Incorporating non-unitary income elasticities, choke prices and choke incomes into applied general-equilibrium models

James R. Markusen
Department of Economics
University of Colorado, Boulder
https://www.colorado.edu/faculty/markusen

Abstract

Traditional applied general-equilibrium (AGE) models have always faced trade-offs between analytical and computational tractability and counter-empirical restrictions. One is the assumption of homothetic preferences implying unitary income elasticities of demand, significantly inconsistent with data. Similarly, there is no "choke" income level, below which a certain good is not purchased and there is no choke price above which a good is not purchased, implying no changes in the extensive margin of trade. Here I analyze three different formulations of preferences which incorporate these features. The first two, Stone-Geary Modified (SGM) and direct explicit additivity (here labeled CRIE - constant relative income elasticity) have been rarely used due to inherent limitations. I avoid these constraints by introducing virtual substitute goods into a standard general-equilibrium framework. While other authors have explored these properties in alternative ways, my approaches have considerable advantages for high-dimension simulation models in that they retain homothetic CES structures and functional forms so that they can slot right into existing modeling formats. I parameterize CRIE using US time-series data, and show that the ratio of the income elasticities of demand for services to goods is estimated at 1.510, and the elasticity of substitution between goods and services estimated as 0.478.

May 2026

Comments and additional references most appreciated. Two of the three numerical models in the paper require a solver for non-linear complementarity problems to compute boundary solutions and regime shifts. I use the solver PATH, accessed from GAMS, but also available through other software including R and Julia, maybe Matlab. See https://pages.cs.wisc.edu/~ferris/path.html

JEL codes: F10, F17, C63, C68

Key words: income elasticities, choke incomes, choke prices, applied general equilibrium, non-homothetic preferences, CES functions.

1. Introduction

High-dimension applied general-equilibrium modeling (AGE) is now 50+ years old, with theoretical and algorithmic roots going back further. A good history of early techniques is Shoven and Whalley (1984). It is fair to say that these early efforts had to impose strict limits as to the types of microeconomic features that they could incorporate. A second major development at about the same time was the formulation of an alternative approach based on complementarity developed by Mathiesen (1985), who extended the mathematical programming (optimization) theorems of Karush (1939) and Kuhn and Tucker (1951) to economic equilibrium problems. This alternative was then computationally operationalized by Rutherford (1985, 1995) in his software mps/ge (mathematical programming system for general equilibrium).

Many limitations remain of course, some due to data restrictions with others due to a lack of insights as to how to convert empirically-relevant features into algebra and code. The purpose of this paper is to try to make progress on two constraining restrictions that are clearly counter-empirical. The first of these, in no particular order, is the continuing reliance on homothetic preferences: all income elasticities of demand are unity. Simple functional forms, CES in particular, allow for solutions to unit cost and expenditure functions depending only on prices, with the application of Shepard's lemma to these costs functions giving unit demands for goods and factors. These features immensely simplify high-dimension general-equilibrium models by neutralizing a critical endogeneity between trade, prices and per-capita income.

Data do not support this convenient assumption. Empirical estimates of income elasticities showing significant deviations from unity are presented in Caron, Fally and Markusen (2014, 2020), with further estimation and counter-factual simulations showing important quantitative effects on trade volume and factor prices from incorporating non-homotheticity. Table 1 is cross-section estimation from Caron, Fally and Markusen (2014) which lists estimated income elasticities for the GTAP data base, sorted in order from smallest to largest. The sample statistics (mean and standard deviations) on the right show a rather substantial departures from unity. Sectors, with an acknowledged bit of arbitrariness, are classified into primary, manufacturing and services. A simple OLS regression of income elasticities on 0-1 dummies for the three aggregate sectors is shown on the right: primary sectors have the lowest income elasticities, manufacturing in the middle, and services having the highest. Below those is the regression for just goods (primary plus manufacturing) and services.

Additional motivation is provided in Figure 1, where the top panel shows the shares of consumer expenditure on goods and services since 1950 and the growth of GDP per capita in constant dollars. While this fits nicely with Table 1, it could be that Figure 1 is explained by relative price changes between goods and services. The lower panel of Figure 1 shows the actual relative price change of services to goods, rising from (normalized) 1.0 to 3.4. Combined with literature estimates of the elasticity of substitution σ between goods and services of 0.5, this secular price change could contribute in part to the top-panel's data. But if we assume that preferences are homothetic CES and the elasticity of substitution is $\sigma = 0.5$, the relative price of services would have to rise by a factor of 11.4 in order to explain the share ratio change (also

Table 1: GTAP Sectors and Income Elasticities:
Caron, Fally, Markusen QJE 2014

GTAP sectors	Income elasticities	primary	manufac	services
Cereal grains nec	0.110	x		
Paddy rice	0.254	x		
Processed rice	0.352	x		
Sugar cane, sugar beet	0.433	x		
Animal products nec	0.444	x		
Bovine cattle, sheep and goats, horses	0.458	x		
Vegetable oils and fats	0.545	x		
Sugar	0.588	x		
Forestry	0.623	x		
Vegetables, fruit, nuts	0.640	x		
Petroleum, coal products	0.664	x	x	
Beverages and tobacco products	0.667		x	
Textiles	0.707		x	
Food products nec	0.777		x	
Dairy products	0.826		x	
Electricity	0.848		x	
Mineral products nec	0.874		x	
Chemical, rubber, plastic products	0.880		x	
Construction	0.880			x
Wheat	0.883	x		
Fishing	0.886	x		
Oil seeds	0.889	x		
Crops nec	0.893	x		
Air transport	0.929			x
Water transport	0.932			x
Machinery and equipment	0.938		x	
Wood products	0.970		x	
Transport equipment nec	0.981		x	
Leather products	0.981		x	
Transport nec	0.990			x
Metal products	0.992		x	
Bovine meat products	1.023		x	
Public Administration and services	1.033			x
Motor vehicles and parts	1.034		x	
Water	1.039			x
Paper products, publishing	1.044		x	
Meat products nec	1.052		x	
Wearing apparel	1.057		x	
Recreational and other services	1.075			x
Electronic equipment	1.094		x	
Manufactures nec	1.095		x	
Trade	1.106			x
Raw milk	1.118	x		
Communication	1.152			x
Business services nec	1.324			x
Financial services nec	1.331			x
Plant-based fibers	1.339	x		
Insurance	1.392			x
Wool, silk-worm cocoons	1.420	x		
Gas manufacture, distribution	2.221		x	x

Unweighted mean and standard deviation

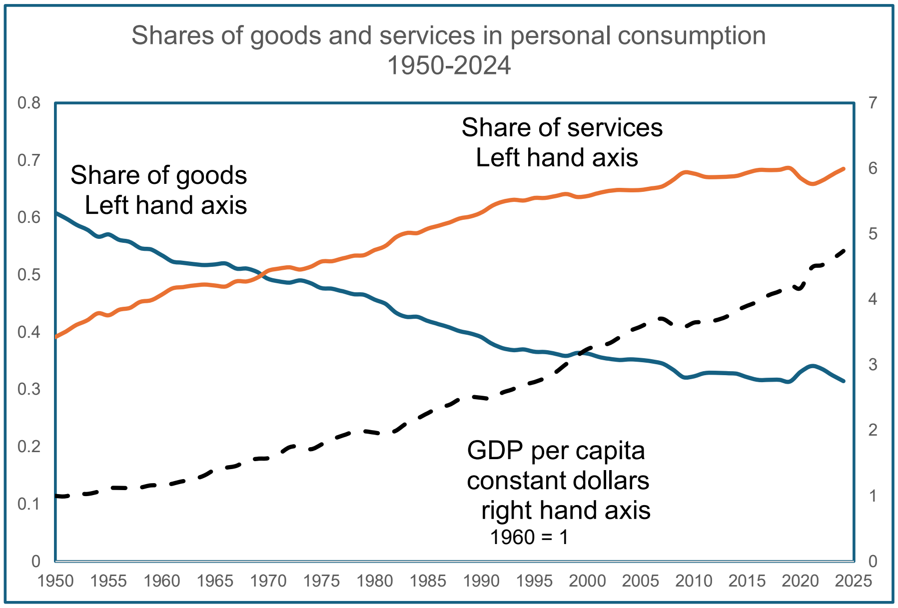
Mean	0.916
Standard deviation	0.342
Mean plus one SD	1.258
Mean minus one SD	0.573

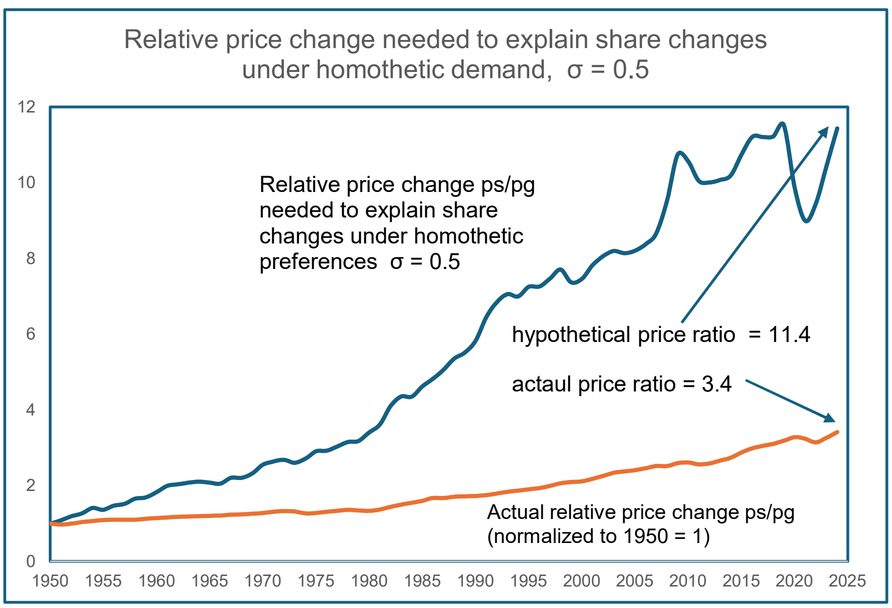
Regression of income elasticity on sector 0-1 dummies

	coefficient	standard error
Primary	0.646	0.066
Manufac	0.903	0.061
Services	1.115	0.078
Adjusted R2		0.899

	coefficient	standard error
Goods	0.828	0.043
Services	1.121	0.074
Adjusted R2		0.907

Figure 1: US personal consumption expenditure:
shares of goods and services in total, 1950-2024
σ = elasticity of substitution between goods and services





The actual relative price ps/pg tracks the relative shares G/S (top panel, not shown) closely.

=> With homothetic preferences, relative price and share changes are only consistent if σ approximately 0: fixed coeifficient preferences

https://www.bea.gov/itable/national-gdp-and-personal-income

National data
National Income and Product Accounts
Table 2.3.5. Personal Consumption Expenditures by Major Type of Product
Publication Category Interactive Data NIPA Tables
Table 1.6.4 Price Indecies for Gross Domestic Purchases

about a factor of 3.4). To put it another way, with homothetic CES, the actual change in the share ratio and the actual change in the relative price (both about 3.4) are only consistent if the elasticity of substitution is approximately $\sigma = 0$: fixed coefficient preferences.

In addition, the homotheticity assumption does not allow for "choke" income levels below which certain goods or services are not purchased. This is clear counter-empirical since the world trade matrix is full of zeros (export from country i to country j in good k). The general tradition in AGE modeling is to hold initially-zero cells fixed at zero when doing counter-factuals. Thus there can be no changes in trade at the extensive margin in simulations. Without in any way demeaning their important contributions, this was embodied in the Armington (1969) assumption and in the popular Eaton-Kortum (2002) approach. These formulations also imply that any good that has a positive entry in the trade matrix can never go to zero. In the terminology of this paper, there is no "choke" income level for produced and traded goods.

The flip side of the absence of choke income levels is the absence, by assumption, of choke prices above which a good is not purchased. With CES preferences with any value for the elasticity of substitution σ, the demand price for a good goes to infinity as the quantity goes to zero. So any good produced anywhere in the world will always be imported be every single country no matter how high the tariff or transport cost. Again, the exceptions are trade cells initially zero and constrained to remain zero in running scenarios.

Figure 2 provides a motivating example of why income elasticities are important for general-equilibrium counter-factuals. Suppose that points A and B are observations on the per-capita income of two households/countries at the same set of prices for goods X_1 and X_2. There are two very different ways of calibrating preferences. One, which is the normal way in general-equilibrium modeling, is to assume that the countries have different, but homothetic preferences. If the countries' per-capita incomes grow in the same proportion, the Engel's curve for the sum of their purchases moves out along the steeper line. On the other hand, if we assume that the countries have identical but non-homothetic preferences, then equal growth in their incomes moves aggregate consumption out along the flatter (not necessarily linear) income curve. Beginning with the same data, the two options generate very different counter-factual results.

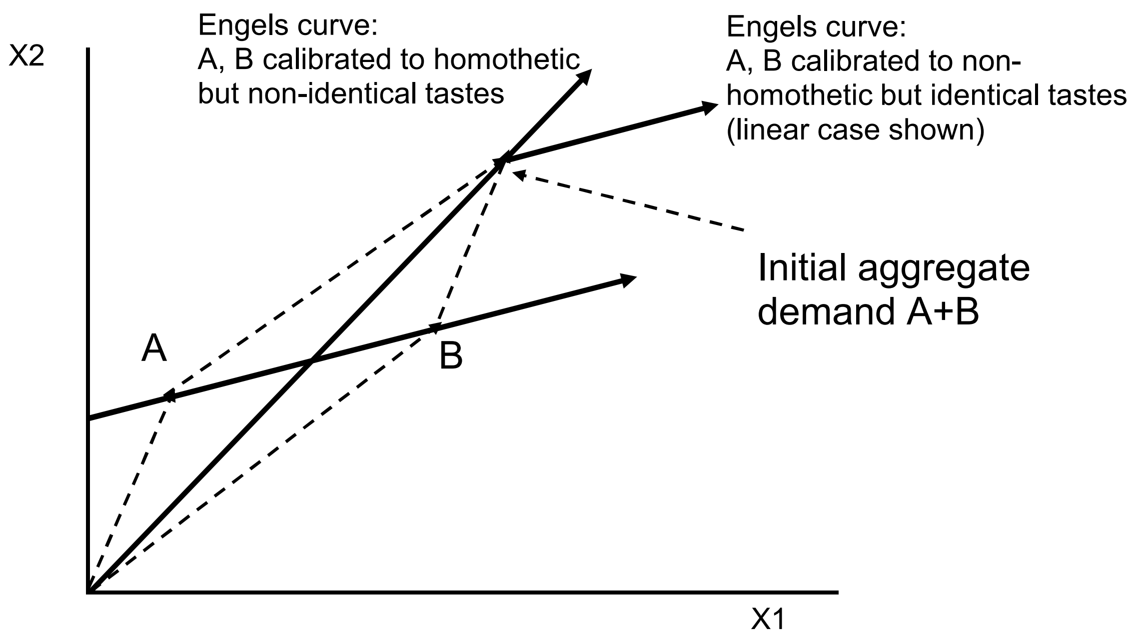
My first alternative proposes and analyzes a simple framework that allows for (a) non-unitary income elasticities of demand, (b) choke income levels below which a good is not purchased, and (c) choke prices above which a good is not purchased. These then allow for rich possibilities in counter-factuals including changes in the extensive marginal of trade (trade from i to j in good k switching on or off), reversals in the direction of trade, and an important role for per capita income in explaining trade flows.

