A Framework for Improving Students' Comprehension of Word Arithmetic and Word Algebra Problems

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ICS Technical Report # 91-05

To appear in
Proceedings of the International Conference on the Learning Sciences, 1991
The Association for the Advancement of Computing in Education

Abstract

We present a text-comprehension based theory of word problem-solving processes that makes a distinction between comprehending the surface structure of a problem, the situation it describes, and its underlying formal mathematical structure. Two empirical studies that applied the theory to an instructional setting in an effort to improve students' problem comprehension are reported. One study was conducted with the domain of arithemtic word problems, while the other was conducted with the domain of algebra story problems. Both involved teaching students to use external, spatial, graphic representation methods and required students, personally, to assess the correctness of their formal, equation representations. In both investigations, students who were trained performed significantly better than controls on problems like those used in training and on problems designed to assess transfer of trained skills. We conclude that teaching students to concretize the relationship between the situations described in problem texts and the mathematical equations needed to express this relationship can be a valuable addition to mathematics curricula.

Students have particular difficulty generating a set of solvable mathematical equations from problem texts (DeCorte, Verschaffel, & De Win, 1985; Dossey, Mullis, Lindquist, & Chambers, 1988; Lewis & Mayer, 1987). Therefore, many researchers have called for more explicit instruction in the representation skills needed for comprehending the mathematical structure of word problems (Hall, 1989; Lewis, 1989a; Nathan, Johl, Kintsch, & Lewis, 1989; Silver, 1987). This paper outlines a text-comprehension based theory of problem solving, then presents the results of two empirical studies, which translated these theoretical ideas to instructional design.

Solving word problems involves two major processes, problem representation--reading the problem and deriving a mathematical structure reflecting the situation described in the text--and problem solution--planning and carrying out the necesary computations to arrive at a numerical answer (Mayer, 1985). Our focus is the representation stage of problem solving, which can be characterized as a special type of text comprehension. Kintsch and Greeno (1985) proposed that when reading a word problem, students first construct a "textbase", or representation of the textual input itself. Along with this textbase, a "situation model" is formed. A situation model represents the actions of the text in everyday, qualitative terms. Finally, in a later version of the theory (Nathan et al. 1989), students must integrate and quantify the situation to create a formal "problem model", which represents the problem situation in mathematical terms, e.g., equations. It should be noted that textbase and situation model construction are similar to the problem-solving stage of "translation" described by Mayer (1985), whereas problem-model formation is parallel to Mayer's "problem-integration" stage.

Thus, students must make a correspondence between their informal understanding of a problem situation and the formal equations needed to mathematically represent that situation. Our studies test the effectiveness of training that is directed at improving students' ability to make this correspondence, by making it more explicit through use of external, graphic representation schemes. Although one study focused on arithmetic word problems and the other focused on algebra story problems, both instructional techniques require that students, themselves, compare a spatial representation with their own understanding of problem situations, and neither instructional method dealt with the problem-solution phase of the problem-solving process.

Study 1: Arithmetic Word Problems

This study was conducted to assess the impact of training students to use a paper-and-pencil diagramming technique to represent the relationship between variables in Compare arithmetic word problems. Compare problems are characterized by the presence of relation statements, sentences that express the value of one variable in terms of another variable, such as "Jeff is 10 cm shorter than Tom". Compare problems were targeted for training because relation statements have consistently proven difficult for students to comprehend (Briars & Larkin, 1984;

Cummins, Kintsch, Reusser, & Weimer 1988; Kintsch & Greeno, 1985; Lewis & Mayer, 1987; Mayer, 1982; Morales, Schute, & Pellegrino, 1985; Riley, Greeno, & Heller, 1983).

Consider the following compare problem:

Problem 1

At ARCO gas sells for $1.13 per gallon.

This is 5 cent less per gallon than gas at Chevron.

How much do 5 gallons of gas cost at Chevron?

To solve this problem, the arithmetic relation between the cost of gas at ARCO and the cost of gas at Chevron must be comprehended. An incorrect selection of subtraction as the necessary operation (i.e., $1.13 - .05 = ?) is called a "reversal" error, the predominant error made on compare problems (Lewis & Mayer, 1987).

