Microeconomic Modeling:
bridging analytical theory and numerical simulation

James R. Markusen
University of Colorado, Boulder
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Preliminary draft of book manuscript with accompanying computer code library. A complete first version including all 11 chapters.

This draft has not been proofread. I am confident that there are many small mistakes and I have already noted some inconsistencies in formatting and style. But the manuscript should be appropriate for evaluation and for providing me with feedback before I start in on refining and revising.

Thanks to all the great people who I have worked with and learned from over the years, but Thomas Rutherford foremost. Tom rescued me from the boredom and frustrations arising from the constraints and limitations of the strictly analytical approach to economic theory.

https://www.colorado.edu/faculty/markusen
https://en.wikipedia.org/wiki/James_Markusen

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Preface

The objective of this book is to construct an interface between analytical theory and applied (numerical or computable) general-equilibrium modeling. Analytical models have long suffered from the limitation in that this approach becomes intractable past quite simple cases. Even in relatively parsimonious cases, they produce only qualitative results that have little use in policy analysis. Numerical models, although not limited to small-dimension restrictions, have often failed to incorporate the richness of theory into their policy simulations. Here, I integrate contributions which I developed over many years, showing how ideas and concepts from analytical theory can be imbedded in numerical models for simulation and policy evaluation.

The book takes theoretical relationships as the raw material, with a particular emphasis on cases that are not analytically solvable, and transforming them in ones that are computationally solvable. These models and materials can be valuable for both use with real data and as a complement to analytical models by extending those basic models to more complex features. Much of my own published research exploits the latter approach which has been dubbed "theory with numbers". In a sense, an analytically solvable case becomes a "model of the model" for a more empirically-relevant numerical specification.

The book proceeds with a series of simple models that I developed over three decades to first teach myself, and then to construct both semester-long and one-week short courses for professionals wishing to develop modeling skills. In each case, an analytical / algebraic model and concept is converted into a computer-solvable one. This is not a textbook in the usual sense since all content is original and developed by myself. But I have in fact taught these techniques to advanced undergraduates and have been surprised at the quality of the projects they produce in the course of the term. Professional have used my model library as a reference manual for adding important features to their work.

I will code the numerical models in GAMS (General Algebraic Modeling System) which is an algebraic language allowing a straightforward translation of analytical models into

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[1]Markusen is a member of the Editorial Advisory Board of the Journal of Global Economic Analysis, the journal of GTAP (Global Trade Analysis Project). He is also a research fellow/associate of the international trade groups at the NBER, CEPR, and CESifo. Personal website and Wikipedia page:
https://www.colorado.edu/faculty/markusen
https://en.wikipedia.org/wiki/James_Markusen

numerically-solvable ones. But the book is not a software manual, nor is it a text on cge modeling: it does not venture into the manipulation, calibration and use of large data sets. There are extensive and well-advanced resources beyond my skill set for these important tasks. The solvers I use can be accessed from a number of alternative software option, and I believe that my algebraic code in GAMS can be transported easily to alternative software of your choice. More will be said in the opening section of Chapter 1. A complete model library will be provided along with the book, with I hope easy accessibility for users. With a new collaboration, all models are being translated into JULIA, a free open-source software.

> "Many branches of both pure applied mathematics are in great need of computing instruments to break the present stalemate created by the failure of the purely analytical approach to nonlinear problems"
>
> --- John Von Neumann, 1945

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James R. Markusen
University of Colorado, Boulder
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University of Colorado, Boulder
https://www.colorado.edu/faculty/markusen

Chapter 1: An Interface between Analytical and Numerical Modeling

1.1 Purpose, Motivation and Objectives

This project is a culmination of three decades of effort, with many revisions along the path. When I began my career in the early 1970s, I was almost exclusively involved in theoretical international trade research using analytical methods. But by the late 1980s, I became frustrated with the constraints and limitations imposed by the requirements that a model must have an analytical solution for it to be considered theory. The Von Neumann quote in the preface, although I believe in reference to partial differential equations, is perfectly applicable to economics.