My formulation is a straightforward modification of traditional Stone-Geary preferences (a great review and extensions to the AIDS formulation is in Deaton and Muellbauer (1980)). Whereas the Stone-Geary version has traditionally used negative parameters subtracted from consumption purchases and interpreted as minimum consumption requirements (e.g., Markusen, 1986), my version uses positive endowment parameters which could be thought of as (unobserved) fixed endowment goods/leisure time which are perfect substitutes for the

Figure 2: Why this is important for counter-factuals

Example: endowments X1E >0, X2E = 0
X1 has income elasticity of demand > 1, X2 income elasticity < 1

Calibrate (per capita) observations A and B
Homothetic but non-identical tastes
Non-homothetic but identical tastes



Alternative calibrations will yield very different counter-factual results
when incomes rise or fall

(capital accumulation, technical
change, lower trade costs)

consumption goods.[1] This implies that goods with large endowment parameters relative to their utility shares have income elasticities great than one, and have maximum choke prices and minimum choke income. This specification has been previously used by Markusen (2013), Simonovska (2015) and Jung, Simonovska and Weinberger (2019). The latter two papers are focused on price elasticities whereas this current paper is more interested in income elasticities.

My second alternative is to modify CRIE (constant relative income elasticities) preferences, an idea that has a long but small history: Hanock, (1975), Chao, Kim and Manne (1982). They have been exploited in their most basic form (direct explicit additivity in Hanock's terminology) in Fieler (2011) and Caron, Fally and Markusen (2014, 2020) in trade models. They do not allow for choke income and prices but offer advantages when there are large differences in per-capita income.. There is some commonality to my SGM version, in that the CRIE version uses endowed sector-specific fixed factors which, combined with purchased goods, map into consumption goods. As in the case of SGM, the resulting cost functions are all homothetic CES. My version also has the advantage of allowing for the independence of income and price elasticities which has not been possible in earlier analyses.

The third alternative is to assume traditional homothetic CES Armington preferences (domestic and foreign goods in an industry are imperfect substitutes), but adds inefficient latent domestic activities that can produce perfect substitutes for foreign varieties if prices are high enough. This introduces choke prices, although not choke incomes, on foreign imports, endogenizing zeros in the world trade matrix. Figure 3 gives a quick look at the three cases which may be useful early on: details to follow in subsequent sections.

I hope my formulations are valuable to researchers working with AGE models, either structural or calibrated. They retain homothetic CES preferences with all their advantages in large simulation models. These include simple solutions to cost and expenditure functions and the easy derivation of closed-form goods and factors demand equations via Shepard's lemma. Another advantage is that all functions are globally regular, making it an attractive choice for simulating large changes in trade costs, technologies, natural disasters etc. The first and third alternative do require that the models are formulated as non-linear complementarity problems, endogenizing corner/boundary solutions, but the software for that is well developed.

The paper includes an empirical exercise using the US time-series data shown in Figure 1 for CRIE preferences. The non-linear estimating equation is solved as an NLP (non-linear programming) problem for the CRIE parameters minimizing the sum of squared errors. Results are that the ratio of the income elasticity of demand for services to that for goods is 1.510. The estimated elasticity of substitution between goods and services is 0.480, which is consistent with literature estimates.

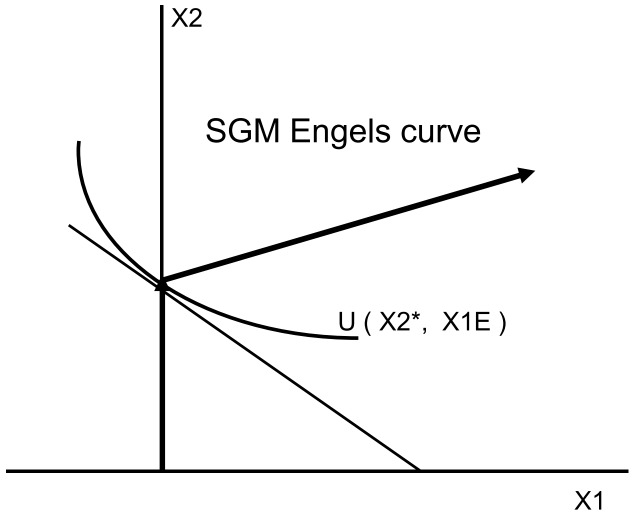
There have been many contributions focusing on non-homothetic preferences over the decades, and a few focusing on choke prices or incomes. But I am not aware of any that treat

---

[1] No one has to watch 3.5 hour American football games. "Free" endowment substitutes like leisure include going for a walk, hike, or cycle, birdwatching, playing with your kids, etc. Other applications involving "virtual" fixed-endowment goods/factors include modeling a large open economy and specific factors of production in order to add strict concavity to some function.

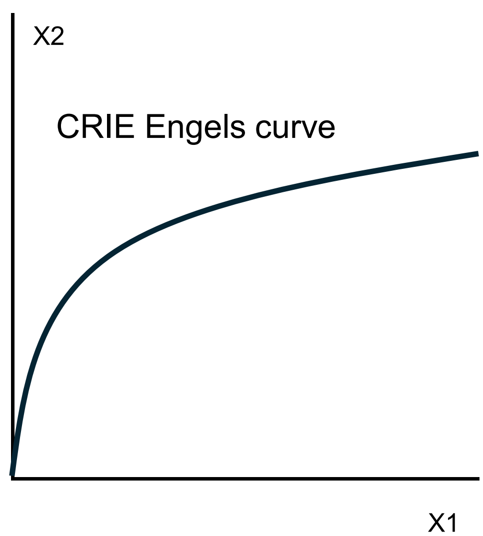
Figure 3: SMG, CRIE and homothetic Armington

X1 income elasticity > 1, X2 income elasticity < 1



Advantage: allows non-unitary income elasticities, choke incomes and choke prices

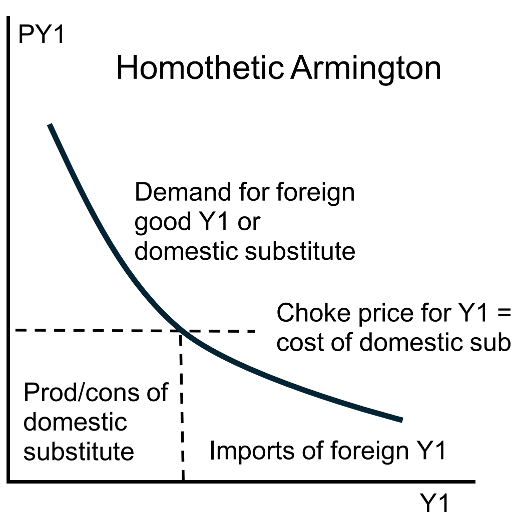
Disadvantage: income elasticities all converge to one as income grows large



Advantage: ratio of income elasticities constant over entire range of incomes

Allows for independence of price and income elasticities

Disadvantage: does not by itself allow for choke prices and incomes



Advantage: allows for choke incomes and choke prices while retaining all the simplicity of homothetic CES

Disadvantage: does not by itself allow for non-unitary income elasticities, but can be combined with one of the other options these simultaneously in the same model framework (references welcome!). Linder (1961) is an important early work that focuses on the role of per capita income in explaining trade in non-primary products, although there is no formal model in his work. Some of the earlier work that did formalize the role of per capita income in general equilibrium were Markusen (1986), Bergstrand (1990) and Hunter (1991). After an apparent gap, interest reappeared in the new millennium with contributions by Matsuyama (2000), Fieler (2011), Markusen (2013) and Caron, Fally and Markusen (2014, 2020). For large-dimension simulation models, CRIE's usefulness has been limited by the fact that there are no close-form solutions for cost or demand functions, but I have found a work-around that retains all the features of homothetic CES.

Some other valuable contributions in this same time period include Simonovska (2015), Fajgelbaum and Khandelwal (2016), and Bertoletti, Federico and Simonovska (2018), Jung, Simonovska and Weinberger (2019) and Matsuyama (2019). These papers generally focus on different issues not closely related to this current paper (e.g., industrial-organization and income-distribution questions) and/or also use alternative preferences (e.g., indirect implicit additivity) which do not lend themselves easily to multi-sector general-equilibrium models. A number of references to papers finding an elasticity of substitution about 0.5 across broad sectors are found in Heblich, Redding, and Zylberberg (2025).

The three papers by Simonovska and co-authors just mentioned do feature choke prices. Wales and Woodland (1985) is a significant early contribution which is unfortunately unknown to trade economists. Wales and Woodland uses quadratic preferences which are also not easy to accommodate in large AGE models. Much more recently, Eaton, Kortum and Sotelo (2013), Gandri, Lu and Shi (2023) and Anderson and Zhang (2025) focus on the econometrics of endognizing zeros. Like Wales and Woodland, these combine theory and estimation, but are perhaps not particularly easy to slot into large simulations models. Fajgelbaum and Khandelwal and Anderson and Zhang use an AIDS formulation derived from Stone-Geary.

2. Stone-Geary-modified preferences, demand, income and price elasticities

First, let's consider the basics in just a partial-equilibrium framework: prices and money income held constant. For each purchased good, there is an unobserved perfect-substitute endowment good (some of which may be zero). This is the notation for the simple case used to derive demand functions, and price and income elasticities of demand.

Non-negative variables

X_{ip}	amount of good i purchased from the market	(*p* for purchased)
X_{ic}	amount consumed: purchase plus endowment	(*c* for consumed)
p_{ic}	implicit consumer price of good i - equal to p_i if $X_{ip} > 0$, less if no i purchased	
e	CES price index	
MT	total "income" including the value of the endowments	
W	welfare (utility)	

Parameters

X_{ie}	perfect substitute endowment good $X_{ie} \geq 0$, can't be traded (*e* for endowment)
p_i	market price of purchased good i - parameter in partial equilibrium model
β	substitution parameter
α_i	share parameters on the goods
σ	elasticity of substitution among goods
M	money income that can be spent in the market - parameter in the PE model

The modified Stone-Geary (SGM) welfare (utility) function is a simple case, but the quantities of each good are the sum of the purchased amount plus the perfect substitute endowment good.

$$W = \left[\sum_i \alpha_i (X_{ip} + X_{ie})^\beta\right]^{1/\beta} \quad 0 < \alpha_i < 1, \quad -\infty < \beta < 1, \quad \sum_i \alpha_i = 1, \quad X_{ie} \geq 0 \qquad (1)$$

Definitional equations for X_i consumed (sum of purchases and endowments), elasticity of substitution σ, and income including the value of endowment.

$$X_{ic} = X_{ip} + X_{ie} \qquad \sigma = \frac{1}{1-\beta} \qquad MT = M + \sum_i p_{ic} X_{ie} \qquad (2)$$

CES price index giving the cost of one unit of welfare.

$$e = \left[\sum_i \alpha_i^\sigma p_{ic}^{(1-\sigma)}\right]^{\frac{1}{1-\sigma}} \qquad (3)$$

Demand function for X_{ic}.

$$X_{ic} = X_{ip} + X_{ie} = (\alpha_i/p_{ic})^\sigma e^{\sigma - 1} MT \qquad X_{ip} > 0 \iff (\alpha_i/p_{ic})^\sigma e^{\sigma-1} MT > X_{ie} \qquad (4)$$

Figure 3 top panel shows this basic two-good case where $X_{1e} > 0$ and $X_{2e} = 0$. X_{1p} is constrained to be non-negative, so the Engel's curve for observed purchases runs up the X_2 axis to the choke point, and then is linear as show holding prices constant. At some set of income and prices M^*, p_1^*, p_2^* the solution is $X_{1p} = 0$, $X_{2p|} = M^*/p_2^*$ and $X_{1c} = X_{1e}$.

The SGM formulation automatically ensures that no endowment is sold. Multiple through by p_{ic} and sum over i.

$$\sum_i p_{ic} X_{ic} = \sum_i (\alpha_i^\sigma p_{ic}^{1-\sigma}) e^{\sigma-1} MT = e^{1-\sigma} e^{\sigma - 1} MT = MT \qquad (5)$$

$$\sum_i p_{ic} X_{ic} = \sum_i (p_{ic} X_{ip} + p_{ic} X_{ie}) = M + \sum_i p_{ic} X_{ie} \implies \sum_i p_{ic} X_{ip} = M \qquad (6)$$

Thus market purchases equal money income so adding up is always (globally) satisfied. Note that it will be the case that $p_{ic} = p_i \quad if \quad X_{ip} > 0$ (perfect substitutes X_{ip} and X_{ie} will have the same price if there are market purchases $X_{ip} > 0$). The implicit price p_{ic} will be less than the market price p_i if no market is purchased: $X_{ip} = 0$.

Equation (4) can be used to solve for the "choke" income level, the value of M below which there is no X purchased at given prices, and also for the choke price above which no X is purchased for the income level and other goods' prices. For good X_1, these values are implicitly

$$(\alpha_1/p_{1c})^{\sigma} e^{\sigma - 1}(M + \sum_i p_i X_{ie}) \leq X_{1e} \qquad => \qquad X_{1p} = 0 \tag{7}$$

The Cobb-Douglas case with $\sigma = 1$ makes this easier to interpret

$$X_{1p} = 0 \quad <=> \quad \alpha_1(M + \sum_i p_i X_{ie}) \leq p_1 X_{1e} \tag{8}$$

with the choke income level M^* give by

$$M_1^* = Max\left[\frac{1}{\alpha_1}(p_1 X_{1e} - \alpha_1 \sum_i p_i X_{ie}),\ 0\right] \qquad M_1^* > 0 \quad <=> \quad \frac{p_1 X_{1e}}{\sum_i p_i X_{ie}} > \alpha_1 \tag{9}$$

M^* is strictly positive for good X_1 iff its value share of the endowment is great than its utility weight, otherwise there is no choke income level. If $X_{ie} = 0$, there is no choke income level.

From (8) (Cobb-Douglas), the choke price for X_1 given M and prices of other goods is

$$p_1^* = Min\left[\frac{\alpha_1(M + \sum_{j \neq i} p_j X_{je})}{(1 - \alpha_1)X_{1e}},\ +\infty\right] \qquad p_1^* < +\infty \quad <=> \quad X_{1e} > 0 \tag{10}$$

A particular simple case for capturing the intuition occurs when good 1 is the only one with a positive endowment.

$$p_1^* = \frac{\alpha_1 M}{(1 - \alpha_1)X_{1e}} \qquad \text{special case with Cobb-Douglas, only } X_{1e} > 0. \tag{11}$$

The top panel of Figure 3 shows this basic case where $X_{1e} > 0$ and $X_{2e} = 0$ (nothing in the picture relies on Cobb-Douglas).

Now turn to income elasticities of demand. The expenditure on all X_i consumption (purchased plus endowment) as a share of all income (MT = money income plus endowment income) is given by

$$\frac{p_{ic}X_{ic}}{MT} = \frac{\alpha_i^{\sigma} p_{ic}^{1-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} = s_i \qquad p_{ic}X_{ic} = p_{ic}(X_{ip} + X_{ie}) = s_i(M + \sum_i p_{ic}X_{ie}) \tag{12}$$

There is one qualifying limitation here: the share defined in this way is the share of total consumption including the endowment quantity $X_{ic} = X_{ip} + X_{ie}$ in total income including the endowment income. That is not the quantity of X and the money income M observed in the data. The amount of purchased, X_{ip}, is given by

$$X_{ip} = \frac{s_i M}{p_{ic}} + \frac{s_i \sum_i p_{ic}X_{ie}}{p_{ic}} - X_{ie} \tag{13}$$

Let $X_{ih} = s_i M / p_{ic}$ denote the purchase of X_i under homothetic demand with no endowments. Note also from (13) that X_{ih} is the asymptotic amount of X_{ip} purchased when M grows arbitrarily large. Alternatively, s_i is the marginal share of added income spent on X_{ip}. Equation (14) then implies that the purchase of a good is less than what it would be under standard homothetic preferences (X_{ih}) when the following holds:

$$X_{ih} = \frac{s_i M}{p_{ic}} > X_{ip} \qquad <=> \qquad \frac{p_{ic}X_{ie}}{\sum_i p_{ic}X_{ie}} > s_i \tag{14}$$

The amount purchased X_{ip} is less than the amount under homothetic preferences (all $X_{ie} = 0$) if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}.