Because students have difficulty relating their equation choice to the situation described in a problem's text, a simple graphic method was developed to serve as an external, spatial representation of the relative values of problem variables. This spatial representation can be easily compared with the problem's text and then converted into a proper equation. Figure 1 outlines this diagramming method and the instruction that accompanyied it.

Method

From a pool of 299 pretested college students, 96 students who made reversal errors were assigned to one of three 32-subject groups in a pretest-posttest control group design. The pretest/posttest presented 8 "target" compare problems, which included one relational statement, 4 "transfer" compare problems, which included two relational statements, and 10 "filler" problems, which did not have relational statements.

After pretesting, subjects participated in two 1/2 hour training sessions and one 1/2 hour posttesting session. Subjects in the "diagram" group were first taught to identify relation statements and then taught how to diagram the relation between variables in problems with one relation statement (target problems), as shown in Figure 1. Subjects in the "statement" group were only taught to identify relation statements and advised to be cautious with them. Subjects in the "control" group received no training but were exposed to the same problems as the other subjects. For a more detailed account of this method, see Lewis (1989b).

Insert Figure 1 about here

Results and Discussion

Comprehension of compare-problem structure was measured by the probability of reversal errors for pre- and posttest target and transfer problems. Figure 2 presents target-problem data for each of the three subject groups. A two-way ANOVA revealed a significant Training Treatment X Time of Test interaction, $\underline{F}(2,93) = 3.28$, $\underline{p} < .05$, $\underline{MS}_e = 0.5$, indicating a difference in the effectiveness of the three groups' training. A Post-hoc Newman-Keuls comparison ($\underline{p} < .05$) showed that the diagram group made significanty less posttest target-problem reversal errors than either the statement or control groups, which did not differ from eachother. Furthermore, the diagram group displayed greater pretest-postest performance gains than both the statement and control groups.

Insert Figures 2 and 3 about here

When solving posttest transfer problems, all 32 diagram-group subjects spontaneously applied the diagramming method in an iterative fashion. Figure 3 presents transfer-problem data for each of the three subject groups. Again, a two-way ANOVA revealed a significant Training Treatment X Time of Test interaction, $F(2,93) = 4.01$, $p < .05$, $MS_e = 0.2$. Supplemental analyses showed that the diagram group made significantly fewer reversal errors at posttesting than at pretesting, $t(31) = 3.40$, p,.001, while no significant improvement was indicated for the statement or control groups.

Thus, subjects in the statement group, who learned to identify troublesome relational statements did not improve their comprehension of the conceptual structure of compare problems beyond the improvement of the control group. On the other hand, subjects in the diagram group, who learned to draw integrated diagrams to represent the relation between compare-problem variables, improved their comprehension more than the other groups. Use of an external, spatial diagram appears to have aided them in making the crucial connection between the situation expressed in a problem's wording and the arithmetic operation, or equation, needed for solution.

Study 2: Algebra Story Problems

Our second study extended the results reported above to the domain of algebra story problems and utilized a computer-based tutor to enable animated, dynamic representation of problem situations. ANIMATE, written in HyperCard^{tm}, runs on Apple^{tm} Macintosh computers and currrently tutors problems in the AMOUNT-PER-TIME RATE family of algebra problems (Mayer, 1981). These problems are characterized by an underlying "distance = rate x time" mathematical structure. For example:

Problem 2

> A huge ant is terrorizing San Francisco. It travels east toward Detroit, which is twenty four hundred miles away, at four hundred miles per hour. The Army learns of this one hour later and sends a helicopter west from Detroit at six hundred miles per hour to intercept the ant. If the ant left at 2 PM, what time will the helcopter and the ant collide (ignoring any time changes)?

In the direct-manipulation ANIMATE environment, a student selects graphics of an ant and a helicopter and specifies their starting locations and directions. The ant and helicopter are then displayed at the top of the screen as shown in Figure 4 and will serve as a representation of the situation described in the problem. Next, the student begins to build a network of equations that express the formal, mathematical structure of the problem situation. This is accomplished by chosing equations from a palate of options. The student chooses one equation as a "stopping condition"--a mathematical relation that will be true when the situation ends. An example would be when distances are equal for overtake problems, such as, when the sum of the distances travelled equals 2400 miles in Problem 2 shown above. Finally, the student directs ANIMATE to put the ant and helicopter into motion in accordance with the newly constructed formal network. In this manner, the student can see the relative motions of the ant and helicopter that the created formal representation dictates. If the characters' movements do not correspond to the student's expectations from his or her understanding of the problem situation, the student learns that the equation network is incorrect and may reorganize the network by changing variables, values, and/or operators until the expected animation is generated.