I was fortunate enough to have colleagues at the University of Western Ontario who were early leaders in the development of software for computing general-equilibrium models, which I refer to as applied general equilibrium (age) but which is interchangeable with the more common term computable general equilibrium (cge). I was fascinated by their work and understood that it was far more useful for policy analysis and counterfactual simulation in general that the simple analytical models I was used to. But I discovered that this type of research was almost completely disjoint from analytical trade theory and econometric estimation, a gap that has largely persisted to this day. There is little overlap between the members of GTAP (Global Trade Analysis Project - the home of age/cge modeling) and mainstream academic international-trade economists.

Passing into the 1990s, I began to develop a niche in which I serve as a bridge between analytical theory and applied (numerical, computable) general-equilibrium modeling. The purpose of this book is to assemble and extend many separate small contributions into a coherent whole, showing how analytical theory can be imbedded in numerical models for counter-factual analysis of large parameter changes and for quantitative comparisons of alternative underlying structural assumptions. The models and techniques presented here have been used in my own research, and form the basis for both graduate and undergraduate courses in modeling, short courses for professionals, and serve as a reference manual for modeling. These models and materials can be valuable for both use with real data and as a complement to analytical models by extending those basic models to more complex features that defy analytical solutions.

Expanding a little on my previous comment, both analytical theory and applied general-equilibrium have strengths and weaknesses. Models that produce analytical solutions must in general be very simple in terms of dimensionality and in the number of features that they can incorporate: they are very far removed from our complex reality. Even the classic two-good, two-factor, two-country Heckscher-Ohlin model has no closed-form analytical solution, although assuming zero trade costs and endowments interior to the factor-price equalization set gets you close. But those restrictions yield a model which is useless for policy analysis.

Many limitations of age/cge modeling arose from the limitations of software and computing. Early models were restricted to perfect competition, constant returns to scale, ad valorem taxes and other restrictions. But they did allow for high dimension problems (numbers of goods, factors, countries, intermediate use matrices). Ironically, because of the incorporation of crucial real-world features, analytical modelers often dismissed the numerical models as "black boxes" without providing any policy-useful alternative. Both analytical modeling and applied general equilibrium have largely avoid boundary solutions or endogenizing zeros in outputs and trade flows although these are crucial features of the real economy.

The difficulty for me was that the limitations of numerical models preventing the incorporation of important theoretical features such as scale economies, imperfect competition, multinational firms and most all boundary solutions persisted as we went forward. So I began the process of developing "toy" models to teach myself and my students how to add theoretical complexity to both standard age/cge models and parsimonious analytical models. Many users in the applied general-equilibrium field have told me over the years that these have proved very useful in learning modeling. And this feedback has encouraged me to finally put the years of work together into this book.

1.2 What is In, What is Out

I will first begin with some general comments about modeling and underlying mathematical foundations, software and solvers. Then the book will proceed with a long series of simple models, grouped together in terms of their economic focus. The process proceeds from the simple to more complex cases, such as starting with a basic competitive models and then adding taxes, quantitative restrictions, externalities, imperfect competition, scale economies, public and infrastructure goods, and so forth. For each model, there is a text analysis of the problem and its algebraic formulation, and then the code itself. As a simple analogy, I suppose that my approach could be described as a "cookbook" for making numerical models from theory as the raw material. While there is a focus on general-equilibrium problems, I also provide examples from industrial organization and games, information, statistics, finance and operations research.

For the numerical models, I will use the software GAMS (General Algebraic Modeling System) as just noted in the preface. GAMS has an attractive advantage for getting started in that it is an algebraic language. I loved it for learning and teaching because the translation of an analytical theory model into code is simple and straightforward. Readers can expect that the code is quickly understandable in that it reads just like we read algebraic equations and models. But for those who have invested in alternative software, the "solvers" I use in GAMS can be accessed from those other platforms and it should be feasible to translate GAMS's algebraic structures into your preferred option. These solvers exploit complementarity formats to easily handle boundary solutions and endogenous zeros for many variables. I use Excel for reading from or writing to data or for making figures from simulation results.

