Fractional bandwidth normalization for optical spectra with application to the solar blackbody spectrum

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Optical spectra are typically normalized per unit wavelength or per unit photon energy, yielding two different expressions or curves. It is advantageous instead to normalize a spectrum to a constant fractional bandwidth, providing a unique expression independent of whether the bandwidth is in dimensions of wavelength or of photon energy. For the Sun, whereas a per-unit-wavelength spectrum peaks in the green and a per-unit-photon-energy spectrum peaks in the IR, when the proposed normalization is used, the output peaks in the red. This approach applies to any spectral source and provides curves of constant spectral resolving power, as produced by many spectrometers. © 2001 Optical Society of America

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1. Introduction

Optical spectra are commonly expressed in one of two ways, per-unit-wavelength interval or per-unit-photon-energy interval. Plots of these two expressions exhibit substantially different shapes and peak at different wavelengths. Because much of terrestrial fauna, flora, and technology is sensitive to the solar spectrum, its intensity distribution and peak wavelength are significant. The conventional ways of expressing spectra are described, and an alternative normalization for the spectral output of a blackbody is developed and generalized to any spectral output. This approach yields a curve that is independent of whether the scale is per-unit-wavelength interval or per-unit-photon-energy interval and is shown to correspond naturally to spectral resolution.

2. Constant-Interval Approach

Because the radiation from a spectrally continuous source is distributed over the spectrum, as any interval in the spectrum approaches zero the intensity at that discrete point goes to zero. The intensity must therefore be specified over defined intervals within the spectrum. Typically, when this spectral intensity is considered as a function of photon energy, the intervals are of photon energy $\hbar\Delta\omega$ or in the limit $\hbar d\omega$, where ω is the optical radial frequency. The dimensions of the function are then power density per-unit photon energy. The constant-photon-energy-interval spectral irradiance for a blackbody of temperature T, expressed as a function of $\hbar\omega$ and of λ, is

$$\frac{dJ_{\omega}}{\hbar d\omega} = \frac{1}{4\pi^2 c^2}\frac{\omega^3}{\exp(\hbar\omega/kT) - 1} = 2\pi c\,\frac{1}{\lambda^3[\exp(2\pi\hbar c/kT\lambda) - 1]}, \tag{1}$$

where k is the Boltzmann constant, c is the speed of light, and wavelength $\lambda = 2\pi c/\omega$.[1] It is worth emphasizing that in plotting Eq. (1) one implicitly assumes that the photon-energy interval $\hbar d\omega$ is constant.

To a good approximation the spectral irradiance of the Sun corresponds to that of a blackbody at a temperature of 5800 K. This is plotted[2] by use of Eq. (1) versus photon energy as the leftmost curve in Fig. 1 and versus wavelength as the rightmost curve in Fig. 2. The peaks in these curves occur where

$$\frac{\hbar\omega}{kT} = \frac{2\pi\hbar c}{kT\lambda} = 2.822.$$

For a blackbody temperature of 5800 K this peak in $dJ_{\omega}/\hbar d\omega$ occurs at a photon energy of 1.41 eV, corre-

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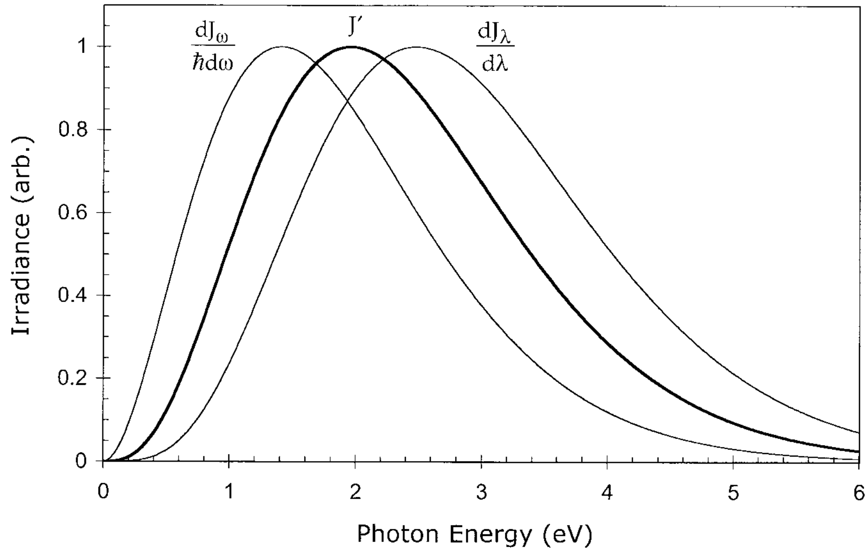

