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Instructional Implications of Research on Problem Solving

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One of the major developments in recent research on cognitive processes has been in the analysis of problem solving. In this chapter we will consider instructional implications of current scientific concepts about problem solving. We will begin with a section reviewing some fairly well-established concepts in the current cognitive theory of problem solving. In this first section we present ideas that form a theory of intellectual skills, and therefore make it possible to begin to analyze instructional goals involving acquisition of skills and procedures in addition to acquisitions of facts, concepts, and principles. Our second section presents some more recent research that has considered the nature of knowledge and skills used by problem solvers who are experts in their domains. This work is beginning to provide an understanding of the nature of knowledge used when a problem solver has definite procedures for representing and understanding problem situations. Finally, we discuss some instructional research and development projects illustrating ways in which some of this developing understanding of psychological mechanisms can be applied to make instruction more effective in strengthening students' abilities in solving problems.

Procedural Knowledge for Problem Solving

Teachers often hope that their students learn to use the concepts and principles of their discipline, in addition to learning the content of the field. Use of the information in a field normally involves skill in solving problems that are related to the field's concepts and principles. Therefore, to a considerable extent, analysis of students' abilities to use information in a field corresponds to analysis of their abilities to solve problems that require that field's information.

Analyses of problem solving during the 1970s have identified major kinds of knowledge structures needed to perform successful problem solving. These analyses have identified more important kinds of procedural and strategic knowledge that students must have or acquire in order to apply knowledge of the content of a course successfully. For example, Greeno (1978a) identified three major kinds of knowledge needed for solving different kinds of problems: (1) induction of structures and patterns; (2) constructing constructive research; (3) performing means-end analysis. Relatively pure cases of problems requiring these various processes can be found.

Induction of structure is required in analogy problems or in series extrapolation problems such as those commonly found in tests of general intelligence. Egan and Greeno (1974), Evans (1968), and Simon and Kotovsky (1963) have identified comprehension of relations and synthesis of relational patterns as important components of cognitive skill in inducing structure.

Constructive search is required for solution of arrangement problems such as anagrams or cryptarithmetic. The major components of knowledge for these problems seem to involve, first, knowledge of possibilities, used to generate trial solutions, and second, knowledge of constraints, used to restrict the generation or acceptance of inappropriate attempts at solution; that is, to prevent generation or acceptance of trial solutions that could not be correct.

Means-end analysis occurs in solutions of simple problems of transformation in which a present situation is compared with the goal and operators are chosen to reduce the differences that are formed (for example, Simon, 1975). The idea of means-end analysis was developed in the General Problem Solver (GPS) (Ernst and Newell, 1969; Newell and Simon, 1972). GPS consists of a set of general methods for performing means-end analysis, and it was shown that these general methods could be used.

In the analysis of transformational problems, the work of Sacerdoti (1977) provided an important advance over simple means-end analysis. He analyzed the structure of knowledge needed for planning solutions of problems. In Sacerdoti's model, Network of Action Hierarchies (NOAH), each action known by the system is stored with preconditions and consequences of the action, as well as the component subactions that must be performed to accomplish the action. In forming a plan for solving a problem, NOAH checks the consequences of actions in its plan to see whether they are consistent with preconditions of actions that occur later; thus, NOAH is able to arrange a sequence...
of actions that avoids blocking of later requirements by earlier actions. The kinds of knowledge that have been identified in these analyses are largely procedural in nature. This contrasts with the way we usually think about the content of a course, as consisting of facts, concepts, and principles stated in the form of procedures, and we are beginning to understand the characteristics of the procedures that are needed. Specific systems that have knowledge of procedures for planning and means-end analysis, for constructive search, and for induction of relational structure have been designed and proposed as hypotheses about the kinds of knowledge involved in successful problem solving.

**Procedural Knowledge for Geometry Problem Solving.** An analysis of problem solving in a school subject has been developed in the form of a computer simulation model called Perdix (Greeno, 1978b; Greeno, Magone, and Chalkin, in press). The simulation of geometry problem solving by Perdix involves three kinds of knowledge. One kind of knowledge is a set of propositions used for making inferences. An example is the proposition "Corresponding angles are congruent," which is used to infer that two angles are congruent if it has been established that they are corresponding angles.

A second kind of knowledge is a set of perceptual concepts that perform pattern recognition. An example is the concept of corresponding angles, consisting of a combination of features that define the pattern of corresponding angles. These are angles with parallel sides, on the same side of a transversal, with one angle between the parallel sides and the other outside the region bounded by the parallels. Pattern recognition is required in order to identify properties and relations that are needed to permit inferences.