In order to keep a relatively narrow focus along my own strengths and specialties, I will not venture into the use of "real" data, particularly the wonderful micro-consistent data bases of GTAP. This data base is very well developed and the foundation of modern applied general-equilibrium modeling. There is extensive documentation about it and many short courses in learning to use it are offered by GTAP. Similarly, the GAMS website has an extensive user guide for the more complex features of the software. But I hope that the basic features of GAMS that I use here are relatively easily understood: almost all the code can be followed with little reference to the user manual.

I will also avoid providing an analysis about details such a nested CES functions, intermediate use (input-output) matrices and calibration issues. While these are important for age/cge modeling, they are beyond the scope of this book and treated only superficially toward the end. Indeed, I will often translate a theory model to a numerical one using simple Cobb-Douglas functions with no intermediate inputs. As per the previous paragraph, there are many other sources which will provide better resources than I could offer. My focus remains on creating an interface between important features of economies analyzed in theoretical papers and examined in econometric estimation, and numerical models for policy and other counter-factual simulations.

1.3 Target Readers and Users

My objective is to provide a set of tools both for applied general-equilibrium modelers and more "mainstream" economists, two groups that have had little overlap. I the Global Trade Analysis Project consortium website notes about 30,000 members. These include economists in academia, government, NGOs and the private sector. I am not sure what the number of GAMS users (and potential users) might be, but likely a few thousand. Mainstream international trade economists, who rely much more on econometric estimation of analytical models, are turning more to numerical counter-factual simulations, the stuff of this proposed book, to explore their models' properties. There are hundreds of government, NGO, and private sector individuals and consulting firms that do at least some simulation modeling.

I believe that the book will appeal to a great many economists in international economics, public economics, industrial organization, environment and resource economics. I have taught a modeling course not only to graduate students and professionals, but also to advanced undergraduates who do not have more than intermediate microeconomics and calculus tools. I believe that the undergraduate classes have been successful, with students producing surprisingly creative computation projects as the course requirement. Many modelers who will want to use the GTAP data set or other off-the-shelf models and data can use this book to understand what is "under the hood" and extend those models to incorporate more advanced/realistic features.

The next section of this chapter details the basics of general-equilibrium modeling. But I do want to emphasize that there are also chapters on problems from industrial organization and games, information, statistics and operations research as noted earlier. A considerable effort is devoted to incorporating industrial-organization features into general-equilibrium formats.

1.4 Introduction to applied general-equilibrium modeling

This is a set of notes to introduce you to applied general-equilibrium modeling and software used to analyze applied GE problems. First some general comments about general-equilibrium modeling. There are many models which are portrayed by their authors' as "general equilibrium". The term assumes different meanings in different fields, so it is probably a good idea to begin with a definition of what this means. When we say general equilibrium, we are normally thinking of models which have the following characteristics.

(1) Multiple interacting agents

(2) Individual behavior based on optimization

(3) Most agent interactions are mediated by markets and prices

(4) Equilibrium occurs when endogenous variables (e.g., prices, quantities) adjust such that
  (i) agents, subject to their constraints, cannot do better by altering their behavior
  (ii) markets (generally) clear so, for example, supply equals demand in each market.
  (iii) incomes of agents balance expenditures

General-equilibrium theory in traditional economics is often quite abstract. A usual introductory formulation consists of a set of markets for goods and factors of production. Agents, which are typically labeled consumers and firms, optimize subject to the constraints they face such as technologies and budget constraints. These optimizations then lead to excess demand functions for each good and factor. Equilibrium is then obtaining by finding a set of prices such that all excess demands are zero. General-equilibrium theory is generally focused on abstract issues such as proving that a set of equilibrium prices and hence equilibrium itself exists.