Fig. 1. Normalized spectral output for a 5800 K blackbody source approximating the Sun versus photon energy. The leftmost curve ($dJ_\omega/\hbar d\omega$) is intensity per-unit-photon energy from Eq. (1), the rightmost curve ($dJ_\lambda/d\lambda$) is intensity per-unit wavelength from Eq. (2) and the central curve J' is intensity per-unit fractional bandwidth from Eq. (3). The respective photon energies and heights of the peaks are 1.41 eV and 2.81 kW cm^{-2} eV^{-1} for $dJ_\omega/\hbar d\omega$, 2.48 eV and 8.44 kW cm^{-2} μm^{-1} for $dJ_\lambda/d\lambda$, and 1.96 eV and 4.72 kW cm^{-2} for J'.

sponding to a wavelength of 879 nm, which is in the near IR.

Usually, when the irradiance is considered as a function of wavelength, the intervals are of wavelength $\Delta\lambda$ or, in the limit, $d\lambda$. The dimensions of the function are then power density per-unit wavelength. The constant-wavelength-interval spectral irradiance for a blackbody of temperature T, expressed as a function of λ and of $\hbar\omega$, is[1]

$$\frac{dJ_\lambda}{d\lambda} = 4\pi^2\hbar c^2 \frac{1}{\lambda^5[\exp(2\pi\hbar c/kT\lambda) - 1]} = \frac{\hbar}{8\pi^3 c^3} \frac{\omega^5}{\exp(\hbar\omega/kT) - 1}. \tag{2}$$

In plotting Eq. (2) one implicitly assumes that wavelength interval $d\lambda$ is constant.

Equation (2) is plotted versus wavelength as the leftmost curve in Fig. 2 and versus photon energy as the rightmost curve in Fig. 1. The peaks in these curves occur where

$$\frac{2\pi\hbar c}{kT\lambda} = \frac{\hbar\omega}{kT} = 4.965.$$

For $T = 5800$ K, this peak of $dJ_\lambda/d\lambda$ occurs at a photon energy of 2.48 eV, corresponding to a wavelength of 500 nm, which is in the green.

We are left with the disconcerting result that the shape of a spectrum and the position of its peak depend on whether the irradiance is considered over a constant interval of photon energy or of wavelength.[3] The difference in spectral shapes is due to the fact that, with increasing wavelength, intervals of constant wavelength correspond to increasingly narrow intervals of photon energy.