The third kind of knowledge needed to solve geometry problems is strategic knowledge, consisting of procedures for planning and for setting and representing goals during work on the problem. Strategic knowledge determines the way in which problem solving performance is organized. It includes planning knowledge that is organized, in the way that Sacerdoti (1977) organized planning knowledge in NAOH, with global actions associated with their preconditions and consequences. It includes some simple associations between goals and procedures that can be used to accomplish these goals. Strategic knowledge also includes tests for the completion of goals, including indefinite goal structures that are represented as pattern recognition systems to detect alternative ways in which the goals can be achieved.

Instruction in geometry includes explicit presentation and explanation of propositions and explicit training in pattern recognition. However, instruction in strategic knowledge apparently is largely implicit. It seems most likely that students acquire procedures for planning and setting goals by inducing them from example problem solutions provided in a text or by their teacher.

**Tactile Knowledge.** Procedural knowledge of problem-solving strategies is not only implicit in instruction; it also appears to be a form of tacit knowledge both for students and for teachers. By tacit knowledge (a term suggested by Polyani, 1967) we mean knowledge that is not recognized by the person who has the knowledge. When a skilled problem solver is asked to describe the process of solving a problem, the person is able to report a fair number of thoughts that occur during problem solving, but very little information is reported about the process that produces the thoughts. Teachers are similarly unaware of the nature of the processes that they or their students use in solving problems. One of us presented to a group of high school mathematics teachers the general features of the Perdix simulation model for problem solving in geometry, together with a student's protocol. The protocol illustrated the absence of strategic knowledge for solving a problem for which the student apparently had the necessary inferential propositions and concepts for pattern recognition. One teacher's reaction to the idea of strategic knowledge was, "What you're talking about is the student's intelligence, isn't it?" Another teacher offered the opinion that students currently in high school have not acquired sufficient persistence in their approach to problems in their elementary school training, and therefore give up on problems that they could solve successfully.

The theories of intelligence and motivation that these teachers proposed are probably widely held, and they probably have reasonable validity. However, they identify the difficulties of students at such a global level that teachers are discouraged from trying to strengthen students' problem-solving skills through specific instruction. The characterization of strategic knowledge provided by recent theories such as Perdix involves relatively specific procedural knowledge. It is reasonable to expect that if students encounter difficulty in solving problems because they lack procedural strategies, then instruction should be structures to meet these needs.

**Problem Solving and Understanding**

The studies of problem solving discussed above concern relatively simple strategies, largely independent of "understanding" any particular subject matter. A number of recent analyses, however, have focused on the nature of processing required for skilled solution of complex problems in specific subject matter domains. Processes of understanding relevant to such problems have been explored in the literature on language comprehension (Anderson, 1976; Norman and Rumelhart, 1976; Schank and Abelson, 1977). These theorists conceptualize understanding as the construction of a complex representation of the various elements in a situation, and the relationships between these elements. Skilled problem solving can be characterized as behavior guided by an understanding of the problem, involving the construction of representations of the elements and relations in the problem. To the extent that this representation is coherent, is connected with other components of a person's knowledge, and corresponds with the actual problem situation, the problem can be said to have been solved "with good understanding" (Greeno, 1977).

A skillful solver's understanding of a problem often progresses during problem solution. That is, in pursuing the problem goal, a person modifies the problem representation by generating new components or relations that are then included in the representation. The final solution pattern includes both initial problem components and the additional components and relations required for achievement of the desired goal state.

The adequacy of the problem representation (the degree of understanding of the problem) in large measure determines the nature and amount of procedural knowledge required to achieve a solution. However, the completeness of the problem representation clearly depends
on the amount and organization of the problem solver's knowledge of concepts and principles. This complex interrelationship between knowledge of concepts and principles and knowledge of problem-solving procedures has been a central concern of current problem-solving studies. This research is beginning to provide an account of comprehension processes, including the mechanisms by which relevant knowledge of concepts and principles is accessed and utilized.

Expert-Novice Distinctions. One method for discovering the ways in which differences in knowledge of content and knowledge of procedures bring about variations in solution procedures is through comparison of highly skilled "expert" problem solvers with less experienced "novices." In addition to allowing theoretical formulation of the nature of skilled problem solving, these analyses can lead to empirically verifiable suggestions for effective instruction. Two examples of the kinds of analyses appearing in the literature are provided here in some detail.