While this is an important task, the theorists rarely bother with analyzing what those equilibrium prices are or how they are related to underlying features of the economy such as preferences, technologies and so forth. And it follows that the abstract theory is of little or no use in answering questions about how changes in policies such as taxes or tariffs influence the equilibrium. Some progress can be made in special theoretical models such as the Heckscher-Ohlin model of international trade. In this model, the direction of trade can be related to underlying technologies and factor endowments, and the effects of policies such as tariffs on welfare and the distribution of income among factor owners (the Stolper-Samuelson theorem) can be derived.

Yet even in the analytical Heckscher-Ohlin model, two problems persist. First, the results are "qualitative"; e.g., they give us the signs of comparative-statics derivatives or tell us that some elasticity is greater than one. But even the simplest HO model has no analytical solution, so analytical results cannot be much more precise than signing derivatives. Second, almost all results are only unambiguous in a version of the model in which there are two goods, two factors, two countries and consumers everywhere have identical and homogeneous preferences over goods. Three goods, three factors, three countries or two distinct consumer groups create problems that cause the elegant results of Heckscher-Ohlin to collapse.

Applied general-equilibrium modeling is the way around these difficulties, such that the concept of general-equilibrium actually becomes useful for analyzing real economies and real policies. Any number of good, factors, household types, and countries may be included. While the field started out with the assumptions of constant returns to scale and perfect competition in all production activities, we have learned how to incorporate scale economies and imperfect competition. We have learned how to include complex tax structures, public goods, externalities, and "rationing constraints" such as price controls or quotas that prevent markets from clearing.

Naturally, there is a price to be paid from the theorist's point of view. We have to assume specific functional forms for preferences, production functions, and so forth. Many parameters of these functions can be drawn from published data or estimated with econometrics, but others remain educated guess work. This exercise draws criticism from both theorists and econometricians alike, but in the end applied GE modeling delivers answers to policy questions, however imprecise those answers might be.

What exactly is an applied GE model? It begins by the following theory: an economy and the equilibrium conditions for that economy are translated into a mathematical formulation. General equilibrium is then represented as the solution to a well-defined mathematical problem. More specifically, there are two general ways of formulating this mathematical problem. The first is to model the economy as an optimization or programming problem. This tend to be the first way a student of economics would approach the problem, since optimization and optimization techniques are a fundamental part of the theory of the consumer and the theory of the firm. Thus general equilibrium could be thought of as the solution to a big linear or non-linear programming problem, in which some objective function is maximized or minimized subject to a set of constraints.

It turns out that representing equilibrium as the solution to an optimization problem becomes awkward when there are several households or countries. What is it that should be optimized? There is no clear objective function to optimize. The second way of approaching the problem follows from formal theory. Individual optimizing behavior and decisions of consumers and firms are embedded in functions describing the agents' choices in response to the values of variables facing them. So, for example, we use individual optimization to derive demand and supply functions that describe how consumers and firms will react to prices, taxes, and other variables.

Once we have done this, finding general-equilibrium is reduced to finding the solution to a square system of n equations in n unknowns. Individual behavior and optimization are embedded in those n equations. That is the approach we take here. An applied general-equilibrium model is a square system of n equations in n unknowns that is formulated in a fashion that permits a numerical solution by computational techniques, finding the actual values of the endogenous variables for given values of exogenous parameters. Endogenous variables include outputs, prices, trade volumes and so forth. Exogenous parameters include preferences, technologies, factor endowments and so forth.

As we will see shortly, the software we use permits a very important generalization of this notion of solving a square system of equations. For many economic problems, equilibrium may involve some goods not being produced or some possible trade links not being actively used. We really would like to formulate the general-equilibrium model as a system of weak inequalities, with each inequality associated with a particular non-negative variable such as a price or quantity. If a particular weak inequality holds as an equation, then the associated variable is strictly positive. If it holds as a strict inequality, then the associated variables is zero.

An example of this for a competitive model is the requirement that, in equilibrium, the profits from a given production activity must be non-positive. The associated variable to this inequality is the output level of that activity. In equilibrium, the weak inequality may hold as a strict equality, in which case there is positive output. If it holds as a strict inequality, (potential) profits from that production activity are negative, and no output is produced.