Insert Figure 4 about here

Method

Fifty-six college students were randomly assigned to one of four groups in a pretest-posttest control group experimental design and participated in a two hour session that encompassed pretesting, training, and posttesting. The pretest and posttest consisted of four paper and pencil problems: two target problems, which were standard distance-rate-time problems, one near-transfer problem, which required students to debug erroneous equation representations of a problem, and one far-transfer problem, which directed students to write a short story corresponding to a set of presented equations.

Each of the four groups was exposed to a different training environment while learning to solve three training problems. The animation group used the ANIMATE tutoring environment, which included: organizing algebraic formalisms in the network graph structure, setting up the qualitative situation (e.g., selecting the characters and stopping condition), and using the equation-driven animation as feedback to assess the correctness of the equation network. The stopping-condition group had the same computer-based environment except that it lacked a running animation as feedback. The network group used the same computer environment as the stopping-condition group, except that they did not set up the qualitative situation, i.e., did not set up a picture or stopping condition. The equation group served as the ultimate control group; all of their training was done with paper and pencil and without the use of the network formalism.

Results and Discussion

As was the case for the first study, what is of interest in this study is how well students comprehended the structure of pre- and posttest problems. The target problems were scored as correct if subjects generated the proper equations. Group performances are shown in Figure 5. A one-way between-subjects ANCOVA was conducted with posttest target-problem performance as the dependent measure, pretest score at the covariate, and treatment group as the between-subjects variable. A significant difference among the performances of the four groups was confirmed, $F(1, 55) = 11.56$, $p < .05$, $MS_e = 2.3$. A post-hoc

Newman-Keuls comparison ($p < .05$) revealed that the animation group performed significantly better than the other three groups, whereas the other groups did not differ from each other. Subjects exposed to the ANIMATE environment had improved ability to make a correspondence between formal mathematical equations and their understanding of problem situations.

Insert Figures 5, 6, and 7 about here

Figure 6 displays subjects' performance on the near-transfer, debugging task. Although improvement was greatest for the animation group, differences among groups were not statistically significant. Animation-group subjects, however, were more likely to repair a buggy equation if they detected it, $F(1,55) = 5.0$, $p < .05$. Finally, Figure 7 presents far-transfer, story writing, performance on pre- and posttest. A one-way, between-subjects ANCOVA with posttest performance as the dependent variable, pretest score as the covariate, and treatment group as the between-subjects variable indicated a significant main effect of treatment, $F(1.\ 55) = 11.8$, $p < .002$, $MS_e = 2.3$. A post-hoc comparison revealed that the animation group performed higher than the equation and network groups but not higher than the stopping-condition group (Newman-Keuls, $p < .05$). Apparently, the greater exposure to situation-level processing provided by the animation and stopping-condition tutors helped subjects in this novel task by providing the graphic and highlighting the role of equations as stopping conditions in problem situations.

Thus, working in an environment that encouraged situation-based reasoning promoted better understanding of the relation between informal situation models and formal equations, even outside of the learning environment, as indicated by posttesting which did not involve tutor use. We suggest that some important changes in conceptual understanding occurred for subjects in the situation-based learning environment that did not occur for subjects in the other training conditions.

Conclusion

The benefits of representation training are evident for both arithmetic word problems and the more complex algebra story problems. Within each domain, utilizing an external, graphic representation scheme aided students in correctly mapping formal equations to informal textual descriptions. Other researchers (e.g., Shalin & Bee, 1985; Fuson & Willis, 1989) have reported on the benefits of diagram use as an aid to problem comprehension, and the future of such endeavors appears to be bright. Of particular interest in the two representation aids reported herein is that they both involved a spatial layout of problem information and both gave students the responsibility for assessing the correctness of the correspondence between the problem situation and the formal equation(s). Furthermore, the addition of animation is a promising direction in representation-skills training (c.f., Rieber, 1990).