The study of scientific problem solving allows examination of the ways in which knowledge of concepts and principles is utilized by, and influences, general problem-solving processes. Analyses in this domain have revealed alternative procedures for achieving problem solutions that are characterized by substantively different procedures and degrees of problem understanding.

Simon and Simon (1978) analyzed the differences between expert and novice solutions of kinematics problems. In terms of overall solution strategies, they found that the expert typically used a "working forward" procedure with little explicit mention of the goals of each solution step. That is, the highly skilled physics problem solver operated from the given of the problem, applying successive equations that could be solved with the givens until the desired values were found. The novice used a "working backward" strategy with more explicit reliance upon means-end analysis. This entailed evoking and applying equations in which the desired quantities were dependent variables and, if not all the independent variables in these equations were known, setting up subgoals to solve them.

Simon and Simon also contrasted the expert's "physical" approach (moving from the problem statement to a representation of the physical situation, and from that representation to equations) with the novice's "algebraic" approach (going directly from the problem statements to the equations). They asserted that knowledge of the physical laws or equations needed for solving these problems comprises only the "algebra of kinematics," to "know physics," an individual must be able to represent complex kinematics problem situations by having those physical laws organized and "indexed" in memory in a meaningful way.

Larkin's (1977a, 1977b) analyses of mechanics problem solving provided additional insights into the nature of expert-novice distinctions and the relationships between organization of scientific knowledge and problem-solving processes. Both experts and novices in these studies were observed to begin work on a problem by constructing an "initial description" of the problem situation from the problem statement. After completing this initial description, which was generally in the form of a labeled sketch, the experts and novices proceeded to solve problems in considerably different ways. The novices immediately evoked and applied equations at this point, as described by Simon and Simon (1978). But the experts engaged in a planning phase before generating mathematical expressions. After constructing the original problem description, experts constructed a "low-detail qualitative physical description" of the problem. During this qualitative analysis, solvers elaborated the problem when necessary by accessing supplementary information required for understanding the problem situation that was not included explicitly in the written statement. This phase of the solution also involved planning in the form of identification of a promising solution method and selection of key aspects of the problem situation to which to apply the method.

Once a promising procedure is identified in conjunction with a coherent physical description of the problem, the expert begins the mathematical description, by applying the method to evoke and apply specific equations. This aspect of solution differs substantially from the novices' equation manipulations in that the experts' generation of equations is systematically guided by the solution method selected and previously explored qualitatively. The expert "applies principles...as part of well-defined methods," in contrast to the novices' sequential access of individual principles.

Thus, novices were once again found to work directly from the problem description to assembly of individual equations, while experts solved problems by a process of "successive refinements," beginning with a loose qualitative analysis and preliminary testing of tentative solution methods, only moving to specific equations when a promising approach to the problem had been identified in an abstract planning space. Experts worked from an elaborate representation of the problem rather than directly from the problem description. The organization of scientific knowledge or principles in related chunks allows this overall solution strategy to be implemented.

Other Problem-Solving Analyses. Additional work on representational processes in problem solving has been conducted in the contexts of artificial intelligence and computer simulation. Novak (1976) has created a computer program ISAAC, that solves physics problems by constructing and elaborating internal problem representations that guide all subsequent solution processes. DeKleer's (1976) program, NEWTON, Bundy's (1978) program, MECHO, and McDermott and Larkin's (1978) program, PH632, model processes of qualitative representation in physics problem solving. A program that models semantic processing (construction of problem representation) in the solution of elementary arithmetic word problems has been developed by Haller and Greeno (1978). Each of these programs focuses on the role of understanding processes and utilization of concepts and principles in achievement of problem solutions.

In a rather different domain, Heller (1979) has studied the problem-solving and representational processes required for the solution of verbal analogies, focusing on both the impact of knowledge of word meanings on analogical problem-solving processes invoked and the ways in which procedural knowledge affects the types of conceptual knowledge accessed.

Procedural Knowledge: A Broader View

A language for conceptualizing procedural and strategic problem-solving behavior was provided in early information-processing analyses of problem solving (Newell and Simon, 1972). The theory of procedural knowledge that grew out of this seminal work has more recently been integrated with concepts from the literature
on language understanding to yield comprehensive theories of problem solving with understanding. In addition, empirical investigations of problem solving have added specific insights into the relations between knowledge structures, representational processes, and general problem-solving procedures. As a result, the theory of problem solving is evolving to encompass a broader range of concepts; a more complete theory of human problem solving is emerging.