The income elasticity of demand for X_{ip} with respect to income M is found from (13).

$$\frac{M}{X_{ip}} \frac{dX_{ip}}{dM} = \frac{s_i M}{s_i M + s_i \sum_i p_{ic}X_{ie} - p_{ic}X_{ie}} \tag{15}$$

$$\frac{M}{X_{ip}} \frac{dX_{ip}}{dM} > 1 \qquad <=> \qquad \frac{p_{ic}X_{ie}}{\sum_i p_{ic}X_{ie}} > s_i \tag{16}$$

Note that the right-hand inequalities in (14) and (16) are the same. The income elasticity of demand for X_{ip} is great than one if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}. For example, with two goods and a positive endowment for only X_1, the income elasticity will be greater than one for X_1 and less than one for X_2. But (15) also implies that all the income elasticities converge to one as (money) income becomes large. This property is specific to the Stone-Geary formulation, but for all non-homothetic preferences the income elasticities of demand for high elasticity goods must fall toward one as income grows large. This is required by the budget constraint, and the algebra will be presented later.

The derivation of the own price elasticity of demand for the purchases of good X_{ip} (money income held constant) is more complex since p_{ic} appears a number of times in both the numerator and denominators of (1). The price elasticity of total consumption including the endowment and holding MT constant (or assume zero endowments, same thing) is given by a fairly well-known formula ($MT = M + \sum_i p_{ic} X_{ie}$).

$$X_{ic} = \frac{\alpha_i^{\sigma} p_{ic}^{-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} MT \qquad - \frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = [\sigma - s_i(\sigma - 1)] \quad (MT \text{ held constant}) \tag{17}$$

The step-by-step derivation for the zero-endowment homothetic case is found in my pedagogic article Markusen (2023). But for data purposes, we want the price elasticity of purchases, X_{ip}, and allow for MT to change with the price of the good. I provide separately a short supplement which shows that price elasticity of purchased X_{ip} allowing MT to change with p_{ic} is given by

$$-\varepsilon_{ip} = \frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = [\sigma - s_i(\sigma - 1)] + (1 - s_i)\sigma \frac{X_{ie}}{X_{ip}} \geq [\sigma - s_i(\sigma - 1)] = -\varepsilon_{ih} \tag{18}$$

where ε_{ih}, the term in square brackets, is the formula for the own price elasticity under zero endowments as noted in (18) (h for homothetic). Perhaps to make it a little more intuitive: the Cobb-Douglas case σ = 1, where from (13) s_i reduces to $s_i = \alpha_i$, (18) is given by

$$- \frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = 1 + (1 - \alpha_i) \frac{X_{ie}}{X_{ip}} \geq 1 = -\varepsilon_{ih} \qquad \text{Cobb-Douglas: } \sigma = 1, \quad s_i = \alpha_i \tag{19}$$

As the price of X_i rises, X_{ip} falls, and so the price elasticity of demand becomes larger (more negative) in both the general and Cobb-Douglas cases . The price elasticity approaches -1 as the price falls toward zero (X_{ip} heads toward infinity) and equals -1 with zero endowment as is well known for Cobb-Douglas. In the general case (18), the formula approaches ε_{ih} as the price falls to zero because X_{ip} heads toward infinity.

3. Small open-economy general-equilibrium model

I begin with a very simple, straight-forward case to illustrate the determination of choke prices and income. There are two goods, X_1 and X_2 with prices p_{x1} and p_{x2}, and two factors of production L and K in inelastic supply, with (endogenous), prices p_l and p_k. One consumer with fixed non-traded endowments X_{1e} and/or X_{2e} with X_{1e} and X_{2e} being perfect substitutes for the produced and trade goods respectively.

The country faces fixed world relative prices for traded goods X_1 and X_2 (no Armington product differentiation). If a good is not purchased (not produced or imported), then its consumer price p_{xic} will be determined by its endowment substitute only and will be less than the world price: the cost inequalities for domestic production or imported X_i are strict. Several other variables and notation are as follows. Over bars indicate fixed values.

$c_{x1}(p_l, p_k)$ $\quad c_{x2}(p_l, p_k)$ $\quad c_w(p_{x1c}, p_{x2c})$ unit cost functions for X_1, X_2, and W
partials give unit demands for X_1, X_2, and W by Shepard's lemma

T_{12}	T_{21}	activities which export X_1 for X_2, X_2 for X_1 respectively the ratio of inputs to output is the world price ratio
tot		parameter which give world price ratio p_{x1}/p_{x2} in the T activities
CONS		consumer income including value of endowments

The model given below consists of 19 weak inequalities with 19 matched non-negative variables. These are divided into three blocks. The first are pricing inequalities derived from optimization conditions, the direction being marginal cost greater-than-or-equal to price, with the complementary variable being the output of that activity. Note that welfare is just treated as a produced good with price p_w, the consumer price index (cost of buying one unit of utility).

Price inequalities (e.g., marginal cost $\geq$ price) quantities as complementary variables

$$c_{x1}(p_l, p_k) \geq p_{x1} \quad \perp \quad X_1 \tag{20}$$

$$c_{x2}(p_l, p_k) \geq p_{x2} \quad \perp \quad X_2 \tag{21}$$

$$p_{x1} \geq p_{x1c} \quad \perp \quad X_{1p} \tag{22}$$

$$p_{x2} \geq p_{x2c} \quad \perp \quad X_{2p} \tag{23}$$

$$p_{x1e} \geq p_{x1c} \quad \perp \quad X_{1e} \tag{24}$$

$$p_{x2e} \geq p_{x2c} \quad \perp \quad X_{2e} \tag{25}$$

$$p_{x2}tot \geq p_{x1} \quad \perp \quad T_{21} \tag{26}$$

$$p_{x1} \geq p_{x2}tot \quad \perp \quad T_{12} \tag{27}$$

$$c_w(p_{x1c}, p_{x2c}) \geq p_w \quad \perp \quad W \tag{28}$$

The second block are market-clearing inequalities, with their direction being supply greater-than-or-equal to demand. The complementary variables are always prices. It is important for the modeler to emphasize that these good and factor demands on the right-hand side must be consistent with and derived from the cost functions above via Shepard's lemma.

Market clearing (e.g., supply $\geq$ demand) prices as complementary variables

$$X_1 \geq X_{1p} - T_{12} + T_{21}/tot \quad \perp \quad p_{x1} \tag{29}$$

$$X_2 \geq X_{2p} + T_{21} - T_{12}tot \quad \perp \quad p_{x2} \tag{30}$$

$$X_{1p} + X_{1e} \geq \partial c_w/\partial p_{x1c} W \quad \perp \quad p_{x1c} \tag{31}$$

$$X_{2p} + X_{2e} \geq \partial c_w/\partial p_{x2c} W \quad \perp \quad p_{x2c} \tag{32}$$

$$\bar{X}_{1e} \geq X_{1e} \quad \perp \quad p_{x1e} \tag{33}$$

$$\bar{X}_{2e} \geq X_{2e} \quad \perp \quad p_{x1e} \tag{34}$$

$$\bar{L} \geq \partial c_{x1}/\partial p_l X_1 + \partial c_{x2}/\partial p_l X_2 \qquad \perp \qquad p_l \tag{35}$$

$$\bar{K} \geq \partial c_{x1}/\partial p_k X_1 + \partial c_{x2}/\partial p_k X_2 \qquad \perp \qquad p_k \tag{36}$$

$$W \geq CONS/p_w \qquad \perp \qquad p_w \tag{37}$$

The final block is income-balance inequalities. Here there is just a single one for the representative consumer. This is effectively just a definitional equation, since it could be substituted into the market-clearing equation for welfare just above.

Income balance income is the complementary variable

$$CONS \geq p_l\bar{L} + p_k\bar{K} + p_{x1e}\bar{X}_{1e} + p_{x2e}\bar{X}_{2e} \qquad \perp \qquad CONS \tag{38}$$

4. A calibrated example with simulations

The following matrix represents a micro-consistent set of initial values for this model. Columns are inputs and output for the sectors of the model including the "production" of welfare and consumer income. Rows are markets with outputs (supplies) positive and inputs (demands) negative. Micro-consistency means that all row and column sums should be zero. There are two features to note. The first is that I have calibrated the model to zero trade initially (no Armington differentiation, so this is possible). Second, X_1 has a large amount of its perfect substitute endowment good, $p_{x1e} = 150$, while $p_{x2e} = 1$. This spread is much greater than the difference in consumption shares, so good X_1 will have an income elasticity of demand greater than one and vice versa for good X_2.

	Production Sectors						Trade		Welfare	Consumer
	X1	X2	X1P	X1E	X2P	X2E	TX21	TX12	W	CONS
Markets										
PX1	100		-100				-0	0		
PX2		100			-100		0	-0		
PL	-25	-75								100
PK	-75	-25								100
PW									351	-351
PX1C			100	150					-250	
PX1E				-150						150
PX2C					100	1			-101	
PX2E						-1				1

These values then allow for the numerical calibration of the cost functions for the two goods and welfare. For simplicity and clarity, I'll just impose Cobb-Douglas functional forms on the goods and welfare. The matrix of values then gives us the numerical versions for producing X_1, X_2 and welfare W.

$$c_{x1}(p_l, p_k) = p_l^{0.25} p_k^{0.75} \qquad\qquad c_{x2}(p_l, p_k) = p_l^{0.75} p_k^{0.25} \tag{39}$$

These give unit factor demand equations in (35) and (36) by Shepard's lemma as follows.

$$\frac{\partial c_{x1}(p_l, p_k)}{\partial p_l} = 0.25 p_l^{0.25-1} p_k^{0.75} \qquad \frac{\partial c_{x1}(p_l, p_k)}{\partial p_k} = 0.75 p_l^{0.25} p_k^{0.75-1} \tag{40}$$

$$\frac{\partial c_{x2}(p_l, p_k)}{\partial p_l} = 0.75 p_l^{0.75-1} p_k^{0.25} \qquad \frac{\partial c_{x2}(p_l, p_k)}{\partial p_k} = 0.25 p_l^{0.75} p_k^{0.25-1} \tag{41}$$

The cost (expenditure) function for producing one unit of utility is given by

$$c_w(p_{x1c}, p_{x2c}) = p_{x1c}^{\alpha_1} p_{x2c}^{\alpha_2} \qquad \alpha_1 = 250/351 \quad \alpha_2 = 101/351 \tag{42}$$

Shepard's lemma give the unit commodity demand functions for producing a unit of W in (43) as

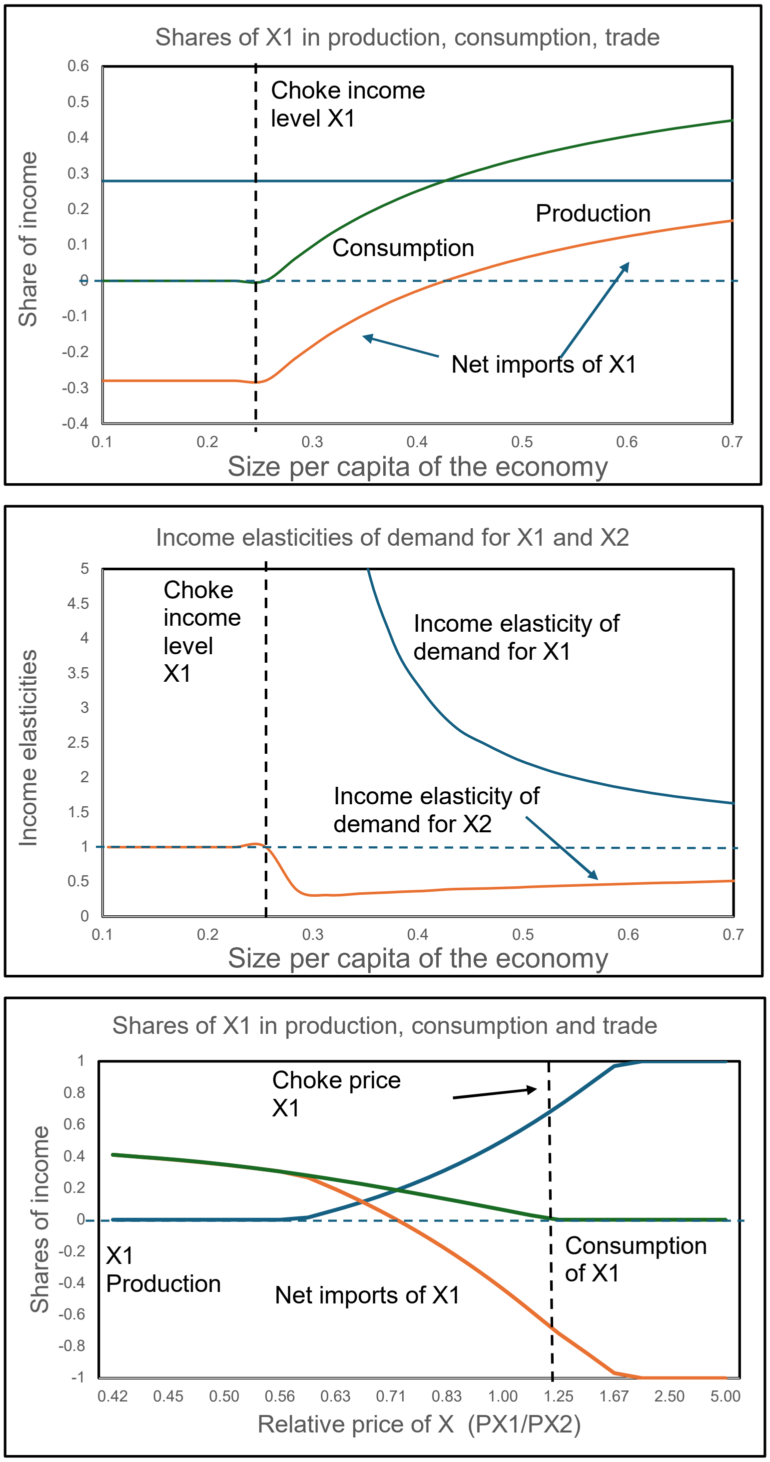
$$\frac{\partial c_w(p_{x1c}, p_{x2c})}{\partial p_{x1c}} = \alpha_1 p_{x1c}^{\alpha_1 - 1} p_{x2c}^{\alpha_2} \qquad \frac{\partial c_w(p_{x1c}, p_{x2c})}{\partial p_{x2c}} = \alpha_2 p_{x1c}^{\alpha_1} p_{x2c}^{\alpha_2 - 1} \tag{43}$$

Figure 4 presents results for this model. Panel A and B loop over the size of the economy, increasing the endowments of labor and capital in the same proportion while holding the levels of the endowment substitutes constant. Panel A focuses on purchased level of good X_{1p} with the high income-elasticity of demand (high endowment substitute), expressing results in shares of income (observed M, labor and capital income). The share of X_1 production is constant due to the small economy assumption. The vertical dashed line is the choke income level for X_{1p} consumption, so in additional to consumption being zero, all production of X_1 is exported. As per-capita income rises, consumption of X_{1p} gradually increases once choke income is passed and trade gradually shifts from exporting all X_1. The curves pass through an autarky point and then the direction of trade switches to importing X_1.

Panel B of Figure 4 graphs the income elasticities of demand for X_{1p} and X_{2p}. When income is below the choke level for X_{1p} the income elasticity for X_{2p} is equal to one. When the choke level is first passed, the income elasticity of demand for X_{1p} is (locally) infinity (at choke income the denominator of (15) is zero) This elasticity falls steeply, and the two income elasticities gradually converge toward one as discussed and shown above.

Panel C of Figure 4 graphs an increase in the relative (world) price of X_1 relative to X_2. At the low price at the left edge, the economy is specialized in the production of X_2. All consumption of X_1 is from imports (which of course equal consumption). As the relative price of X_1 continues to increase that good is produced and eventually the direction of trade reverses. Those results are completely "normal". What is different is that we eventually reach a choke price level in which the consumption of X_1 goes to zero. All production of X_1 is exported.