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Acknowledgements

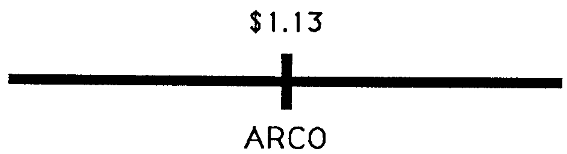
This work was supported in part by Grant MH15872 from the National Institutes of Mental Health to Walter Kintsch and in part by a National Science Foundation graduate fellowship research grant to Anne Bovenmyer Lewis. We would also like to thank Paul Johl and Emilie Young for their contributions in developing the ANIMATE system.

Sample Problem

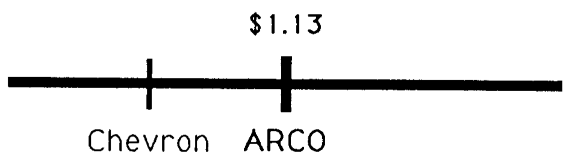
At ARCO gas sells for $1.13 per gallon. This is 5 cents less per gallon than gas at Chevron. How much do 5 gallons of gas cost at Chevron?

Diagramming Steps

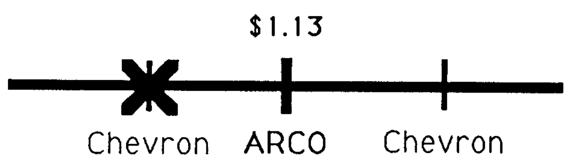
1. Draw a number line and place the variable and value from the assignment statement at the middle of the line.



2. Tentatively place the unknown variable (Chevron's gas price) on one side of the middle.



3. Compare your representation with the information in the relation statement, checking to see if your representation agrees with the meaning of the relation statement. If it does, then you can continue. If not, then try the other side.



4. Translate your representation into an arithmetic operation. If the unknown variable is to the right of the center, then the operation is an increase, such as addition or multiplication. If the unknown variable is to the left, then the operation is a decrease, such as subtraction or division.

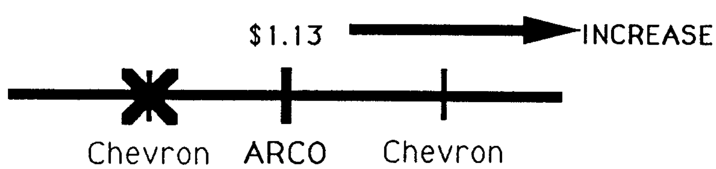


Figure 1. Diagramming procedure for the sample problem.

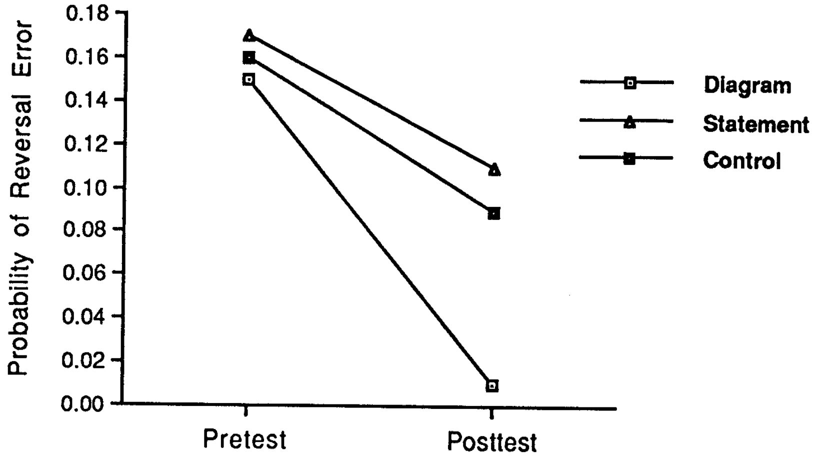


Figure 2. Probability of pretest and posttest target-problem reversal errors for the three groups in Study 1.