Thus we will formulate a general equilibrium model as a square system of weak inequalities, each with an associated non-negative variable. This is referred to as a complementarity problem in mathematics, and the associated variables are referred to as complementary variables.

Software other than that used here (GAMS and MPS/GE) generally do not allow the user to solve complementarity problems, greatly limiting model formulation and the range of comparative statics questions analyzed by the modeler.

Steps in Applied General-Equilibrium Modeling

Here are the "normal" steps in applied general-equilibrium modeling.

(1) Specify dimensions of the model.
- Numbers of goods and factors
- Numbers of consumers
- Numbers of countries
- Numbers of active markets

(2) Chose functional forms for production, transformation, and utility functions; specification of side constraints.
- Includes choice of outputs and inputs for each activity
- Includes specification of initially slack activities

(3) Construct micro-consistent data set.
- Data satisfies zero profits for all activities, or if profits are positive, assignment of revenues
- Data satisfies market clearing for all markets, income balance for agents

(4) Calibration – parameters are chosen such that functional forms and data are consistent.
- By "consistent" we mean that the data represent a solution to the model

(5) Replication – run model to see if it reproduces the input data.

(6) Counter-factual experiments.

Steps (3) and (4) are not strictly speaking necessary. The software can be used for pure simulation analysis, in which there is no data initially. However, in learning the software, it is very valuable to start by writing down a micro-consistent data set and then transform that into code such that the solution to the model reproduces the initial data.

Let's now turn to a concrete example of a simple general-equilibrium model.

Example: 2-good, 2-factor closed economy with fixed factor endowments, one representative consumer.

Take a very simply economy, two sectors (X and Y), two factors (L and K), and one representative consumer (utility function W). L and K are in inelastic (fixed) supply, but can move freely between sectors. p_x, p_y, p_l, and p_k are the prices of X, Y, L and K, respectively. CONS is consumer's income and p_w will be used later to denote the price of one unit of W. These are the equations of the model with L^* and K^* fixed endowment parameters.

(1) $X = X(L_x, K_x)$

(2) $Y = Y(L_y, K_y)$

(3) $L^* = L_x + L_y$

(4) $K^* = K_x + K_y$

(5) $W = W(X, Y)$

(6) $CONS = p_l L^* + p_k K^* = p_x X + p_y Y$

How do we find equilibrium, which in this case is a set of prices, and factor allocations to the two sectors? Many economists' first reaction would be to formulate equilibrium as the solution to an optimization problem. Equilibrium could be solved for by a constrained optimization problem: Max (5) subject to the constraints (1), (2), (3), (4), and (6).

While this would work, the usefulness of this approach breaks down quickly as the model becomes more complicated. Suppose, for example, there are two different consumer types with different preferences and different factor endowments. What do we maximize? You could maximize the utility of one consumer subject to an arbitrary fixed level of the utility of the other consumer, exploiting the first theorem of welfare economics. But unless you are extraordinarily lucky, the solution will give each consumer an implied expenditure level which is not equal to the consumer's income. Thus there is an inconsistency in the proposed solution.

The alternative approach is to convert the problem to a system of equations that embodies optimization at the level of the sector and consumer, and solve that system. First, solve the underlying cost minimization problems for producers and consumers, so that individual optimizing behavior is embedded in the model. In the present model, we want to solve for cost functions for each sector, which embody efficient and optimizing cost-minimizing behavior. These give the minimum cost of producing a good at given factor prices.

Similarly, we can solve for a cost function for the consumer, commonly called an expenditure function, which gives the minimum cost at given commodity prices of buying one unit of utility or welfare (W). These functions are given as follows.