In part, these changes have been influenced by consideration of problem solving in specific subject matter domains. Because solution of complex problems generally requires knowledge of the world, the amount, accuracy, and organization of an individual's knowledge must influence the effectiveness of problem-solving efforts. That is, regardless of the sophistication of a solver's procedural and strategic knowledge, deficiencies in knowledge of facts, concepts, and principles constrain the scope of that individual's problem-solving ability. As demonstrated by Larkin (1977a, 1977b), the organization of knowledge also influences a problem solver's efficiency. The functioning of planning processes in skilled scientific problem solving is a prime example of this phenomenon: the availability of solution methods comprised of principles stored in coherent related clusters facilitates finding optimal solution procedures. While this explanation of planning is consistent with concepts such as those embodied in the General Problem Solver (Newell and Simon, 1972) and NOAH (Sacerdoti, 1977), it includes specific references to the nature of knowledge required to accomplish effective planning in problem situations encountered during schooling. Thus, procedural knowledge cannot be considered in isolation from the knowledge structures on which processes must operate.

Efforts to identify characteristics of highly skilled problem solvers have revealed the importance of representational processes in problem solving. Rather than merely demonstrating that experts are proficient means-end analysts, for example, these studies indicate that it is reliance upon qualitatively elaborated problem representations that distinguishes skilled problem solvers. In effect, experts understand problem situations better than novices, and it is this understanding that guides their more effective and efficient solutions. While both experts and novices may be observed to break problems into subparts, or set subgoals to deal with difficulties, these procedures are apparently executed by experts with an emphasis on the problem "Fractal" while novices tend to solve problems with more piecemeal approaches. These findings emphasize the importance of construction and transformation of problem representations throughout solution, as well as the mechanisms by which problem representations influence and are affected by knowledge of concepts and principles, as well as procedures.

Applied Work in Practical Instruction

We now turn to some examples of practical instruction in problem solving. These can be roughly grouped into three kinds, the first stressing direct teaching of procedural knowledge, the second the development of qualitative representations, and the third the teaching of general strategies.

Teaching Procedural Knowledge. A classic example of teaching of procedural knowledge is the course developed by Bloom and Broder (1956) at the University of Chicago. They worked with undergraduates who had failed the university's comprehensive examination, which required solving problems using knowledge integrated from several years of study. Their instructional technique was to have a student solve a problem, while thinking aloud, observed by a second student. Then the two students together looked at a written solution process produced by a skilled solver also thinking aloud while solving the same problem. The two students then worked together to try to identify similarities and differences between the solution process of the student solver and that of the skilled solver. Bloom and Broder found that students began to adopt techniques that were at least qualitatively similar to those used by the skilled solvers. Furthermore, the course was a practical success, ultimately resulting in students' increasing their scores on the comprehensive examination from an average of 0.74 to an average of 1.34 on a 4-point scale.

The success of Bloom and Broder's course is almost certainly not due entirely to direct instruction in procedural knowledge. In particular, the motivational effects of contact with other students and individual attention from the instructors almost certainly played an important role. Indeed, the effects of the course were considerably less dramatic when the course was expanded to involve more students, and correspondingly automated so that students received less standardized instruction and less individual interaction with the instructors.

The work of Schoenfeld (1978a) at the University of California, Berkeley provides an example of a course based on somewhat more explicit analysis and direct teaching of procedural knowledge. He developed a course for university-level students majoring in mathematics designed to teach them "heuristics," techniques for selecting a promising approach from the various mathematical techniques they already know. His instructional technique is to specify heuristics corresponding to various aspects of the problem-solving process (analysis, exploration, verification). For example, heuristics for analysis of a problem include drawing a diagram, examining special cases, and trying to simplify the problem. Each heuristic includes some further detailed suggestions for application (for example, examining special cases is elaborated by the suggestion "Choose special values;", "Examine limiting cases;" and "Set integer parameters equal to 1, 2,3,. . ."). In addition to exposing students directly to these explicitly formulated heuristics, Schoenfeld's course involved large amounts of practice and illustration. Thus one could infer that his students acquired useful procedural knowledge both from the direct explanation of the heuristics and from the variety of practice in doing problems.