Figure 4: Small open economy - fixed world prices
SGM preferences - loop over size per capita, relative price of X1



Panel A: increase per-capita income holding world prices constant shares of production, consumption and trade

Panel B: increase per-capita income holding world prices constant income elasticities of demand

Panel C: increase relative price of X1 factor income fixed

C:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\SG-open.xlsx

5. CRIE preferences, demand, income and price elasticities

Now we turn to a similar model in many respects, but now preferences are changed to what we have labeled CRIE, constant relative income elasticity. In simple terms, a CRIE function is like a CES with different exponents on different goods: $W = \sum_i X_i^{\gamma_i} \quad 0 < \gamma_i \leq 1$.

As noted in the introduction, these preferences have been around for a very long time. CRIE has the advantage that income elasticities do not all converge to one even with large differences in per-capita income. But it has been the view that they are awkward to use in general-equilibrium models since there are no closed form solutions for cost/expenditure and demand functions. Second, this simple version used in Fieler (2011) and Caron-Fally-Markusen (2014, 2020) does not allow for elasticities of substitution across sectors to be less than one. Third, income elasticities and price elasticities are bound together. These difficulties were noted in Hanock's (1975) original investigation, but all three will be overcome here.

I present a very simple version of the model in order to develop the intuition and clearly show its principal properties. The model is a two-final-good, two (conventional, observed) primary factors. It is a small, open economy facing fixed world prices, which can be adjusted as counter-factuals. Each produced and traded good is a perfect substitute in trade for same good from the (not-modeled) rest of world.

I model CRIE preferences in a way that retains constant-returns, homothetic CES functions so that these preferences slot right into AGE models. There are parameters which we will refer to as sector-specific factors that create the non-homotheticity. They are not observed in the data. The observed goods are combined with the specific factors/goods in fixed supply to produce "utility goods". It may be useful to think of these fixed endowments as leisure, which must be combined with the physical goods to produce welfare. Results are qualitatively similar if there is a single endowment good (leisure) which has to be divided between sectors, but the algebra is slightly more involved. CRIE preferences do not allow for choke income and prices. However, the latter can be done by adding the SGM endowments to this formulation.

Notation for quantity variables:

X_1	production of good 1 - observed in data
X_2	production of good 2 - observed in data
X_{1p}	domestic purchases of good 1 - observed in data
X_{2p}	domestic purchases of good 2 - observed in data
TX_{12}	trade activity exports X_1 in exchange for X_2
TX_{21}	trade activity exports X_2 in exchange for X_1
L_1 K_1	labor and capital used to produce X_1
L_2 K_2	labor and capital used to produce X_2
L, K	total fixed labor and capital - scaled up/down to change the size of the economy
R_1	specific factor used by the Y_1 sector - fixed supply - not observed in data
R_2	specific factor used by the Y_2 sector - fixed supply - not observed in data

Y_1	utility good produced X_1 and R_1 - Cobb-Douglas - not observed in data
Y_2	utility good produced X_2 and R_2 - Cobb-Douglas - not observed in data
W	welfare produced from Y_1 and Y_2 - CES

The following are the key properties of the production, consumption, and trade with the rest of world for the two-sector model.

$$X_1 = L_1^{\beta_{1l}} K_1^{\beta_{1k}} \qquad X_2 = L_2^{\beta_{2l}} K_2^{\beta_{2k}} \qquad L = L_1 + L_2 \quad K = K_1 + K_2 \tag{44}$$

$$X_{1p} = X_1 - TX_{12} + TX_{21} \qquad X_{2p} = X_2 - TX_{21} + TX_{12} \tag{45}$$

$$Y_1 = X_{1p}^{\alpha_{x1}} R_1^{\alpha_{r1}} \qquad Y_2 = X_{2p}^{\alpha_{x2}} R_2^{\alpha_{r2}} \qquad \alpha_{xi} + \alpha_{ri} = 1 \tag{46}$$

$$W = (Y_1^{\rho} + Y_2^{\rho})^{1/\rho} \qquad -\infty \le \rho \le 1 \qquad \sigma = \frac{1}{1-\rho} \qquad \rho = \frac{\sigma - 1}{\sigma} \tag{47}$$

All production activities have constant returns to scale (CRS), which greatly facilitates computation in large models, especially using Rutherford's mps/ge.

The relationship between α_{x1} and α_{x2} determines which good has the higher income elasticity. Letting $R_1 = R_2 = 1$ for simplicity, W can be written as

$$W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \qquad \gamma_1 = \rho\alpha_{x1} \qquad \gamma_2 = \rho\alpha_{x2} \tag{48}$$

If $\rho > 0$ or equivalently $\sigma > 1$, then the good with the *higher* α will have the higher income elasticity of demand. However, if $\rho < 0$ or equivalently $\sigma < 1$, then the good with the *lower* α will have the higher income elasticity; e.g., X_1 has the higher elasticity with a lower α because its exponent is less negative: $\rho\alpha_{x2} < \rho\alpha_{x1} < 0$.

Notation for price and income $(tot = p_{x1}/p_{x2})$:

p_{x1}	price of production good X_1
p_{x2}	price of production good X_2
p_{x1p}	price of the observed consumption good X_1
p_{x2p}	price of the observed consumption good X_2
p_{r1}	price of the endowment good R_1
p_{r2}	price of the endowment good R_2
p_{y1}	price of consumption good Y_1
p_{y2}	price of consumption good Y_2
p_w	consumer price index (unit expenditure function)
p_l	price of labor
p_k	price of capital
CONS	representative consumer income

CRIE, as its name suggests, has the nice feature that the ratio of income elasticities between two goods is constant, which could aid greatly in calibration. Let p_{x1p} and p_{x2p} denote goods price, held constant here. Maximize utility subject to a budget constraint with income M and Lagrangean multiplier λ.

$$\max\ W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \quad s.t. \quad M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \tag{49}$$

$$\delta\gamma_1 X_{1p}^{\gamma_1 - 1} - \lambda p_{x1p} = 0 \quad \delta\gamma_2 X_{2p}^{\gamma_2 - 1} - \lambda p_{x2p} = 0 \quad \delta = (1/\rho)(X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho - 1} \tag{50}$$

An increase in M is proxied here by a change (fall) in λ. Hold prices constant. Forming the ratio, λ and δ cancel out, giving

$$\frac{dX_{1p}/d\lambda}{dX_{2p}/d\lambda} = \frac{p_{x1p}}{p_{x2p}}\frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 2}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 2}} \qquad \frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{p_{x1p}}{p_{x2p}}\frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 1}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 1}} \tag{51}$$

$$\frac{p_{x1p}}{p_{x2p}}\frac{\gamma_2 X_{2p}^{\gamma_2 - 1}}{\gamma_1 X_{1p}^{\gamma_1 - 1}} = 1 \text{ from the first-order conditions (51)}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{1-\gamma_2}{1-\gamma_1} \tag{52}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \qquad \mathit{if}\ \gamma_i > 0 \quad \mathit{and} \quad \gamma_1 > \gamma_2 \quad \mathit{or} \quad \gamma_i < 0 \quad \mathit{and}\ \gamma_1 < \gamma_2$$

or alternatively

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \qquad \mathit{if}\ \rho > 0 \quad \mathit{and} \quad a_{x1} > a_{x2} \quad \mathit{or} \quad \rho < 0 \quad \mathit{and} \quad a_{x1} < a_{x2} \tag{53}$$

This relationship is verified in the simulations, both for $\sigma > 1$ ($0 < \rho < 1$) and $\sigma < 1$ ($\rho < 0$).

Next, consider the effect on demands of an increase in income M holding the prices of the two goods constant.

$$M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \qquad 1 = p_{x1p}\frac{dX_{1p}}{dM} + p_{x2p}\frac{dX_{2p}}{dM} \tag{54}$$

$$1 = \left[\frac{p_{x1p}X_{1p}}{M}\right]\frac{dX_{1p}/X_{1p}}{dM/M} + \left[\frac{p_{x2p}X_{2p}}{M}\right]\frac{dX_{2p}/X_{2p}}{dM/M} \tag{55}$$

Let s_{xi} denote the expenditure share on good i and let η_i denote the income elasticity of demand for good i. Equation (55) reduces to

$$\sum_i s_{xi}\eta_i = 1 \qquad i = 1, 2 \tag{56}$$

Note that, although the ratio of income elasticities is constant as noted above, there levels cannot be constant as M increases (unless all income elasticities are one - homothetic preferences). As the expenditure share of the good with the high η increases with income, the η's cannot remain constant or (57) is violated. Both income elasticities must fall to maintaining their ratio as income grows. Note that this coming exclusively from the budget constraint: it has nothing to do with the particular specification of preferences. Equation (56) is verified in the simulations.

Now consider the effect of increasing the price of good 1 holding income and the price of good 2 constant. While the algebra that follows here is not specific to any type of preferences, it is useful to help interpret simulation results to follow.

$$M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \qquad \frac{dM}{dp_{x1p}} = X_{1p} + p_{x1p}\frac{dX_{1p}}{dp_{x1p}} + p_{x2p}\frac{dX_{2p}}{dp_{x1p}} = 0 \tag{57}$$

$$1 + \frac{p_{x1p}}{X_{1p}}\frac{dX_{1p}}{dp_{x1p}} + \left[\frac{p_{x2p}X_{2p}}{p_{x1p}X_{1p}}\right]\frac{p_{x1p}}{X_{2p}}\frac{dX_{2p}}{dp_{x1p}} = \frac{1}{X_{1p}}\frac{dM}{dp_{x1p}} = 0 \tag{58}$$

$$\varepsilon_{11} + \frac{s_{x2}}{s_{x1}}\varepsilon_{21} = -1 \qquad \varepsilon_{ij} \text{ - elasticity of demand for } X_{ip} \text{ wrt } p_{xjp} \tag{59}$$

$$\sum_i s_{xi}\varepsilon_{i1} = -s_{x1} \le 0 \qquad M \text{ constant or economy specialized in producing } X_2 \tag{60}$$

In general, we expect the own-price elasticity to be negative. However, the cross-price elasticity could be negative as well: with a low elasticity of substitution between goods, the effect of the price increase holding income constant could lead to a fall in demand for both goods and indeed this must be the case with a zero elasticity of substitution.

Suppose that the economy is specialized in X_2 so that $M = p_{x2p}\bar{X}_2$, where the over bar is the end of the production frontier. The (59) and (60) remain valid in that a change in the price of X_1 has no effect on M.

Suppose instead that the economy is specialized in X_1 so that $M = p_{x1p}\bar{X}_1$. Now the right-hand side of (60) is no longer zero.

$$M = p_{x1p}\bar{X}_1, \quad \frac{dM}{dp_{x1p}} = \bar{X}_1 \qquad \frac{1}{X_{1p}}\frac{dM}{dp_{x1p}} = \frac{\bar{X}_1}{X_{1p}} = \frac{p_{x1p}\bar{X}_1}{p_{x1p}X_{1p}} = \frac{1}{s_{x1}} \tag{61}$$

$$\varepsilon_{11} + \frac{s_{x2}}{s_{x1}}\varepsilon_{21} = -1 + \frac{1}{s_{x1}} \tag{62}$$

$$\sum_i s_{xi}\varepsilon_{i1} = -s_{x1} + 1 \geq 0 \qquad \text{economy specialized in producing } X_1 \tag{63}$$

As seems intuitive, the response of demands to an increase in the price of good 1 are larger - less negative or even turning positive - if the economy is specialized in good 1 that if it is specialized in good 2. The effect of the increase in the price of good 1on income when the economy is specialized in good 1 is positive while it is negative if specialized in good 2. Note that in this case, the price elasticities of both goods can be positive.

The middle diagram in Figure 3 gives a (hand drawn) picture of the Engels curve for the CRIE case. As noted, these preferences by themselves do not allow for choke income and prices, but they have significant advantages in that the income elasticities do not converge to one, particularly relevant for goods with low income elasticities. It is hard to swallow these goods having unitary income elasticities at high levels of income.

The next section gives the inequalities and complementary variables for the small-economy general-equilibrium model in the same format as the SGM model in (20)-(38). I have tried to construct the model in a manner that stay close to the SGM formulation and indeed, as I'll comment later, the two can be combined with both endowment goods and specific fixed factors.

5. CRIE small open-economy general-equilibrium model

Price inequalities — quantities as complementary variables

$$c_{x1}(p_l, p_k) \geq p_{x1} \quad \perp \quad X_1 \tag{64}$$

$$c_{x2}(p_l, p_k) \geq p_{x2} \quad \perp \quad X_2 \tag{65}$$

$$p_{x1} \geq p_{x1p} \quad \perp \quad X_{1p} \tag{66}$$

$$p_{x2} \geq p_{x2p} \quad \perp \quad X_{2p} \tag{67}$$

$$p_{x2}tot \geq p_{x1} \quad \perp \quad T_{21} \tag{68}$$

$$p_{x1} \geq p_{x2}tot \quad \perp \quad T_{12} \tag{69}$$

$$c_{y1}(p_{x1p}, p_{r1}) \geq p_{y1} \quad \perp \quad Y_1 \tag{70}$$

$$c_{y2}(p_{x2p}, p_{r2}) \geq p_{y2} \quad \perp \quad Y_1 \tag{71}$$

$$c_w(p_{y1}, p_{y2}) \geq p_w \quad \perp \quad W_1 \tag{72}$$

Market clearing   prices as complementary variables

$$X_1 \;\geq\; X_{1p} - T_{12} + T_{12}/tot \qquad \perp \qquad p_{x1} \tag{73}$$

$$X_2 \;\geq\; X_{2p} + T_{21} - T_{12}tot \qquad \perp \qquad p_{x2} \tag{74}$$

$$X_{1p} \;\geq\; \partial c_{y1}/\partial p_{x1p} Y_1 \qquad \perp \qquad p_{x1p} \tag{75}$$

$$X_{2p} \;\geq\; \partial c_{y2}/\partial p_{x2p} Y_2 \qquad \perp \qquad p_{x2p} \tag{76}$$

$$L \;\geq\; \partial c_{x1}/\partial p_l X_1 + \partial c_{x2}/\partial p_l X_2 \qquad \perp \qquad p_l \tag{77}$$

$$K \;\geq\; \partial c_{x1}/\partial p_k X_1 + \partial c_{x2}/\partial p_k X_2 \qquad \perp \qquad p_k \tag{78}$$

$$R_1 \;\geq\; \partial c_{y1}/\partial p_{r1} Y_1 \qquad \perp \qquad p_{r1} \tag{79}$$

$$R_2 \;\geq\; \partial c_{y2}/\partial p_{r2} Y_2 \qquad \perp \qquad p_{r2} \tag{80}$$

$$Y_1 \;\geq\; \partial c_w/\partial p_{y1} W \qquad \perp \qquad p_{y1} \tag{81}$$

$$Y_2 \;\geq\; \partial c_w/\partial p_{y2} W \qquad \perp \qquad p_{y2} \tag{82}$$

$$W \geq\; CONS/p_w \qquad \perp \qquad p_w \tag{83}$$

Income balance   consumer income as complementary variable

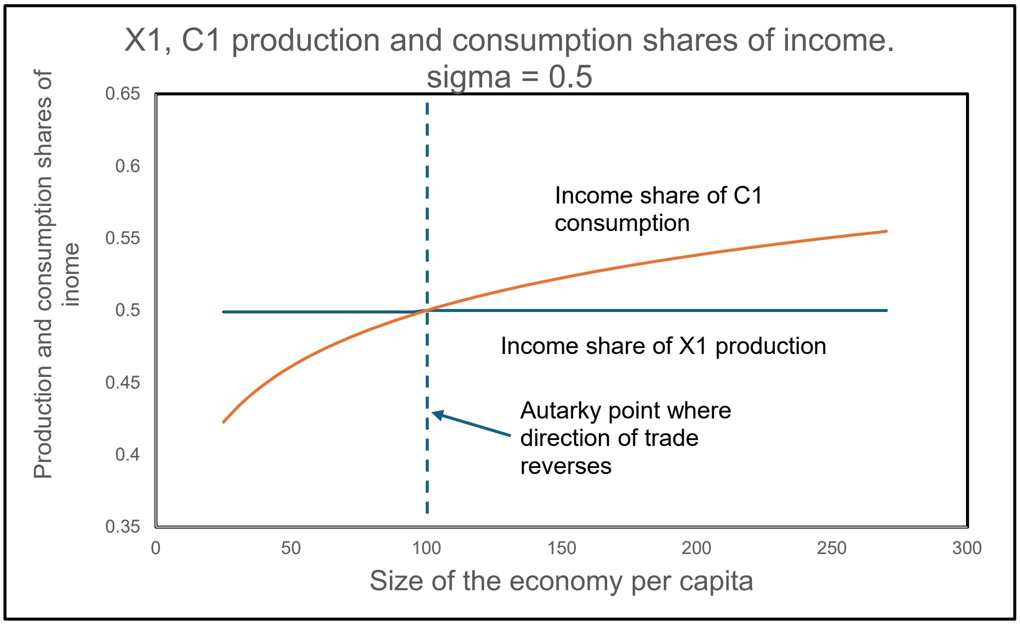
$$CONS \;\geq p_l L + p_k K + p_{r1} R_1 + p_{r2} R_{r2} \qquad \perp \qquad CONS \tag{84}$$

