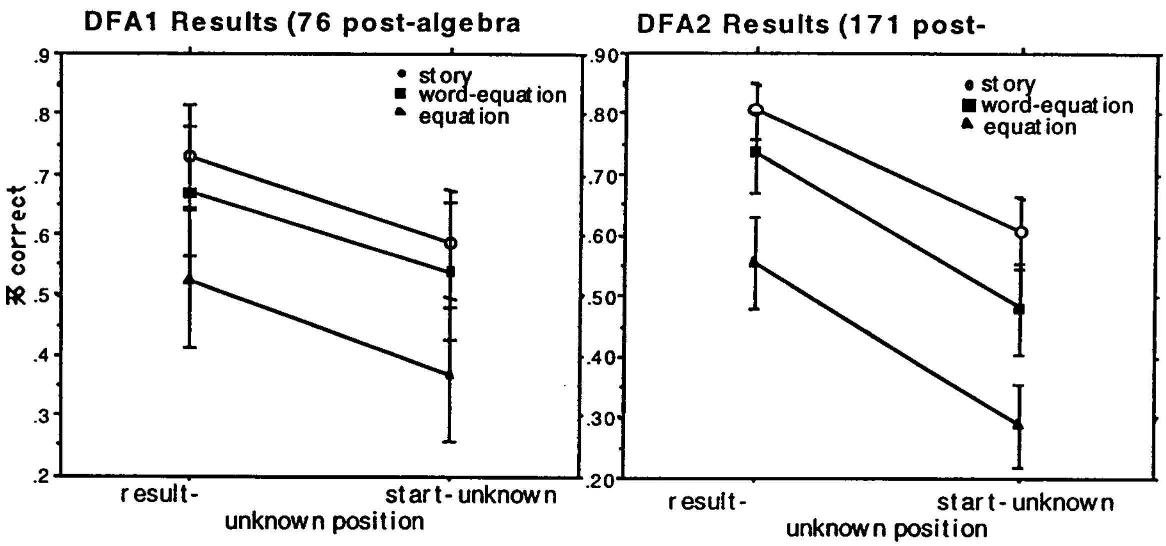



Figure 2f.

**Figure 3**




(a) (b)

**Figure 3**. The effect of the difficulty factors or presentation format and unknown position on student problem-solving performance. (a) The initial study (n=76), and (b) the replication (n=171). Taken from Tabachneck, Koedinger & Nathan (1994).

. Amanda brought home a pizza cut into 14 pieces. She gave 2 pieces of pizza to the dog, and then she handed out the rest of the pizza equally to the 4 family members. How many pieces did each family member get?

$$14 - 2 = 12 \qquad 12 \div 4 = 3,\; 12 - 12 = 0 \qquad \boxed{3}$$

(a) Arithmetic strategy used by Subject #113.

Starting with some number, if I subtract 40 and then divide by 3, I get 20. What number did I start with?

$$(x-40)\div 3 = 20$$
$$x-40 = 20\times 3$$
$$x-40 = 60$$
$$x = 60+40$$
$$x = 100$$

$20 \times 3 = 60$, $60 + 40 = 100$

Check
$100-40=60$
$60\div 3=20$

$\boxed{100}$

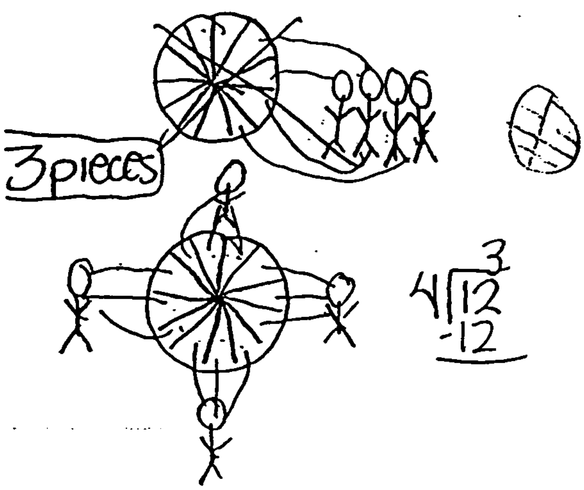
$100 - 40 = 60$, $60 \div 3 = 20$

(b) Algebra strategy used by Subject # 17.

**Figure 3**. Students' uses of various solution strategies. (a) Arithmetic, (b) algebra, (c) diagrams, (d) guess-and-test, and (e) unwinding.

**Figure 3**, continued.

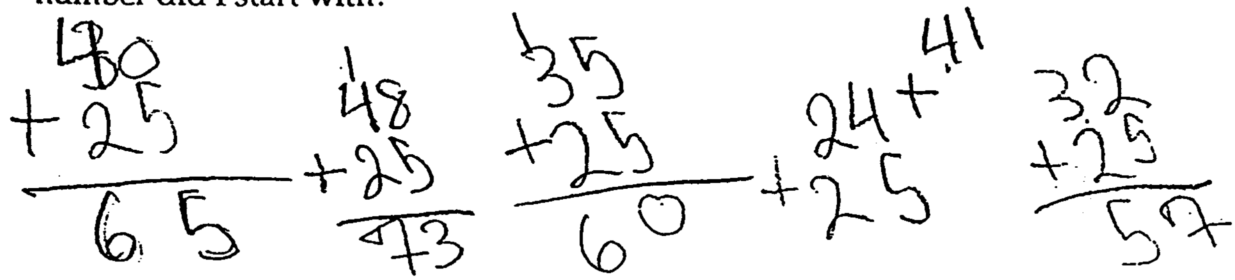
3. After bringing home a pizza, Amanda divided the 12 pieces in the pizza by the 4 family members and found the number of pieces each person got. How many pieces did each family member get?