Unit cost functions for X and Y $\quad cx = cx(_{pl}, p_k), \quad cy = cy(p_l, p_k)$

Unit cost (expenditure) function for W $\quad e = e(p_x, p_y)$

The next crucial step is provided by theory, Shepard's lemma in particular. This result, which in turn relies on the envelope theorem, states that the partial derivatives of these functions are quantities. In particular, we have:

$$\frac{\partial cx}{\partial p_l} = cx_{pl} = \text{ X producer's demand for labor per unit of output (similarly for Y)}$$

$$\frac{\partial cx}{\partial p_k} = cx_{pk} = \text{ X producer's demand for capital per unit of output (similarly for Y)}$$

$$\frac{\partial e}{\partial p_x} = cx_{px} = \text{ Consumer's demand for X per unit of utility (similarly for Y)}$$

Now we are in a position to specify general equilibrium as the solution to a square system of 9 weak inequalities in 9 unknowns. Formulating the system as weak inequalities with matched non-negative variables allows for endogenous boundary solutions in which some variables are zero. Analysis is temporarily postponed until the following section on complementarity problems. For the moment, we can ignore the direction of the inequalities and their corresponding complementary variables: just consider the inequalities as strict equalities.

General-equilibrium formulated as a square system:

	Inequality		Complementary Variable
(1)	Non-positive profits for X	$cx(_{pl}, p_k) \geq p_x$	X
(2)	Non-positive profits for Y	$cy(_{pl}, p_k) \geq p_y$	Y
(3)	Non-positive "profits" for W	$e(p_x, p_y) \geq p_w$	W

(4)	Supply $\geq$ Demand for X	$X \geq e_{px}(p_x, p_y)W$	p_x,
(5)	Supply $\geq$ Demand for Y	$Y \geq e_{py}(p_x, p_y)W$	p_y
(6)	Supply $\geq$ Demand for W	$W \geq CONS / p_w$	p_w
(7)	Supply $\geq$ Demand for L	$L^* \geq cx_{pl}X + cy_{pl}Y$	p_l
(8)	Supply $\geq$ Demand for K	$K^* \geq cx_{pk}X + cy_{pk}Y$	p_k
(9)	Income balance	$CONS = p_l L^* + p_k K^*$	$CONS$

These weak inequalities can be solved for the unknowns: X, Y, W, p_x, p_y, p_w, p_l, p_k, and I.

Note that these inequalities are of three types, and this is generally true of a very large class of general-equilibrium models. These three types are:

- Zero-profit conditions, inequalities (1)-(3) in the above example.
- Market clearing conditions, inequalities (4)-(8) in the above example
- Income balance, equation (9) in the above example.

Now let's turn to the issue of starting with a micro-consistent data set, a set of numbers which are in fact consistent with the above problem formulation. That is, let's start with a set of numbers that satisfy zero profits, market clearing, and income balance.

The above problems can be thought of as consisting of three production activities, X, Y, and W, and four markets, X, Y, L, and K.

In what follows, we will represent the initial data for this economy by a rectangular matrix. This matrix is related to the concept of a "SAM" – social accounting matrix, which is discussed later. But the term SAM has been used in a rather different sense, so we will just refer to our rectangular matrix as a "MCM" – micro-consistency matrix.

In the present example, there are two types of columns in the rectangular MCM, corresponding to production sectors and consumers. In the model outlined above, there are three production sectors (X, Y and W) and a single consumer (CONS). Rows correspond to markets in the present example. Complementary variables are prices, so we have listed the price variables on the left to designate rows.

In the MCM, there are both positive and negative entries. A positive entry signifies a receipt (sale) in a particular market. A negative entry signifies an expenditure (purchase) in a particular market. Reading down a production column, we then observe a complete list of the transactions associated with that activity. Here is our matrix of initial values.

	Production Sectors			Consumers	
Markets	X	Y	W	CONS	Row sum
P	100		-100		0
PY		100	-100		0
PW			200	-200	0
Pl	-25	-75		100	0
PK	-75	-25		100	0
Column sum	0	0	0	0	

A rectangular MCM is "balanced" or "micro-consistent" when row and column sums are zeros. Positive numbers represent the value of commodity flows into the economy (sales or factor supplies), while negative numbers represent the value of commodity flows out of the economy (factor demands or final demands).