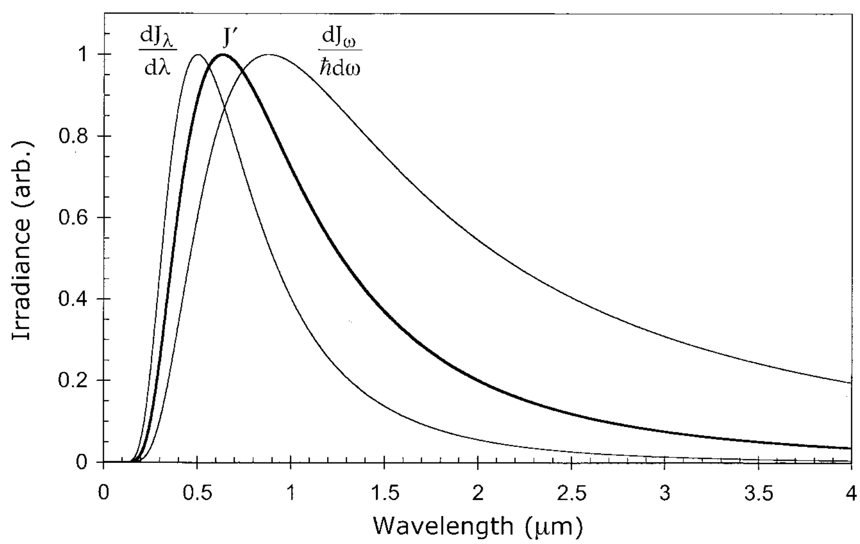

Fig. 2. Normalized spectral output for a 5800 K blackbody source approximating the Sun versus wavelength. The rightmost curve ($dJ_\omega/\hbar d\omega$) is intensity per-unit-photon energy from Eq. (1), the leftmost curve ($dJ_\lambda/d\lambda$) is intensity per-unit wavelength from Eq. (2), and the central curve J' is intensity per-unit fractional bandwidth from Eq. (3). The respective wavelengths and heights of the peaks are 879 nm and 2.81 kW cm^{-2} eV^{-1} for $dJ_\omega/\hbar d\omega$, 500 nm and 8.44 kW cm^{-2} μm^{-1} for $dJ_\lambda/d\lambda$, and 633 nm and 4.72 kW cm^{-2} for J'.

3. Constant-Fractional-Bandwidth Approach

It is desirable to find a unique expression for the spectrum that is independent of whether the intervals under consideration are of photon energy or of wavelength. Furthermore, this approach should correspond naturally to how a spectrum is measured. I present a method that provides a spectrum having a unique peak position and then justify the approach.

Instead of the constant photon-energy interval of Eq. (1), an interval having a constant fractional bandwidth of the spectrum can be chosen. The fractional bandwidth is $\Delta\omega/\omega$ or, in the limit, $d\omega/\omega$. The term $\hbar d\omega$ in Eq. (1) is replaced by $\hbar d\omega/\hbar\omega$. The resulting expression for the irradiance per unit fractional photon energy, $dJ_\omega/(\hbar d\omega/\hbar\omega)$, expressed as a function of $\hbar\omega$ and of λ, is

$$J' = \frac{\hbar}{4\pi^2c^2}\frac{\omega^4}{\exp(\hbar\omega/kT)-1} = 4\pi^2\hbar c^2\frac{1}{\lambda^4[\exp(2\pi\hbar c/kT\lambda)-1]}. \tag{3}$$

As with the above constant-photon-energy-interval case, instead of the constant wavelength interval of Eq. (2), an interval with a constant fractional bandwidth of the spectrum can be chosen. The term $d\lambda$ in Eq. (2) is replaced by $\Delta\lambda/\lambda$ or, in the limit, $d\lambda/\lambda$. The resulting expression for the irradiance per-unit fractional wavelength, $dJ_\lambda/(d\lambda/\lambda)$, is identical to Eq. (3).

Thus curves of the irradiance per-unit fractional photon energy, $dJ_\omega/(\hbar d\omega/\hbar\omega)$, and per-unit fractional wavelength, $dJ_\lambda/(d\lambda/\lambda)$, have the same shape and the same peak wavelength. The J' of Eq. (3) may therefore be identified as a general expression for blackbody irradiance per unit fractional bandwidth. The dimensions for J' are simply power density.

Equation (3) is plotted as the central curves in Figs. 1 and 2. The peak of J' occurs where

$$\frac{2\pi\hbar c}{kT\lambda} = \frac{\hbar\omega}{kT} = 3.921.$$

For $T = 5800$ K this peak is at a photon energy of 1.96 eV, corresponding to a wavelength of 633 nm, which is in the red.[4]

This equivalence of irradiance per unit fractional photon energy and per unit fractional wavelength for blackbody radiation generalizes to any spectrum. This can be shown by taking the derivative of both sides of $\omega = 2\pi c/\lambda$. The result is $d\omega/\omega = -d\lambda/\lambda$. Replacing $\pm d\omega$ by the interval $\Delta\omega$ and $\pm d\lambda$ by the interval $\Delta\lambda$ yields the general result

$$\Delta\omega/\omega = \Delta\lambda/\lambda. \tag{4}$$

Thus the fractional photon energy is equivalent to the fractional wavelength at any part of a spectrum.