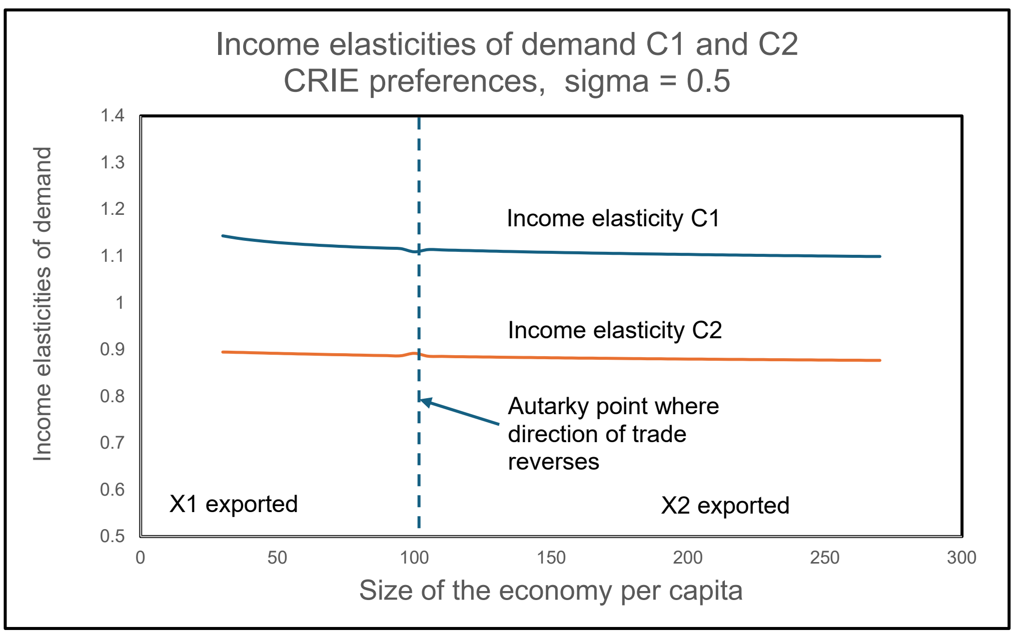
The initial calibration of the model is as follows, with a micro-consistent set of numbers requiring row and column sums equal to zero. As in the SGM case, I calibrated it to zero trade initially. The share of the specific factor in Y_1 is lower than in Y_2 so that, with an elasticity of substitution greater than one, Y_1 is the sector with the higher income elasticity of demand.

	Production Sectors						Trade		Welfare	Consumer
	X1	X1P	X2	X2P	Y1	Y2	TX21	TX12	W	CONS
Markets										
PX1	100	-100					-0	0		
PX1P		100			-100					
PX2			100	-100			0	-0		
PX2P				100		-100				
PR1					-90					90
PR2						-10				10
PY1					190				-190	
PY2						110			-110	
PW									300	-300
PL	-25		-75							100
PK	-75		-25							100

Figure 5 presents a simulation of a case with the elasticity of substitution σ between Y_1 and Y_2 in (47) equal to 0.5. Panels A and B correspond to the results shown in panels A and B of Figure 4, looping over the endowment size of the economy. Endowments of the specific factors

Figure 5: Small open economy - fixed world prices
CRIE preferences - loop over size (income) per capita





c:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\CRIE-open-m-05 mcp.gms
c:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\crie-open.xlsx

are held fixed as L and K increase, similar to holding the endowment goods constant in SGM.

Panel A of Figure 5 shows how the income shares of X_1 consumption and production change with per-capita size. The production share of X_1 is constant by virtue of the small-country assumption. The dashed line is the autarky size and also the calibration point as just noted. At a small size, X_1 is exported while at large size it is imported. The vertical distance between the production and consumption lines is the share of imports or exports. Panel B of Figure 5 give the income elasticities of demand for the same per-capita income experiment. As noted a couple of times, a feature of CRIE is that these income elasticity differences persist over a much larger range of income and the good with the low income elasticity of demand never rises (and of course doesn't approach one) contrary to SGM.

It is certainly not a surprise that a high-income country specializes in consuming the income elastic good. I should note that the result here that this maps directly into the direction and balance of trade is not general however. Fieler (2011) and especially Caron-Fally-Markusen (2014, 2020) show that in models with much richer production structures, high income countries are relatively specialized in the production of skilled-labor-intensive goods that are also the high income-elasticity goods, so those countries may export the high income elasticity goods.

My CRIE formulation allows for elasticities of substitution between sectors to be less than one, which I believe is supported by empirical evidence (literature references in Heblich, Redding and Zylberberg (2025)). The results for an elasticity of substitution of 2.5 don't differ much from those I just presented for σ = 2.5, so I won't show them here. The code and figures for the case with σ = 2.5 are available from myself. The only qualitative different between the two cases occurs experiment when changing prices, which we will look at in section 7..

6. Estimation using US Time-Series Data

The data used to construct Figure 1 can be use for a simple estimation of the income elasticities and the elasticity of substitution between goods and services using the CRIE formulation of preferences. Drawing on different sources, the knowns for 1950-2024 are

Expenditures on goods and services in total consumer expenditures
    from this, the ratio gives expenditure shares
Prices of goods and services
    from these two, real quantity indices of goods and services
Population data
    from these, real consumption quantities per capita

The consumer optimization problem is give as follows, with sector/subscript 1 being services and sector/subscript 2 being goods.

$$W = \left[\beta X_1^{\alpha_1 \rho} + (1-\beta) X_2^{\alpha_2 \rho}\right]^{1/\rho} + \lambda(M - p_1 X_1 - p_2 X_2) \tag{85}$$

The first-order conditions can form a ratio which eliminates λ

$$\frac{p_1 X_1}{p_2 X_2} = \frac{\beta}{(1-\beta)} \frac{\alpha_1}{\alpha_2} \left(\frac{X_1^{\alpha_1}}{X_2^{\alpha_2}} \right)^{\rho} \qquad \text{Ratio of first-order conditions} \tag{86}$$

Prices, quantities and therefore market shares are observed in the data as just noted. Parameters to be estimated are β, α_1, α_2, and ρ. I estimate this by solving a non-linear programming problem (NLP) minimizing the sum of squared errors between the ratio of observed market shares and the predicted value of market shares given by the right-hand side of (86) using the observed X_i. However, the software taught me what now seems apparent: there is a continuum of solutions among the three variables β, α_1, and α_2. Fixing β allows for a (unique) solution for the α_i, or fixing the ratio of the α_i allows for a solution for beta (including identical α_i - homothetic preferences). Since the goal is to estimate the differences in the α_i along with the elasticity of substitution $\sigma = 1/(1-\rho)$, I chose to fixed the value of β and write the NLP as (*t* for time/year):

$$Min_{\alpha_1, \alpha_2, \rho} \sum_t \left[\frac{SH_{1t}}{SH_{2t}} - C \frac{\alpha_1}{\alpha_2} \left(\frac{X_{1t}^{\alpha_1}}{X_{2t}^{\alpha_2}} \right)^{\rho} \right]^2 \qquad C = \frac{\beta}{(1-\beta)} \qquad SH_i = \frac{p_i X_i}{M} \tag{87}$$

The chosen value of *C* is the average of the ratio of expenditure shares on services to goods over the sample period, which has a value of 1.445. This is admittedly rather arbitrary. The key results, which are the ratio of the income elasticities η_1/η_2 and the elasticity of substitution σ, do change with *C* but are not very sensitive to the value of *C*. I believe that *C* is just a weighting parameter perhaps analogous to weighted least squares with a higher value giving more weight to services (sector 1). The GAMS program using the NLP solve is attached at the end of the paper.

Table 2: Estimation of income and substitution elasticities from US time-series data, 1950-2024, Services =1, Goods = 2 using non-linear programming

Parameter estimates: $\alpha_1 = 0.225$ $\alpha_2 = 0.826$ $\rho = -1.051$ $R^2 = 0.852$
(C = $\beta/(1-\beta)$ held fixed at average share ratio services/goods)

Ratio of income elasticities η_1/η_2 (same all years)	1.510
Average η_1 services over the time series	1.170
Average η_2 goods over the time series	0.775
Elasticity of substitution σ	0.478

$$W = \left(0.59\, X_1^{-0.237} + 0.41\, X_2^{-0.868}\right)^{-0.951} \qquad \text{Calibrated utility function}$$

With reference to Figure 2 on the importance for counter-factuals, what would be the "prediction error" on a simulated increase in per-capita income? Counterfactual: increase per-capita income by 50 percent holding prices constant. Normalize or index the initial consumption of goods and services to one.

New levels/ratio of goods to services consumption, homothetic preferences $\eta_i = 1$:
$X_1 = 1.500 \qquad X_2 = 1.500 \qquad X_1/X_2 = 1.000$

Predicted values under estimated preferences: average values $\eta_1 = 1.170$, $\eta_2 = 0.775$
$X_1 = 1.607 \qquad X_2 = 1.369 \qquad X_1/X_2 = 1.174$

This "prediction error" is fairly substantial, and a 50 percent increase is not a crazy number over a couple of decades. Imagine that government private or public-sector planners were trying to estimate the future demand for goods and services (the latter both public goods and private services) thirty or forty years ago? A 100 percent increase would give a ratio of 1.315 whereas homothetic preferences would stay stuck at 1.000 under the constant price assumption.

I do not try to compute standard errors of the estimated parameters, in part because it is not clear how to do this in the NLP formulation (Caron, Fally, and Markusen (2013) use bootstrapping) and second, there are many "heroic assumptions" used in any case with the maintained assumptions that consumption preferences don't change over 75 years and similarly that the categories goods and services are comparable overtime.

7. Independence of price and income elasticities in CRIE

As noted at the beginning of Section 5 on CRIE preferences, the third "nail in the coffin" of the simplest form of CRIE preferences,

$$W = \sum_i X_i^{\gamma_i} \quad 0 < \gamma_i \leq 1, \text{ labled "direct explicit additivity" by Hanock (1975),} \tag{88}$$

is that this functional form binds together price and income elasticities. An analogous problem exists in the commonly-used intertemporally additive utility function in macroeconomics: the intertemporal elasticity of substitution and the degree or relative risk aversion are tied together. This is particularly troublesome in that literature estimates referred to above of the elasticity of substitution between broad sectors in less than one (around 0.5) and the second nail in the coffin is that (88) does not permit this.

We have already seen that my formulation avoids this second problem and this section constitutes a brief note that it also avoids the third. Because of the difficulties in defining price elasticities in the open economy discussed in Section 5 (the income effects depend on the direction of trade), I simulated a very simple two-good, partial-equilibrium model holding money income constant. Recall that, letting $R_1 = R_2 = 1$ for simplicity, W can be written as

$$W = (X_{1p}^{\rho\alpha_{x1)}} + X_{2p}^{\rho\alpha_{x2}})^{1/\rho} \quad \text{allowing } \rho \text{ to take on positive or negative values} \tag{89}$$

The relative α_{xi} values primarily determine the relative income elasticities while the ρ value primarily determines the prices elasticities. Like the income elasticities, the substitution and price elasticities are not constant, but they can be calibrated independently at a benchmark. The procedure for separating price and income elasticities is as follows.

(a) Pick a value of σ (e.g., from a literature estimate) and set $\rho = \sigma/(\sigma - 1)$ in (89).

(b) Using literature estimates of the relative income elasticities η and expenditure shares, use the procedure outlined in the appendix to determine the α_{xi} values.

(c) Equation (56) then determines the initial level values of the η (they are not constant).

Figure 6 presents simple simulations of a partial-equilibrium case holding money income constant to illustrated this (code available from the author includes parameter values). The top two panels are for $\sigma = 2.5$ ($\rho = 0.6$) and $\sigma = 0.5$ ($\rho = -1$), giving own and cross-price elasticities for and increase in p_1. Other than the difference in the levels of the elasticities, the important difference in the two cases is that with $\sigma > 1$, the share of expenditure on X_1 is falling in own price as we expect, while with $\sigma < 1$, the expenditure share on X_1 rises with own price.

The bottom panel of Figure 6 notes that the results for the ratio of the income elasticities in these two simulations are identical. In short, these results are illustrating that the second and third problems with the simple direct implicit additivity version are “solved” in my version: substitution elasticities less than one are permitted and price and income elasticities can be set independently.

8. A Short Comment on Homothetic Armington with Choke Prices

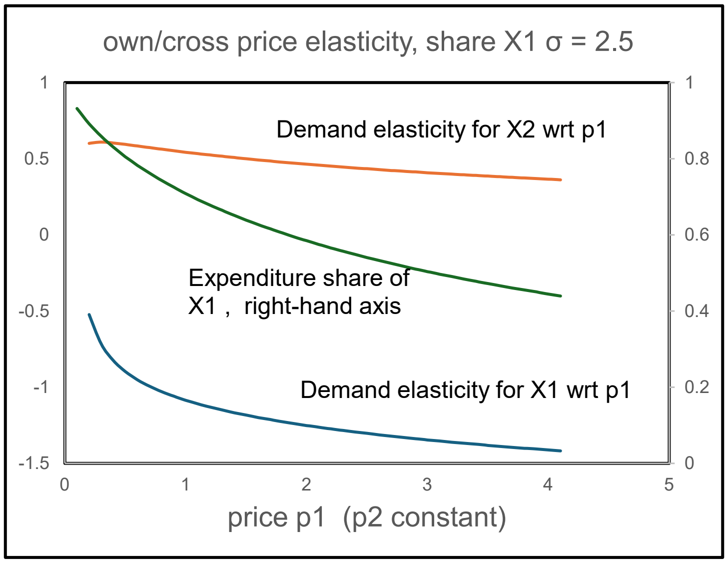
There are many other cases and situations that may be of interest to general-equilibrium modelers that can be formulated and simulated using more or less the same methodologies used above. For example, a modeler might want to retain homothetic CES preferences but introduce choke prices in order to endogenize initial zeros in the trade matric and/or allow initially-positive values to go to zero at higher trade costs.