Although this course itself has not been extensively evaluated, Schoenfeld himself (1978b) conducted a teaching experiment designed to assess the role of heuristics. He had seven students work a variety of mathematics problems over a total of five two-hour sessions. For four of these students, the problems were grouped so that each group could be solved quite readily through the application of a particular heuristic. After solving each problem, a student received feedback in the form of a complete solution with the use of any heuristics indicated. For the remaining three students, the problems were presented in random order, and the feedback had the comments on heuristics omitted. Results of a posttest and pretest showed significant gains in the number of problems solved for the heuristics group, but not for the group that had only experienced practice in solving problems. This experiment supports Schoenfeld's view
that analysis and direct teaching of procedural knowledge has benefits beyond any associated practice.

The work of Schoenfeld and of Bloom and Broder suggests both the feasibility and the potential utility of analyzing the procedural knowledge of skilled solvers and then teaching this knowledge to students directly. Furthermore, their work provides some ideas about how to go about teaching procedural knowledge in a practical setting.

**Development of Qualitative Knowledge.** Clement (1978) at the University of Massachusetts, Amherst, has studied in detail college students' natural, qualitative models for physical situations. For example, he describes for students a rocket ship moving "sideways" (perpendicular to the thrust of its rockets) in outer space. He then asks students to describe the motion of the ship when the rockets are turned on briefly and then shut off again. A large proportion of students in this situation show no intuitive understanding of Newton's laws (that the force due to the rockets will change the ship's acceleration), but instead use an Aristotelian world model, saying that the force due to the rockets changes the ship's position.

Based on insights like these, Lochhead (1978), also working at the University of Massachusetts, has developed a variety of problem-solving courses for first- and second-year engineering students. These courses provide students with extensive experience in manipulating and observing objects relevant to the physical theories they are learning. Lochhead further stresses the connections between quantitative and qualitative representations of the world by having students make qualitative observations of their objects and derive for themselves the correspondence with the theories studied in class.

**Teaching of General Strategies.** A final group of problem-solving courses include as a major component the teaching of general domain-independent strategies. Such courses include those of Rubenstein (1974), Hayes (1978), and Woods (1976). All three of these courses attempt to acquaint students as broadly as possible with what is known about problem-solving strategies. Students then see how these strategies apply in a wide variety of particular domains.

For example, Hayes applies general psychological principles of memory to the task of learning phyia in biology, the strategy of means-end analysis to solving algebra problems, and techniques of problem representation to solving word problems in mathematics. With Flower, Hayes has observed in detail the strategies used by experienced writers (writing freely and then editing, tailoring the writing for the audience), in contrast to the limited strategies used by most students (relying on inspiration or on a disciplined making and "flexing out" of an outline). Then they work to teach these strategies directly (Flower and Hayes, 1977).

Woods has compiled a vast assortment of problem-solving strategies largely developed by educators working in the classroom. For example, students learn to "brainstorm" freely when stuck on a problem, and also to use very structured strategies for learning and problem solving, such as described by Reif, Larkin, and Brackett (1976). Woods exposes his students to many of these strategies, discusses their similarities and differences and ways in which each might be useful, and works with students in discussion groups to develop personal applications of these methods to current problems in the students' engineering curriculum.

The course of Rubenstein stresses the role of motivational problem solving by employing carefully selected and dynamic instructors. The conceptual content of the course includes both some basic information about the psychology of problem solving (the use of state spaces and operators to describe the progress from the current state of knowledge to the goal), together with a broad survey of special techniques used particularly in scientific problem solving (probabilistic reasoning, computer modeling, optimization). The generality of these courses is apparent from the wide audiences they address. Woods' course may be taken by any engineering student; the courses of Hayes and Rubenstein are intended for any university undergraduate.

**Summary and Conclusion**

Current work in the cognitive psychology of problem solving becomes increasingly relevant to teachers as it begins to address explicitly what we would most like our students to learn—the ability to use their knowledge of a discipline powerfully in solving a variety of problems. For the beginning, efforts to understand problem solving have necessarily concerned procedural knowledge, the knowledge needed for doing things. Initial efforts focused on general knowledge; for example, the strategic knowledge needed to solve puzzles, independent of any particular subject matter.

More recently, however, there has been an increasing appreciation that solving real problems requires an immense amount of "understanding." Thus current studies of problem solving emphasize modeling a large quantity of real-world knowledge, and elucidating how this knowledge is organized and brought to bear on various problems.

While one must always be cautious in trying to transfer the results of controlled psychological studies into a complex setting such as a classroom, in fact there begins to be evidence suggesting that some of the ideas reviewed here might be a good basis for instruction. Suggestions of such utility come both from what we have called teaching experiments, and from a few promising efforts in developing practical instruction.

**References**


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