(c) Diagram strategy used by Subject #174.

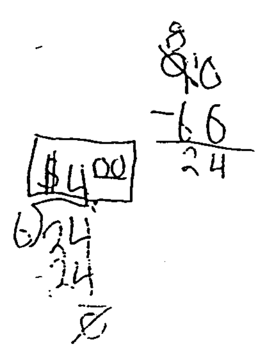
**Figure 3**, continued.

. Starting with some number, if I multiply it by 4 and then add 25, I get 66.40. What number did I start with?



(d) Guess-and-test strategy used by Subject #103.

When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $90. How much per hour does Ted make?



(e) Constraint untangling strategy used by Subject #99.

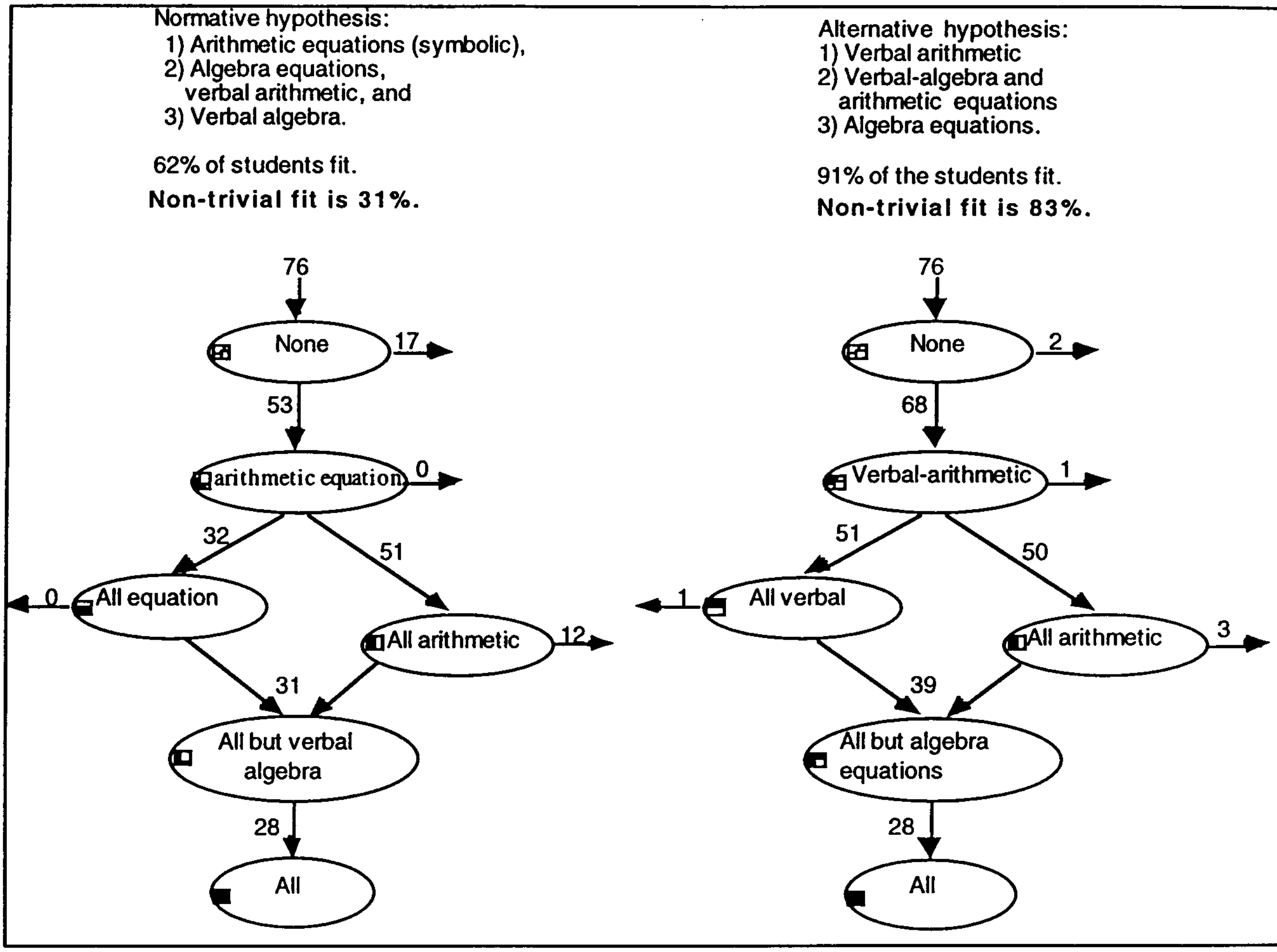



**Figure 5**. Two competing developmental models of early algebra. (a) The model that follows from the normative view favors development or arithmetic before algebra, and symbolic problem solving over verbal reasoning. (b) An alternative model suggested by our research that favors the development of verbal problem solving over symbolic problem solving. The fit of model (b) is superior for students in the original study (n=76). Taken from Koedinger, Nathan & Tabachneck, 1996.