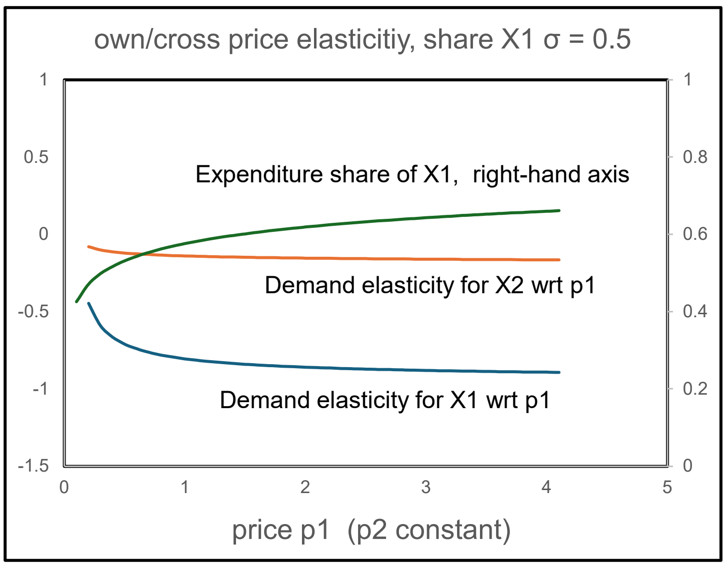
As always, the problem with homothetic CES is that the demand price for a good goes off to infinity as the quantity consumed of that good goes to zero. Thus there can be no endogenous zero values in the trade matrix: anything zero initially is dropped from the model and thus remains zero, and any initially positive value cannot go to zero. But suppose instead of assuming the existence of fixed endowment substitutes as we have done above, there are domestic technologies that can produce perfect substitutes for foreign (Armington differentiated) goods, which are inefficient at benchmark prices such that those technologies are inactive initially. I don’t find this implausible when thinking about a broad range of goods such as clothing, footwear, kitchen goods and implements, etc. High trade costs can induce foreign firms to produce inside the domestic country, although it is beyond the scope of this paper to get into endogenizing multinational firms.

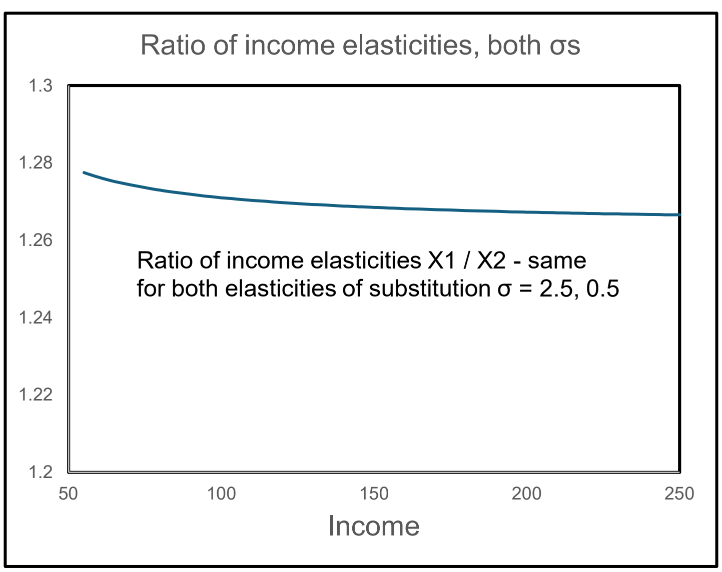
At this point, perhaps the algebra of my approaches is well understood, so let me just

Figure 6: Independence of price and income elasticities: Partial equilibrium example

Holding ratio of income elasticities constant, contrast σ = 2.5 with σ = 0.5







$W = (X_1\text{^}\rho\alpha_{x1} + X_2\text{^}\rho\alpha_{x2})\text{^}(1/\rho)$

Level of ρ determines the elasticity of substitution and price elasticities of demand

Relative values of α_{x1} and $\rho\alpha_{x2}$ determine the income elasticities of demand

Pick σ set α_{x1} α_{x2} to match ratios of income elasticities

c:\jim\RESEARCH\SGM and CRIE\SG CRIE PE elasticities\INCELAS.xlsx
\Elasticities-p-h.gms, \Elasticities-p-m.gms

illustrated a simple model the makes the point. The computer code is available from myself. There are two industries 1 and 2 which, the domestic and foreign varieties are X_1 and Y_1 in industry 1 and X_2 and Y_2 in industry 2. In the usual homothetic CES Armington case, the domestic economy will always import some Y_1 and Y_2 no matter how high the price. But suppose that in the benchmark, there are latent technologies which can produce perfect substitutes for Y_1 and Y_2 , but are inefficient at benchmark prices.

Figure 7 presents a simulation of a small economy calibrated to a symmetric initial benchmark with the rest of world with foreign prices equal to one for Y_1 and Y_2. The upper nest of utility is Cobb-Douglas while there is an elasticity of substitution equal to 4 between the domestic and foreign varieties in a sector. At initial world prices of one, both Y_1 and Y_2 are imported. But when either price gets 15 percent higher, production of the domestic substitute kicks in. Figure 7 shows the resulting trade pattern and the world choke prices for the two foreign goods. In the mid-section, the model is and behaves exactly like the classic cge homothetic Armington model: preferences and all production technologies are homothetic of degree one. But the (latent) technologies impose choke prices for foreign goods.

As discussed above and will be discussed a little more in the following section, calibration will be a challenge as will be setting or solving for the degree of unprofitability of initially inactive trade links, or the upper cutoff for initially actively traded goods.

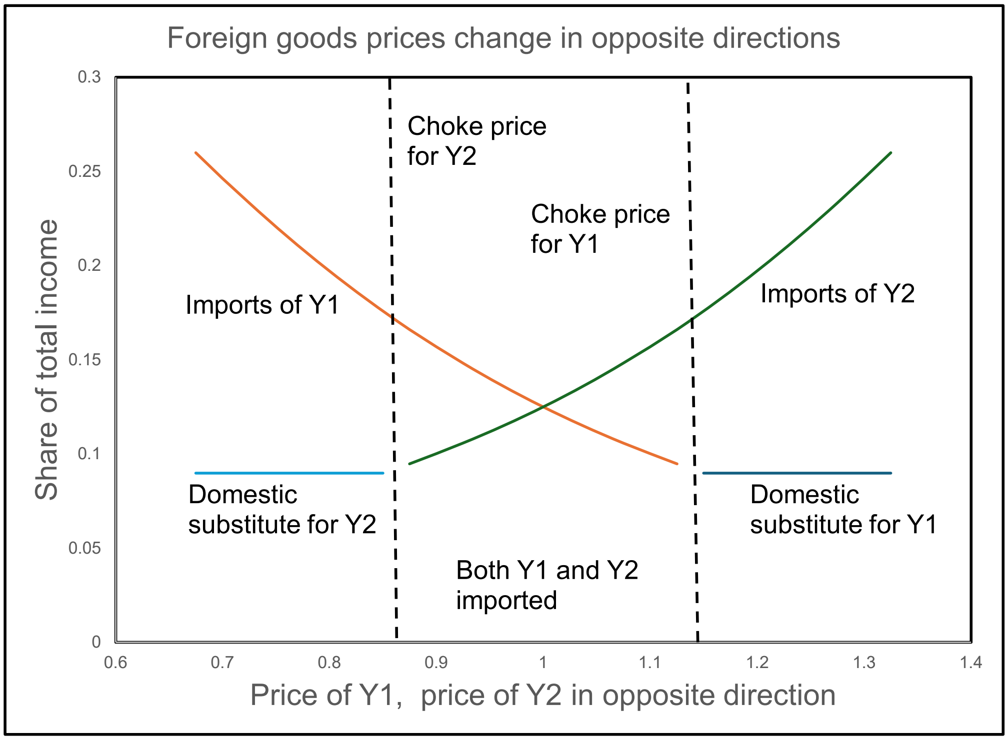
9. Strengths, Weaknesses and Challenges

As noted early on, a number of other authors have considered adding non-homothetic preferences to general-equilibrium trade theory and empirics. Many or most of these are significantly more technically complex than this present paper and proposal. Some have analytical solutions and some do not. Having an analytical solution is a great strength, but the restrictive assumptions needed have a cost for modelers seeking to formulate empirically-relevant models calibrated to large, multi-sector, multi-country, multi-factor and multi-household-type data sets. Other formulations without analytical solutions are equally awkward for high-dimension models. The CRIE preferences used in Caron, Fally and Markusen (2014, 2020) do not yield analytical solutions.

I am arguing that the simplicity of my formulation is a strength and an attractive feature for AGE modelers. In particular, it retains all the valuable features of CES preferences which generate solutions for cost and expenditure functions depending only on prices. This in turn generates straightforward and simple derivations of factor and goods-demand equations via Shepard's lemma. These features then allow a model to be specified in a complementarity format where pricing (optimization) inequalities have quantities as complementary variables and market clearing inequalities have prices as complementary variables. The final step is that these characteristics allow computation of solutions with choke income and prices levels.

However, I am the first to acknowledge that there are two significant problems to implementation on general-equilibrium data sets. The first of these is calibration as discussed in the appendix to follow: we have to solve for the level of the unobserved endowment goods in the SGM version in order to get the income elasticities. Imagine that we have a set of estimates of

Figure 7: Choke prices for foreign goods in a traditional Armington model



Foreign goods can be replaced by inefficiently proudced (in the benchmark) domestic goods
which are perfect substitutes for the imported goods
this example: in the benchmark, domestic substitutes 15 percent more costly

C:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\Armington-choke.xlsx
\Armington-choke.gms

income elasticities of demand such as those from Caron, Fallly and Markusen (2014, 2020) shown earlier. We could make the assumption that these apply to every country regardless of their per-capita income (the theory says these elasticities converge to one as income becomes large). But prices differ substantially in the benchmark data. Perhaps running a simulation which eliminates all trade costs could form a basis for setting the endowment parameters by country by good? But the implied income elasticities may then not be consistent with price elasticities if there are estimates of those.

The second difficulty lies in assessing the choke prices and incomes for goods not traded or not produced initially as just discussed above. How unprofitable are these trade links for latent goods at the benchmark? This is another difficult problem but one which I think AGE modelers can crack. I should point out that we have always had this problem. The standard procedure is to simply hold latent (zero) trade links in the benchmark constant at zero. In effect, there is an unbounded unobserved and implicit trade barrier on these links. This comes at a huge cost in that there cannot be an expansion of trade at the extensive margin following a liberalization. Conversely, an initially-active link cannot go to zero given standard CES preferences. The other approach is the Eaton-Kortum formulation in which (as I understand it) there are no zeros initially, just extremely small but positive trade flows on all links. So again, there is no change in trade at the extensive margin. When trade costs fall just a little, there will be immediate positive or negative (but still non-zero) responses on all links.

The CRIE formulation with its unobserved factor (leisure?) not only has theoretical advantages (the non-convergence of income elasticities to one), but is probably a better candidate for calibration to existing data sets. Given a set of income elasticities as in Caron, Fally and Markusen shown earlier, these allow for relatively easy parameterization of expenditure and demand functions. Further, the CRIE preferences permit the independent calibration of price and income elasticities: in the notation of the paper: the ρ parameter can be combined with the differences in the α_i across sectors to achieve independence in the two elasticities. The drawback of CRIE, given the goals of the paper, is that the basic formulation does not allow for choke income and prices. SGM and CRIE can be combined (just add the endowment goods to CRIE), but this then revisits the calibration issues.

****
Available from the author directly

.gms and excel files for all models, a couple which have both mcp and mps/ge formats. Also code for the partial equilibrium examples of price versus income elasticities, and code focusing on price elasticities in addition to income elasticities.

Computer code in GAMS:

I have attached the GAMS program for the SGM model (no Armington differentiation). This is in MPC format (mixed complementarity problem), in which all weak inequalities are written out, and the model statement gives the complementary variable associated with each inequality.

Then the code for the basic CRIE models follow in mcp format. Again, no Armington differentiation to keep things simple. As noted at the end of the concluding section, more version are available from myself. The equivalent code in Rutherford's mps/ge format is available from myself.

The final program is for the calculation of the income and substitution elasticities in the empirical example. As noted above, this formulates the estimation problem as an NLP, bi-passing the tradition econometrics' approach. I am unsure how common this might be, but I have been told it is used in non-parametric statistics. Treating estimation as a non-linear programming problem has considerable advantages that might be of interest. First, almost any estimating objective function can be used: there is no need to restrict it to linear or log linear. Second, inequalities bounds can be used on variables in complementarity format, such as restricting certain coefficients t non-negative values. Third, other side constraints can be imposed on the optimization problem, such as restricting a technology to constant returns to scale (restriction on the sum of input coefficients).

REFERENCES

Anderson, James E. and Penglong Zhang (2025), "Latent Exports: Almost Ideal Gravity and Zeros", *Reivew of Economics and Statistics* 107(1), 221-239.

Armington, Paul S. (1969), "A Theory of Demand for Products Distinguished by Place of Production." *IMF Staff Papers*, 159–176.

Baldwin, Richard and James Harrigan (2011), "Zeros, Quality, and Space: Trade Theory and Trade Evidence", *American Economic Journal: Microeconomics* 3, 60-88.

Berstrand, Jeffrey H. (1990), "The Heckscher-Ohlin-Samuelson Model, the Linder Hypothesis and the Determinants of Bilateral Intra-Industry Trade", *Economic Journal* 100, 1216-1229.

Bertoletti, Paolo, Federico Etro and Ina Simonovska (2018), "International Trade with Indirect Additivity", *American Economic Journal: Microeconomics* 10(2), 1-57.

Caron, Justin, Thibault Fally and James R. Markusen (2014), "International Trade Puzzles: a Solution Linking Production and Preferences", *Quarterly Journal of Economics* 129, 1501-1552.

Caron, Justin, Thibault Fally and James R. Markusen (2020), "Per-Capita Income and the Demand for Skills", *Journal of International Economics* 123, article 103306.

Chao, Hung-po, Sehun Kim, and Alan S. Manne (1982), "Computation of Equilibria for Nonlinear Economies: Two Experimental Model", *Journal of Policy Modeling*, 4, 23–43.

Eaton, Jonathan and Samuel Kortum (2002), "Technology, Geography and Trade", *Econometrica* 70, 1741-1779.

Eaton, Jonathan, Samuel Kortum and Sebastian Sotelo (2013), "International Trade: Linking Micro and Macro", in Daron Acemoglu, Manuel Arellano, and Eddie Dekel, *Advances in Economics and Econometrics, Tenth World Congress, Volume II, Applied Economics*, Cambridge University Press, 329-370.

Fajgelbaum, Pablo and Amit Khandelwal, (2016), "Measuring the Unequal Gains from Trade", *Quarterly Journal of Economics* 131, 1113–1180.

Fieler, Ana Cecília (2011) "Nonhomotheticity and Bilateral Trade: Evidence and a Quantitative Explanation", *Econometrica* 79 (4), 1069–1101.

Gandhi, Amit, Zhentong Lu, and Xiaoxia Shi (2013), "Estimating Demand for Differentiated Products with Zeroes in the Market Share Data", *Quantitative Economics* 14, 381-418.

Hanoch, Giora (1975), "Production and Demand Models with Direct or Indirect Implicit Additivity", *Econometrica*, 43, 395–419.

Harris, Richard (1984), "Applied General Equilibrium Analysis of Small Open Economies with Scale Economies and Imperfect Competition", *American Economic Review* 74(5), 1016-1032.

Heblich, Stephan, Stephen J. Redding, and Yanos Zylberberg (2025), "The Distributional Consequences of Trade: Evidence from the Grain Invasion", working paper.

Hunter, Linda (1991), "The Contribution of Non-homothetic Preferences to Trade", *Journal of International Economics* 30, 345-358.

Jung, Wook, Ina Simonovska and Ariel Weinberger (2019), "Exporter Heterogeneity and Price Discrimination: A Quantitative View", *Journal of International Economics* 116, 103-124.

Karush, William (1939), *Minima of Functions of Several Variables with Inequalities as Side Constraints*, M.Sc. thesis, Department of Mathematics, University of Chicago.

Kehoe, Timothy J. and Kim J. Ruhl (2013), "How Important is the New Goods Margin in International Trade?", *Journal of Political Economy* 121(2), 358-392.

Kuhn, Harold W. and Albert W. Tucker (1951), "Nonlinear Programming", *Proceedings of 2nd Berkeley Symposium*. Berkeley: University of California Press, 481-492.

Linder, Staffan B. (1961), *An Essay on Trade and Transformation*, Uppsala: Almqvist and Wiksells.