4. Discussion

From the discussion above, expressing the irradiance per-unit fractional bandwidth appears to be advantageous. A unique expression is obtained whether the spectrum is considered over wavelength or over photon energy. I now examine whether this approach is consistent with how a spectrum is measured using common instruments.

The conventional way to define the chromatic discrimination of a detector is in terms of the chromatic resolving power $\lambda/\Delta\lambda$,[5] where $\Delta\lambda$ is the smallest resolvable wavelength difference measurable at a mean wavelength λ. This resolving power is just the reciprocal of the fractional bandwidth. Thus there is a natural link between the spectral discrimination of a detector and the spectral output of a source when the output is expressed per constant interval of the fractional bandwidth.

For a grating spectrometer the chromatic resolving power is

$$\lambda/\Delta\lambda = |m|N, \tag{5}$$

where m is the order number for the diffraction (constant for a particular measurement) and N is the number of grooves illuminated uniformly on the grating.[6] Therefore, for a grating spectrometer, $\lambda/\Delta\lambda$ and hence the fractional bandwidth are constant.

For a prism spectrometer a similar argument can be made within limits. The prism chromatic resolving power is

$$\frac{\lambda}{\Delta\lambda} = t\left|\frac{dn}{d\lambda}\right|, \tag{6}$$

where t is the greatest thickness of the prism through which the rays pass and n is the refractive index.[6] Within a limited spectral range the dispersion is often close to being constant, and, consequently, $dn/d\lambda$ is approximately constant. Therefore, to the extent that the dispersion is constant within the spectral range of interest, for a prism spectrometer, $\lambda/\Delta\lambda$ and hence the fractional bandwidth are constant.

When measuring the solar spectrum with conventional spectrometers, one obtains the constant-fractional-bandwidth curve J' of Fig. 2 rather than the more commonly plotted constant-wavelength-interval curve $dJ_\lambda/d\lambda$. It appears to be advantageous to leave the curve in its natural form, per-unit fractional bandwidth, rather than renormalizing the data per-unit wavelength.

Obtaining the integrated irradiance over a finite interval of the spectrum is straightforward. If the integration is over wavelength, one must integrate $(J'/\lambda)d\lambda$, or, if the integration is over photon energy, one must integrate $(J'/\omega)d\omega$.

5. Conclusions

Any optical spectrum can be expressed as irradiance per-unit fractional bandwidth rather than per-unit wavelength or per-unit photon energy (or frequency). With this normalization an expression for the spectral output of a source then corresponds to constant spectral resolving power. The relative magnitude at each point in these spectra is directly proportional to that which would be measured with common spectrally resolving detectors.

Furthermore, expressions of irradiance per-unit fractional bandwidth provide unambiguous curves that do not depend on whether the output is considered per-unit wavelength or per-unit photon energy and simply have dimensions of power density. For the case of the Sun this approach yields a spectrum that peaks in the red at a wavelength of 633 nm.

The author thanks Kelvin Wagner and W. Thomas Cathey for helpful comments on constant-fractional-bandwidth normalization.

References and Notes

1. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 3rd ed. (Butterworth-Heinemann, Oxford, 1980), pp. 184–187.
2. This is the irradiance at the surface of the Sun. To obtain the irradiance intercepting the Earth, this expression must be divided by the square of the ratio of the Earth–Sun distance to the radius of the Sun.
3. B. H. Soffer and D. K. Lynch, "Some paradoxes, errors, and resolutions concerning the spectral optimization of human vision," Am. J. Phys. **67,** 946–953 (1999).
4. Although the spectrum is centered in the red, we do not perceive the Sun as being red. This can be explained by the fact that the spectrum covers our visible range and as a result of our physiological ability to redefine perceived colors with reference to the illumination source. See R. Mausfeld, "Color perception: from Grassmann codes to a dual code for object and illumination colors," in *Color Vision, Perspectives from Different Disciplines*, W. G. K. Backhaus, R. Kliegl, and J. S. Werner, eds. (de Gruyter, Berlin, 1998), Chap. 12.
5. M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1980), p. 333.
6. Ref. 5, pp. 406–407.