With this interpretation, a row sum is zero if the total amount of commodity flowing into the economy equals the total amount of commodity flowing out of the economy. This is market clearance, and one such condition applies for each commodity in the model.

Columns in this matrix correspond to production sectors or consumers. A production sector column sum is zero if the value of outputs equals the cost of inputs. A consumer column is balanced if the sum of primary factor sales equals the value of final demands. Zero column sums thus indicate zero profits or "product exhaustion" in an alternative terminology.

I emphasize that the numbers of the matrix are values, prices times quantities. The modeler is free as to how to interpret these as prices versus quantities. A good practice is to choose units so that as many things initially are equal to one as possible. Prices can be chosen as one, and "representative quantities" for activities can be chosen such that activity levels are also equal to one (e.g., activity X run at level one produces 100 units of good X). In the case of taxes, both consumer and producer prices cannot equal one of course, a point we will return to in a later section.

The final step is to choose specific functional forms and calibrate the model, so that running the model reproduces the benchmark data. Here, I illustrate this by choosing Cobb-Douglas functional forms for all three activities. I will present the ultra-basics of CES functions including the Cobb-Douglas specific case in an appendix. But this is something you should be familiar with from MA level microeconomics. The numerical coefficients in the following are the representative quantities, so that the solution to the model will involve all variables except *CONS* equal to one (the latter could also equal one if it is multiplied by 200 wherever it appears).

(1) Non-positive profits X $\quad 100\,(p_l^{0.25}\; p_k^{0.75}) \;\geq\; 100\, p_x$

(2) Non-positive profits *Y* $\quad 100\,(p_l^{0.75}\; p_k^{0.25}) \;\geq\; 100\, p_y$

(3) Non-positive profits W $200\,(p_x^{0.50}\,p_y^{0.50}) \ \geq\ 200\,p_w$

(4) Supply ≥ Demand for X $100\,X \ \geq\ 100\,W p_w / p_x$

(5) Supply ≥ Demand for Y $100\,Y \ \geq\ 100\,W p_w / p_y$

(6) Supply ≥ Demand for W $200\,W \ \geq\ CONS / p_w$

(7) Supply ≥ Demand for L $100\,Lendow \ \geq\ 25\,X(p_l^{0.25}\,p_k^{0.75})\,/p_l \ +\ 75\,Y(p_l^{0.75}\,p_k^{0.25})\,/p_l$

(8) Supply ≥ Demand for K $100\,Kendow \ \geq\ 75\,X(p_l^{0.25}\,p_k^{0.75})\,/p_k \ +\ 25\,Y(p_l^{0.75}\,p_k^{0.25})\,/p_k$

(9) Income balance $CONS \ =\ 100\,Lendow\,p_l \ +\ 100\,Kendow\,p_k$

This is now the complete model calibrated to the benchmark data with *Lendow* and *Kendow* = 1. After converting this to code, running the model should then pass the replication check, with all variables taking on a value of 1 expect *CONS*. Simple counter-factuals could be changing the values of the parameters *Lendow* and/or *Kendow*.

1.5 Complementarity: Boundary (Corner) Solutions, Regime and Technology Switches, Endogenous Zeros

The solvers used in GAMS allow a solution to one nagging issue of general-equilibrium modeling that has generally persisted throughout its history. This is the difficulty of endogenizing corner solutions and regime switches. These usually involve the fact that most economic variables such and prices and quantities are by nature non-negative. But with existing older mathematical formulations, it was difficult to allow for variables to take on either positive or zero values. If assumptions could be made such that positive benchmark values stayed positive or constrained to remained zero, then the model could be formulated as a set of equations and we knew how to solve large systems of equations. Allowing for endogenous zeros required formulating a model as a system of weak inequalities.