Markusen, James R. (1986), "Explaining the Volume of Trade: An Eclectic Approach," *American Economic Review* 76, 1002-1011.

Markusen, James R. (2013), "Putting Per-Capita Income back into Trade Theory", *Journal of International Economics* 90, 255-265.

Markusen, James R. (2023), "Incorporating theory consistent endogenous markups into applied general-equilibrium models", *Journal of Global Economic Analysis* 8(2), 60-99. https://www.jgea.org/ojs/index.php/jgea/article/view/206/237

Mathiesen, Lars (1985), "Computation of Economic Equilibria by a Sequence of Linear Complementarity Problems", *Mathematical Programming* 23, 144-162.

Matsuyama, Kiminori, (2000), "A Ricardian Model with a Continuum of Goods under Non-Homothetic Preferences: Demand Complementarities, Income Distribution, and North-South Trade", *Journal of Political Economy* 108. 1093-2000.

Matsuyama, Kiminori (2019), "Engel's Law in the Global Economy: Demand-Induced Patterns of Structural Change, Innovation, and Trade." *Econometrica* 87 (2): 497–528

Rutherford, Thomas F. (1985), *MPS/GE user's guide*, Department of Operations Research, Stanford University.

Rutherford, Thomas F. (1995), "Extensions of GAMS for complementarity problems arising in applied economic analysis", *Journal of Economic Dynamics and Control* 19(8), 1299-1324.

Shoven, John B. and John Whalley (1984), "Applied General-Equilibrium Models of Taxation and International Trade: An Introduction and Survey", *Journal of Economic Literature* 22(3), 1007-1051.

Simonovska, Ina (2015), "Income Differences and Prices of Tradables: Insights from an Online Retailer", *Review of Economic Studies* 82, 1612-1656.

Wales, Terry J. And Alan D. Woodland (1985), "Estimation of Consumer Demand Systems with Binding Non-negativity Constraints", *Journal of Econometrics* 21, 263-285.

Appendix: A Note on Calibration Challenges

Potential difficulties for calibrating either SGM or CRIE preferences to large multi-sector models seem formidable. A thorough analysis of these challenges is beyond the scope of this paper. However, this section will present a brief outline of how this might be done for CRIE preferences using available data and literature estimates for some parameters. Calibrating the endowment parameters in the SGM is a straightforward task of solving the system of equations in (15) where the left-hand side income elasticities are known from literature estimates and the endowment parameters are the unknowns. The shares s_i can be calibrated from expenditure data. However, for the SGM case, the important task is determining the degree of unprofitability of initially-zero trade links, something left for future work.

The CRIE example I will use here assumes that there are five sectors. After discovering a number of "roadblocks", I found I need to break out one sector and treat it as a "numeraire" sector in order to remove degeneracy of the solution. In math programming terms, degeneracy is multiple equilibria and are all the same in real terms but causes computational difficulties and indeterminacy. We are all familiar with this insofar as we know that fixing one price in general-equilibrium models is needed to avoid indeterminacy of the price level. Let j denote a set indexing the five sectors. Set i(j) is a sub-set, consisting of sectors 2-5 with sector one considered the numeraire.

Sets j sectors {1...5}, i(j) equal subset of sectors {2...5}, sector 1 "numeraire"

The following are the parameters of the calibration model. While made up here, these are knows from data or literature estimates: a vector of income elasticities, the expenditure share across sectors, the elasticity of substitution among the five final goods, plus one parameter that is simply picked by the modeler in order to avoid an indeterminacy. This parameter is the total share of the fixed factors *R* in benchmark income.

Parameters

η_{i1}	literature estimate of income elasticity for sector i, divided by the literature estimate for sector 1
s_i, s_1	expenditure share of sector i
σ	elasticity of substitution across sectors, $\rho = (\sigma - 1)/\sigma$
s_r	share of fixed factor *R* in total income (picked by modeler)

Variables to be solved for are as follows, using the same notation as in the CRIE section.

Variables to be solved for

η_i, η_1	income elasticities consistent with Engels aggregation
γ_i, γ_1	given by $\gamma_i = \rho\alpha_{xi}$
α_{xi}, α_{x1}	share of good X_i in producing Y_i

The first difficulty is that the literature estimates of income elasticites as in Caron, Fally and Markusen (2014) will not satisfy the Engels aggregation condition (from the budget constraint) in (56) for most/all countries given their different expenditure shares. So the task is to modify the literature income elasticities for a country to (a) retain the relative values of those elasticities from the estimated ones, and (b) set the levels of those elasticities to satisfy Engels aggregation given each country's vector of expenditure shares. This is done by solving five equations in the five modified income elasticities using the Engels condition to solve for the income elasticity of the sector 1.

$$\eta_i/\eta_1 = \eta_{i1} \qquad \perp \qquad \eta_i \tag{90}$$

$$\sum_i s_i\eta_i + s_1\eta_1 = 1 \qquad \perp \qquad \eta_1 \tag{91}$$

Note that these five equations can be solved first and separately from the other unknowns. From the ratio of now-known income elasticities, the values of γ for industries 2-5 can now be found.

$$\eta_i/\eta_1 = (1 - \gamma_1)/(1 - \gamma_i) \qquad \perp \qquad \gamma_i \tag{92}$$

But an indeterminacy remains, and needs to be pinned down by specifying some relationship between the γ_i and ρ in the utility function (48) (with CES, all the $\gamma_i = \rho$). I have chosen to pin down the levels of the γ_i by setting the expenditure-weighted sum of the shares of the specific factor in each industries (equal to $\alpha_{ri} = (1 - \alpha_{xi})$) equal to the aggregate shares of the specific factors in total income denoted s_r. In the Table that follows, I arbitrarily use s_r = 1/3 which is its value in the simulations in the previous section.

$$\sum_i (1 - \alpha_{xi}) s_i + (1 - \alpha_{x1}) s_1 = s_r \qquad \perp \qquad \gamma_1 \tag{93}$$

Note that the value of s_r does not affect the modified income elasticities of demand. The final step need for the calibration is to set the α_{xj} equal to the formula in (48).

$$\alpha_{xj} = \gamma_j(\sigma - 1)/\sigma \qquad \perp \qquad \alpha_j \tag{94}$$

Equations (92), (93) and (94) are solved as a simultaneous system (in the code, I solve all equations (90)-(94) together as a single solve).

Table 2 gives some simulation results to show how this procedure works. In all sets of numbers, the three right-hand columns are made-up parameter values, but all could come from data and literature estimates in real data. The top two sets of results use a value $\sigma = 2.5$. The first solutions and the second for $\sigma = 2.5$ use different expenditure shares across sectors. The first use equal expenditure shares across the five sectors while the second has expenditures concentrated on he lower income-elasticity goods. We could imagine that the first set represents a high-income country and the second represents a low-income country.

The results in Table 2 for $\sigma = 2.5 > 1$ are consistent with the theory developed above. The η values that are consistent with the expenditures shares to satisfy Engels aggregation preserve their relative values η_{i1} (RELETA in the Table). The η values are somewhat higher for the "poorer" country consistent with our theory results. The values of γ and α are increasing in their sector's income elasticity η. The α are the key values for calibration. The inputs into *Y* production are composed of the observed *X* values and the (unobserved) *R* values, the latter given values according to the calculated α values and then added to the country's endowment.

The lower two sets of numbers in Table 2 give results for $\sigma = 0.5$, which is a generally-accepted value for the elasticity of substitution across broad sectors, although that doesn't mean it is a good guess at this intermediate level of disaggregation (literature references presented in Heblich, Redding and Zylberberg (2025)). Now the γ values are negative as noted in (52) and (53) above. With $\sigma = 0.5$, $\rho = (\sigma - 1)/\sigma = -1$, implying $\alpha = -\gamma$ for this particular value of σ. The γ values are increasing with the income elasticity (becoming less negative) and the α are decreasing with increases in η as discussed in (52) and (53).

Concluding this section, the point is that adding non-homothetic CRIE preferences to general-equilibrium is possible, and it will certain improve the accuracy of counter-factual experiments in which incomes are growing. SGM preferences add in the important features of choke prices and incomes. However, the important and challenging question of how to calibrate the unprofitability of initially inactive trade links leads me to leave that to a future project.

Table 3: Example of calculating/calibrating ALPHA input shares from known data for CRIE preferences

Known from data and/or literature estimates

SIGMA = 2.5

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.833	0.290	0.484	1.00	0.20	2.50
Sector 2	0.917	0.355	0.591	1.10	0.20	2.50
Sector 3	1.000	0.409	0.681	1.20	0.20	2.50
Sector 4	1.083	0.454	0.757	1.30	0.20	2.50
Sector 5	1.167	0.493	0.822	1.40	0.20	2.50

alternative expenditure shares

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.866	0.315	0.525	1.00	0.20	2.50
Sector 2	0.952	0.377	0.629	1.10	0.40	2.50
Sector 3	1.039	0.429	0.715	1.20	0.15	2.50
Sector 4	1.126	0.473	0.788	1.30	0.15	2.50
Sector 5	1.212	0.511	0.851	1.40	0.10	2.50

EXPSH	Observed data on expenditure shares across sectors
RELETA	Relative income elasticities (sector 1 numerarie) from literature estimates
SIGMA	Elasticity of substitution across sectors from literature estimates
ETA	Income elasticities calculated from EXPSH and RELATA satisfying Engles aggregation
GAMMA	Parameters calculated from ETAs, EXPSH, and simultaneous with ALPHAs
ALPHA	Shares of X in producing Y, calculated from SIGMAs and simulteneously from GAMMAs

SIGMA = 0.5

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.833	-0.972	0.972	1.00	0.20	0.50
Sector 2	0.917	-0.793	0.793	1.10	0.20	0.50
Sector 3	1.000	-0.644	0.644	1.20	0.20	0.50
Sector 4	1.083	-0.517	0.517	1.30	0.20	0.50
Sector 5	1.167	-0.409	0.409	1.40	0.20	0.50

alternative expenditure shares

	ETA	GAMMA	ALPHA	RELETA	EXPSH	SIGMA
Sector 1	0.866	-0.904	0.904	1.00	0.20	0.50
Sector 2	0.952	-0.731	0.731	1.10	0.40	0.50
Sector 3	1.039	-0.587	0.587	1.20	0.15	0.50
Sector 4	1.126	-0.465	0.465	1.30	0.15	0.50
Sector 5	1.212	-0.360	0.360	1.40	0.10	0.50