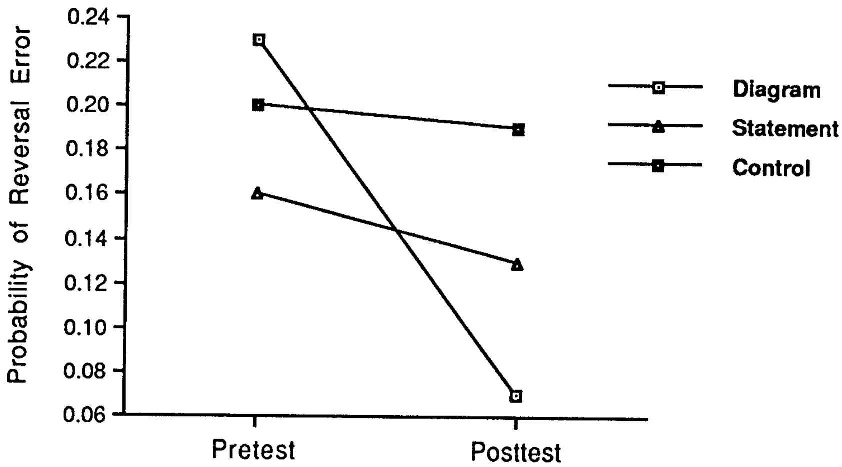


Figure 3. Probability of pretest and posttest transfer-problem reversal errors for the three groups in Study 1.

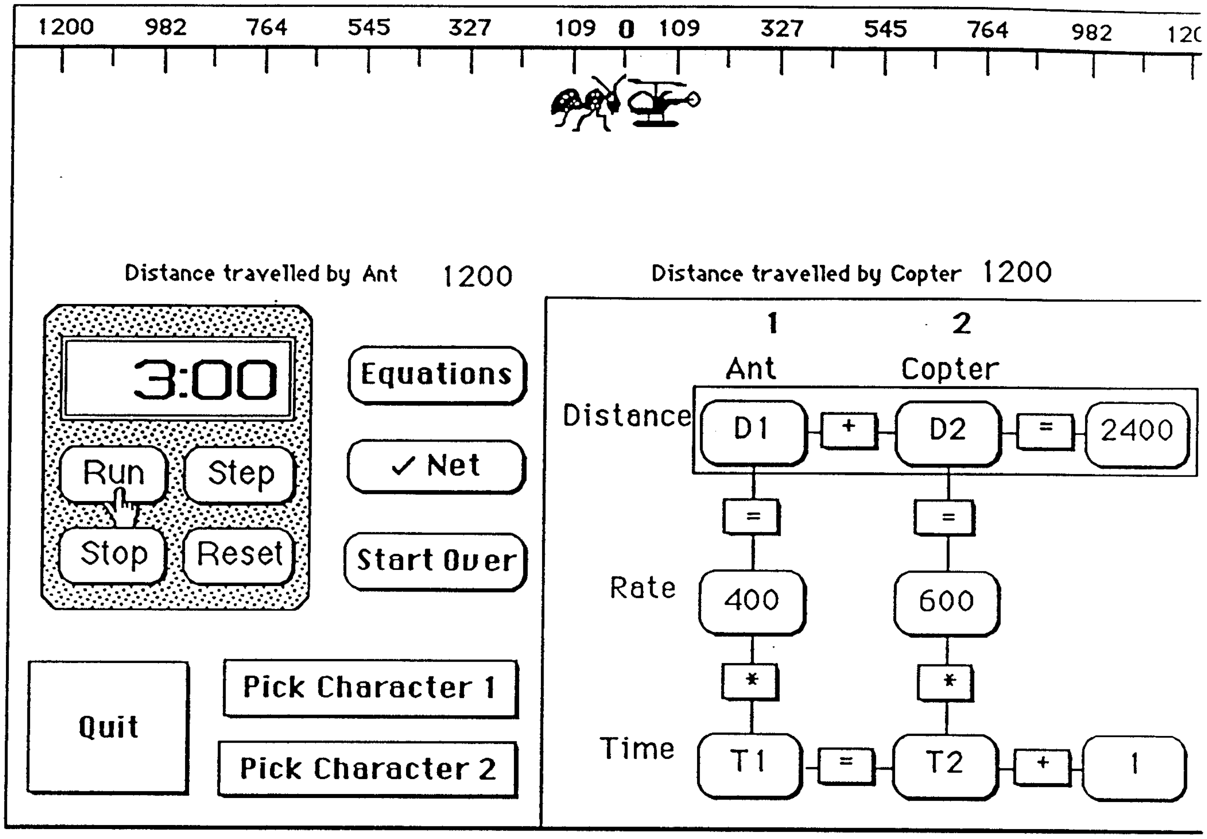


Figure 4. The ANIMATE environment. This example shows a collision situation in accordance with expectations. Time and distance gauges show solution values of the network formalism.