c:\\jim\research\SGM and CRIE\SG CRIE gms xlsx\CRIE calibration 2.gms
c:\\jim\research\SGM and CRIE\SG CRIE gms xlsx\CRIE CALIB2.xlsx

```
$TITLE: SG-model 1 text version  Stone-Geary Modified model MCP format
*       C:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\SG-model 1 text mcp.gms
*  small open economy
*  homogeneous goods, no Armington differentation

$ontext
Small open economy model with positive endowment of a good which is a
perfect substitute for good X1, denoted X1E, small endowment of X2E

  The benchmark accounting matrix:
              Production Sectors                  Trade      Welfare Consumers

  Markets|    X1    X2   X1P  X1E    X2P   X2E   TX21   TX12    W     CONS1
  ----------------------------------------------------------------------
    PX1   |  100        -100                      -0      0
    PX2   |       100             -00              0     -0

    PL    |  -25  -75                                                 100
    PK    |  -75  -25                                                 100

    PW    |                                                    351   -351
    PX1C  |              100  150                             -250
    PX1E  |                  -150                                     150
    PX2C  |                          100     1                -101
    PX2E  |                                 -1                          1
  ----------------------------------------------------------------------

$offtext

PARAMETERS
   SCALEX1  scales up or down the endowment X1E /1/
   SCALEX2  scales up or down the endowment X2E /1/
   ENDOW    scales up or down the endowment of L and K/1/
   TOT      world price PX1 over PX2 /1/;

NONNEGATIVE VARIABLES

    X1        Output of sector X1
    X2        Output of sector X2
    X1P       Output of X1P (produces PX1C from PX1)
    X1E       Output of X1E (produces PX1C from endowment PX1E)
    X2P       Output of X2P (produces PX2C from PX2)
    X2E       Output of X2E (produces PX2C from endowment PX2E)
    W         Produces welfare from PW1 and PX2
    TX21      Export X2 in eX1change for X1
    TX12      Export X1 in eX1change for X2

    PX1       Price of commoditX2 X1
    PX2       Price of commoditX2 X2  (used as numeraire here)
    PL        Price of primarX2 factor L
    PK        Price of primarX2 factor K
    PX1C      Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
    PX2C      Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
    PW        Price of welfare
    PX1E      Price of the endowment good - perfect substitute for X1 cannot be traded
    PX2E      Price of the endowment good - perfect substitute for X2 cannot be traded

    CONS      Consumer;

EQUATIONS

    COSTX1     Marginal cost - price X1
    COSTX2     Marginal cost - price X2
    COSTX1P    Cost of producing X1 consumption from X1 production
    COSTX1E    Cost of producing X1 consumption from X1E endowment
    COSTX2P    Cost of producing X2 consumption from X2 production
    COSTX2E    Cost of producing X2 consumption from X2E endowment
    COSTW      Cost of producing a unit of welfare
```

```
    COSTTX21   Cost of exporting X2 in exchange for X1
    COSTTX12   Cost of exporting X1 in exchange for X2

    MKTPX1     Supply-demand for X1
    MKTPX2     Supply-demand for X2
    MKTPL      Supply-demand for labor
    MKTPK      Supply-demand for capital
    MKTPX1C    Market for X1 consumption of good 1
    MKTPX2C    Market for X2 consumption of good 2
    MKTPW      Supply demand for welfare
    MKTPX1E    Supply-demand for endowment good 1
    MKTPX2E    Supply demand for endowment good 2

    INC        Consumer income balance;

COSTX1..   PL**0.25*PK**(0.75) =G= PX1;
COSTX2..   PL**0.75*PK**(0.25) =G= PX2;
COSTX1P..  PX1 =G= PX1C;
COSTX1E..  PX1E =G= PX1C;
COSTX2P..  PX2 =G= PX2C;
COSTX2E..  PX2E =G= PX2C;

COSTW..    PX1C**(250/351)*PX2C**(101/351) =G= PW;

COSTTX21.. PX2*TOT =G= PX1;
COSTTX12.. PX1*1.001 =G= PX2*TOT;

MKTPX1..   X1 =G= X1P + TX12 - TX21/TOT;
MKTPX2..   X2 =G= X2P + TX21 - TX12*TOT/1.001;

MKTPL..    100*ENDOW*PL =G= 0.25*(PL**0.25*PK**(0.75))*100*X1 +
                  0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..    100*ENDOW*PK =G= 0.75*(PL**0.25*PK**(0.75))*100*X1 +
                  0.25*(PL**0.75*PK**(0.25))*100*X2;

MKTPX1C..  100*(X1P + X1E)*PX1C =G= (250/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;
MKTPX2C..  100*(X2P + X2E)*PX2C =G= (101/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;

MKTPW..    W*351*(PX1C**(250/351)*PX2C**(101/351)) =G= CONS*351;

MKTPX1E..  1.5*SCALEX1 =G= X1E;
MKTPX2E..  (1/100)*SCALEX2 =G= X2E;

INC..      CONS*351 =G= PL*100*ENDOW + PK*100*ENDOW + PX1E*150*SCALEX1 + PX2E*1*SCALEX2;

MODEL SG1 /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX1E.X1E, COSTX2P.X2P, COSTX2E.X2E,
           COSTW.W, COSTTX21.TX21, COSTTX12.TX12,
           MKTPX1.PX1, MKTPX2.PX2, MKTPL.PL, MKTPK.PK, MKTPX1C.PX1C, MKTPX2C.PX2C,
           MKTPW.PW, MKTPX1E.PX1E, MKTPX2E.PX2E, INC.CONS/;

PL.L = 1; PK.L = 1; PX1C.L = 1; PX2C.L = 1; PX1.L = 1; PX2.L = 1; PX1E.L = 1; PX2E.L = 1;
PW.L = 1; X1.L = 1; X1P.L = 1; X1E.L = 1.5; X2.L = 1; X2P.L = 1; X2E.L = 1/100; W.L = 1;
PX2.FX = 1; CONS.L = 1;

OPTION ITERLIM = 0;
SOLVE SG1 USING MCP;

OPTION ITERLIM = 1000;
SOLVE SG1 USING MCP;


SETS I /I1*I25/;

PARAMETERS RESULTS1(I, *) economX2 size, RESULTS2(I, *) terms of trade,
           X10, X20, M, M0, PX10;

X10 = 0; X20 = 0; M0 = PL.L*ENDOW +PK.L*ENDOW;
SCALEX1 = 1; SCALEX2 = 1; TOT = 1/1.25;
```

```
SOLVE SG1 USING MCP;

LOOP(I,

ENDOW = 0.03*ORD(I) + 0.045;

SOLVE SG1 USING MCP;

M = PL.L*ENDOW +PK.L*ENDOW;

RESULTS1(I, "ENDOW") = ENDOW;
RESULTS1(I, "X1") = MAX(EPS, PX1.L*X1.L/M);
RESULTS1(I, "TX21") = PX1.L*TX21.L/TOT/M;
RESULTS1(I, "TX12") = PX1.L*TX12.L*1.001/M;
RESULTS1(I, "SHX1") = PX1.L*X1P.L/M;
RESULTS1(I, "INCELASX1")$(X10 GT 0) = ((X1P.L-X10)/X10)/((M-M0)/M0);
RESULTS1(I, "INCELASX2")$(X20 GT 0) = ((X2P.L-X20)/X20)/((M-M0)/M0);

X10 = X1P.L; X20 = X2P.L; M0 = PL.L*ENDOW +PK.L*ENDOW;

);
DISPLAY RESULTS1;
```

```
$TITLE: CRIE IN MCP USING FIXED FACTOR TRICK    SIGMA = 0.5
*       C:\jim\RESEARCH\SGM and CRIE\SG CRIE gms xlsx\CRIE-open-m-05 mcp.gms
*       THIS VERSON CALCULATES INCOME ELASTICITIES OF DEMAND

$onText
Small open economy model with a produced good X which is combined
with a fixed factor/good R to produce a utility good Y

      The benchmark accounting matrix:

                 Production Sectors                       Consumers
 Markets   X1     X1P     X2     X2P      Y1     Y2     W      CONS      alpha1x = 100/190
  PX1     100    -100                                                    alpha1r = 90/190
  PXIP            100                   -100                             alpha2x = 100/110
  PX2                    100    -100                                     alpha2r = 10/110
  PX2P                          100            -100                      beta11  = 25/100
  PR1                                    -90                    90       beta1k  = 75/100
  PR2                                           -10             10       beta21  = 75/100
  PY1                                    190           -190              beta2k  = 25/100
  PY2                                           110    -110
  PW                                                    300    -300
  PL      -25            -75                                   100
  PK      -75            -25                                   100

$offText

Parameters
 SIGMA   elasticity of substitution between Y1 and Y2 /0.5/,
 SIZE    multiplier on endowment of L and K primary factors /1/,
 TOT     world relative price p1 over p2 /1/;

NONNEGATIVE VARIABLES

   X1    production of X1 from L K
   X2    production of X2 from L K
   X1P   consumption of X1
   X2P   consumption of X2
   Y1    produces consumption good Y1 from X1 and R1
   Y2    produces consumption good Y2 from X2 and R2
   TX21  Export X2 in exchange for X1
   TX12  Export X1 in exchange for X2
   W     Produces welfare from PW1 and PX2

   PX1   price of X1
   PX2   price of X2
   PX1P  consumper price of the X1
   PX2P  consumer price of X2
   PR1   price of endowment of R1
   PR2   price of endowment of R2
   PY1   price of consumption good Y1
   PY2   price of consumption good Y2
   PW    consumer price index
   PL    price of labor
   PK    price of capital

   CONS  !representative consumer's income

EQUATIONS

   COSTX1     Marginal cost - price X1
   COSTX2     Marginal cost - price X2
   COSTX1P    Cost of producing X1 consumption from X1 production
   COSTX2P    Cost of producing X2 consumption from X2 production
   COSTY1     Cost of producing X1 consumption from X1E endowment
   COSTY2     Cost of producing X2 consumption from X2E endowment
   COSTTX21   Cost of exporting X2 in exchange for X1
   COSTTX12   Cost of exporting X1 in exchange for X2
   COSTW      Cost of producing a unit of welfare
```

```
   MKTPX1      Supply-demand for X1
   MKTPX2      Supply-demand for X2
   MKTPX1P     Supply-demand for X1
   MKTPX2P     Supply-demand for X2
   MKTPR1      Supply-demand for R1
   MKTPR2      Supply-demand for R2
   MKTY1       Supply-demand for Y1
   MKTY2       Supply-demand for Y2
   MKTPW       Supply demand for welfare
   MKTPL       Supply-demand for labor
   MKTPK       Supply-demand for capital

   INC         Consumer income balance;

COSTX1..   PL**0.25*PK**0.75 =G= PX1;
COSTX2..   PL**0.75*PK**0.25 =G= PX2;
COSTX1P..  PX1 =G= PX1P;
COSTX2P..  PX2 =G= PX2P;

COSTY1..   PR1**(90/190)*PX1P**(100/190) =G= PY1;
COSTY2..   PR2**(10/110)*PX2P**(100/110) =G= PY2;

COSTTX21.. PX2*TOT =G= PX1;
COSTTX12.. PX1*1.001 =G= PX2*TOT;

COSTW..    ((190/300)*PY1**(1-SIGMA) + (110/300)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= PW;

MKTPX1..   X1 =G= X1P + TX12 - TX21/TOT;
MKTPX2..   X2 =G= X2P + TX21 - TX12*TOT/1.001;

MKTPX1P..  PX1P*90*X1P =G= (90/190)*PR1**(90/190)*PX1P**(100/190)*190*Y1;
MKTPX2P..  PX2P*10*X2P =G= (10/110)*PR2**(10/110)*PX2P**(100/110)*110*Y2;

MKTPR1..   PR1*90 =G=  (90/190)*PR1**(90/190)*PX1P**(100/190)*190*Y1;
MKTPR2..   PR2*10 =G=  (10/110)*PR2**(10/110)*PX2P**(100/110)*110*Y2;

MKTY1..    190*Y1 =G= (190/300)*PY1**(-SIGMA)*((190/300)*PY1**(1-SIGMA) + (110/300)*PY2**(1-SIGMA))
                      **(SIGMA/(1-SIGMA))*300*W;
MKTY2..    110*Y2 =G= (110/300)*PY2**(-SIGMA)*((190/300)*PY1**(1-SIGMA) + (110/300)*PY2**(1-SIGMA))
                      **(SIGMA/(1-SIGMA))*300*W;

MKTPW..    300*W*((190/300)*PY1**(1-SIGMA) + (110/300)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= CONS;

MKTPL..    100*SIZE*PL =G= 0.25*(PL**0.25*PK**(0.75))*100*X1 +
                           0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..    100*SIZE*PK =G= 0.75*(PL**0.25*PK**(0.75))*100*X1 +
                           0.25*(PL**0.75*PK**(0.25))*100*X2;

INC..      CONS =G= PL*100*SIZE + PK*100*SIZE + PR1*90 + PR2*10;

MODEL CRIEMCP /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX2P.X2P, COSTY1.Y1, COSTY2.Y2,
           COSTTX21.TX21, COSTTX12.TX12, COSTW.W,
           MKTPX1.PX1, MKTPX2.PX2,  MKTPX1P.PX1P, MKTPX2P.PX2P, MKTPR1.PR1, MKTPR2.PR2,
           MKTY1.PY1, MKTY2.PY2, MKTPW.PW, MKTPL.PL, MKTPK.PK,INC.CONS/;


PL.L = 1; PK.L = 1; PR1.L = 1; PR2.L = 1; PX1P.L = 1; PX2P.L = 1; PX1.L = 1; PX2.L = 1;
PW.L = 1; X1.L = 1; PY1.L = 1; PY2.L = 1;
X1.L = 1; X1P.L = 1;  X2.L = 1; X2P.L = 1; Y1.L = 1; Y2.L = 1; W.L = 1;
PX2.FX = 1; CONS.L = 300;

OPTION ITERLIM = 0;
SOLVE CRIEMCP USING MCP;

OPTION ITERLIM = 1000;
SOLVE CRIEMCP USING MCP;
```

```
SIZE = 1.3;
TOT = 1.2;
SOLVE CRIEMCP USING MCP;

*$EXIT

SETS I /I1*I50/;
PARAMETERS
 RESULTS1(I, *) L = SIZE  SIGMA = 0.5
 RESULTS2(I, *) INC and SHARES
 SIZE0, X10, X20, INCOME, INC0
 INCELAS1, INCELAS2, CONS0, Y10, Y20;

SIZE0 = 0.15; X10 = .1; X20 = .1; INC0 = 1;
INCELAS1 = 1; INCELAS2 = 1; CONS0 = .1;

TOT = 1.;
SIGMA = 0.5;

LOOP (I,

SIZE =  0.2 + 0.05*ORD(I);

SOLVE CRIEMCP USING MCP;

INCOME = PL.L*100*SIZE + PK.L*100*SIZE;

INCELAS1 = (((X1P.L - X10)/X10)/((INCOME-INC0)/INC0))$(X10 GT 0);
INCELAS2 = (((X2P.L - X20)/X20)/((INCOME-INC0)/INC0))$(X20 GT 0);

RESULTS1(I, "L") = SIZE*100;
RESULTS1(I, "X1") = X1.L;
RESULTS1(I, "X2") = X2.L;
RESULTS1(I, "T12") = TX12.L;
RESULTS1(I, "T21") = TX21.L;
RESULTS1(I, "C1") = X1P.L;
RESULTS1(I, "C2") = X2P.L;
RESULTS1(I, "C1/C2") = (X1P.L/X2P.L)$(X2.L GT 0);
RESULTS1(I, "X1/X2") =  X1.L/X2.L;
RESULTS1(I, "INELAS1") = INCELAS1$(ORD(I) GT 1);
RESULTS1(I, "INELAS2") = INCELAS2$(ORD(I) GT 1);
RESULTS1(I, "RATIOINELAS") = (INCELAS1/INCELAS2)$(ORD(I) GT 1);

RESULTS2(I, "INCOME") = INCOME;
RESULTS2(I, "X1") = MAX(EPS, PX1.L*X1.L*100/INCOME);
RESULTS2(I, "TX21") = PX1.L*TX21.L*100*TOT/INCOME;
RESULTS2(I, "TX12") = PX1.L*TX12.L*100*TOT*1.*1.001/INCOME;
RESULTS2(I, "SHX1") = PX1.L*X1P.L*100/INCOME;

SIZE0 = SIZE;  X10 = X1P.L; X20 = X2P.L;
Y10 = Y1.L; Y20 = Y2.L; CONS0 = CONS.L;
INC0 = PL.L*100*SIZE + PK.L*100*SIZE;
);

DISPLAY RESULTS1, RESULTS2;


$exit

Execute_Unload 'CRIE-open.gdx' RESULTS1
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS1 rng=SHEET2!A3:M55'

Execute_Unload 'CRIE-open.gdx' RESULTS2
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS2 rng=SHEET2!A56:F109'
```

```
$TITLE: Calculate 2.gms James Markusen, University of Colorado, Boulder
* uses the BEA and other data to create an excel file calculate.xlsx
* this excel file must be used with this GAMS file
* it is load into the program, which solves for alpha1, alpha2 and rho
* using non-linear programming to minimize the sum of squared erros

SETS I /1*75/
     J /1*3/;

PARAMETERS
  MATRIX(I,J);

execute 'gdxxrw calculate.xlsx par=MATRIX rng=Sheet2!A1:D76'
execute_load 'calculate.gdx' MATRIX;

DISPLAY MATRIX;

SETS K /1*2/;
PARAMETERS
  GOODS(I), SERVICES(I), SHRATIO(I), ETA12, SHARES(I,K), ETARAT(I), SIGMA;

GOODS(I) = MATRIX(I,"1");
SERVICES(I) = MATRIX(I,"2");
SHRATIO(I) = MATRIX(I,"3");
DISPLAY GOODS, SERVICES, SHRATIO;

* minimize the sum of square errors solving for the ALPHA parameters

VARIABLES
  OBJ      objective function minimize the sum of squared errors
  RHO      elasticity parameter sigma = 1 over (1-rho);

NONNEGATIVE VARIABLES
  C   average ratio of expenditure shares services over goods,
  ALPHA1, ALPHA2;

EQUATIONS
  MINSSE    sum of squared errors;

MINSSE..  OBJ =E= SUM(I, POWER(SHRATIO(I) - (C*ALPHA1/ALPHA2)*(SERVICES(I)**(RHO*ALPHA1))
                    /(GOODS(I)**(RHO*ALPHA2)), 2));
MODEL
  calculate /MINSSE/;

* fix C as mean of share ratio from calculate.xlsx Sheet1
ALPHA1.L = 0.25; ALPHA2.L = 0.75; RHO.L = -1; C.L = 1;
C.FX = 1.445;

SOLVE calculate USING NLP MINIMIZING OBJ;

* N.B.  this is the formula when rho is negative (sigma < 1)
* if sigma > 1, change +alpha to -alpha

ETA12 = (1 - RHO.L*ALPHA2.L)/(1 - RHO.L*ALPHA1.L);
SIGMA = 1/(1-RHO.L);

DISPLAY ALPHA1.L, ALPHA2.L, ETA12, RHO.L, SIGMA;

execute 'gdxxrw calculate.xlsx par=SHARES rng=Sheet3!A1:C76'
execute_load 'calculate.gdx' SHARES;

DISPLAY SHARES;

* calculateulate the sigma levels using the budget adding up condition

NONNEGATIVE VARIABLES
  ETA1(I), ETA2(I);
```

```
EQUATIONS
  RATIO(I)
  ADDING(I);

RATIO(I)..  ETA1(I)=E= ETA12*ETA2(I);
ADDING(I).. SHARES(I,"2")*ETA1(I)+ SHARES(I,"1")*ETA2(I)=E=1;

MODEL LEVELS /RATIO.ETA1, ADDING.ETA2/;

ETA1.L(I) = 0.5; ETA2.L(I) = 0.5;

SOLVE LEVELS USING MCP;
ETARAT(I) = ETA1.L(I)/ETA2.L(I);
DISPLAY ETA1.L, ETA2.L, ETARAT;

PARAMETERS
 RESULTS(I,*), AVETA1, AVETA2;

RESULTS(I, "ETA1") = ETA1.L(I);
RESULTS(I, "ETA2") = ETA2.L(I);
RESULTS(I, "RATIO") = ETARAT(I);
AVETA1 = SUM(I, ETA1.L(I))/75;
AVETA2 = SUM(I, ETA2.L(I))/75;
DISPLAY RESULTS, AVETA1, AVETA2;

*$ontext
execute_unload 'calculate.gdx' RESULTS
execute 'gdxxrw.exe calculate.gdx par=RESULTS rng=SHEET7!A1:D76'
*$offtext

PARAMETERS
 YHAT(I), YBAR, SST, SSR, RSQR;

YHAT(I) = (C.L*ALPHA1.L/ALPHA2.L)*(SERVICES(I)**(RHO.L*ALPHA1.L))/(GOODS(I)**(RHO.L*ALPHA2.L));
YBAR = SUM(I, SHRATIO(I))/75;
SST = SUM(I, POWER(SHRATIO(I) - YBAR, 2));
SSR = SUM(I, POWER(SHRATIO(I) - YHAT(I), 2));
RSQR = 1 - OBJ.L/SST;
*RSQR = 1 - SSR/SST;
DISPLAY YBAR, YHAT, OBJ.L, SSR, SST, RSQR;
```