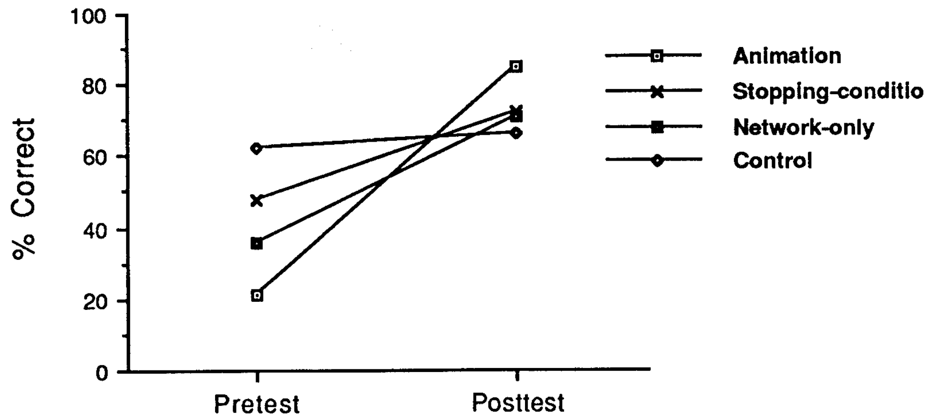


Figure 5. Percent correct on pretest and posttest target problems for the four groups in Study 2

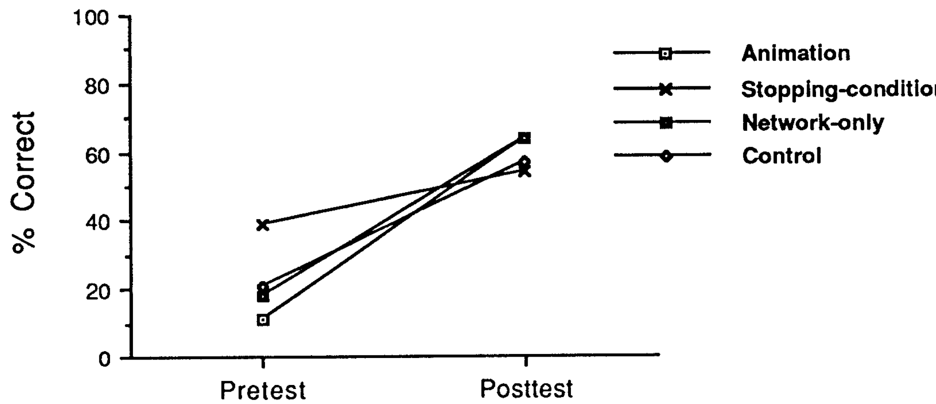


Figure 6. Percentage correct on pretest and posttest near-transfer problems for the four groups in Study 2.

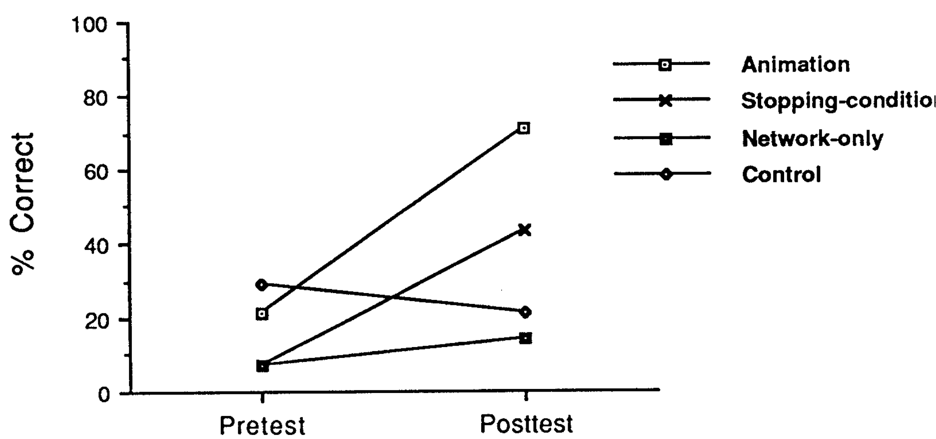


Figure 7. Percent correct on pretest and posttest far-transfer problems for the four groups in Study 2.