An early and still-used tactic for ruling out zeros is referred to as the Armington assumption: goods in a sector are differentiated by country of origin and, with CES preferences, every country will always produce and export in all sectors (unless some sector or trade link is constrained to be inactive throughout). If I understand it correctly, restrictions ensuring no zeros persists today in the popular and highly cited Eaton and Kortum (2002) model which relies on probabilistic production. But the world trade matrix, exports by country i to country j in industry k, is full of zeros, as much as 80 percent depending on the degree of aggregation.

Traditional approaches either require a flow that is initially zero to remain zero under counter-factuals, or else the zero is replaced by a very small positive number, with assumptions such that it can never go to zero (the demand price goes to infinity as the quantity goes to zero). So there is no choke price above which a good is not purchased and no choke income below which it is not purchased. There can be no change in the extensive margin of trade. In addition to the issue of the large share of zeros in the trade matric just mentioned, there cannot be switches in the direction of trade or switches in which available technology is used to produce a good, electricity being a common example.

The approach throughout this book is to model general equilibrium as a complementarity problem. This approach was first formulated by Mathiesen (1985) and implemented in numerical models by Rutherford (1985, 1995). Deriving its intuition from the Karush-Kuhn-Tucker theorem (KKT: Karush 1939, Kuhn and Tucker (1951)), a model is specified as a system of weak inequalities, each with a complementary non-negative variable. If a weak inequality holds as an equation (e.g., marginal cost greater-than-or-equal to price), then the complementary variable is positive (output). If it holds as a strict inequality, then the complementary variable is zero. Of course, the complementarity approach works perfectly well in models in which there can be no endogenous zeros: it will find the interior solution.

Suppose we have an equation or set of equations in implicit form $F(Z) = 0$, that we wish to solve. But there is a complication in that there are inequality side constraints $G(Z) \geq 0$. It may be or likely be that any solution to $F(Z) = 0$ violates one (or more) of the inequality constraints $G(Z) > 0$. A typical complementarity problem reformulates the set of equations as weak inequalities, $F(Z) \geq 0$. Each of these is matched with a specific weak inequality side constraint. The complementarity condition is that the product of the each weak inequality F times the matched weak inequality G equals zero. A complementarity problem is thus written as

$$\text{Solve} \quad F(Z) \geq 0 \qquad s.t \quad G(Z) \geq 0 \qquad F(Z)^T G(Z) = 0 \tag{1}$$

In economics, the archetype situation is that the Z are constrained to be non-negative. In such cases, $G(Z) \geq 0$ reduces to $Z \geq 0$. Later, I introduce a heterogeneous firm model in which there are two-sided bounds on Z, but for now let's stick with $Z \geq 0$. The formulation in (1) looks simple, but it does require the correct direction of the inequalities and the correct association of each member of the set $F(Z)$ with a member of $G(Z)$. These must come from the modeler's knowledge of the underlying problem.

We can develop the intuition with a simple supply-demand model, good X with price p_x, an intuition that carries through to much more complex general-equilibrium models. First, there is the optimization condition that price equals marginal cost, $mc(X)$. But a non-negativity restriction on X may mean that the solution is that marginal cost is greater than the demand price p_x at $X = 0$ and the good is not produced. Our economics tells us that the inequality cannot run the other way, price greater than marginal cost leaves an unexploited profit opportunity. So the correct formulation of the problem is

$$mc(X) - p_x \geq 0 \qquad X \geq 0 \qquad (mc(X) - p_x)X = 0 \tag{2}$$

If $mc > p$ in equilibrium, X is not produced. There are many examples of this in economics. One is that there are multiple technologies for producing X (electricity), but some are unprofitable under current economic conditions ((2) refers to a specific one of those technologies). Second, the world trade matrix is full of zeros: it is unprofitable at the equilibrium to export good i from country j to country k.

The other type of situation that requires a complementarity condition due to a non-negativity constraint is a market clearing equation, supply equal demand, $D(p_x)$. But it may be that the solution involves excess supply in equilibrium, implying that the price is zero.

$$X - D(p_x) \geq 0 \qquad\qquad p_x \geq 0 \qquad\qquad (X - D(p_x))p_x = 0 \tag{3}$$

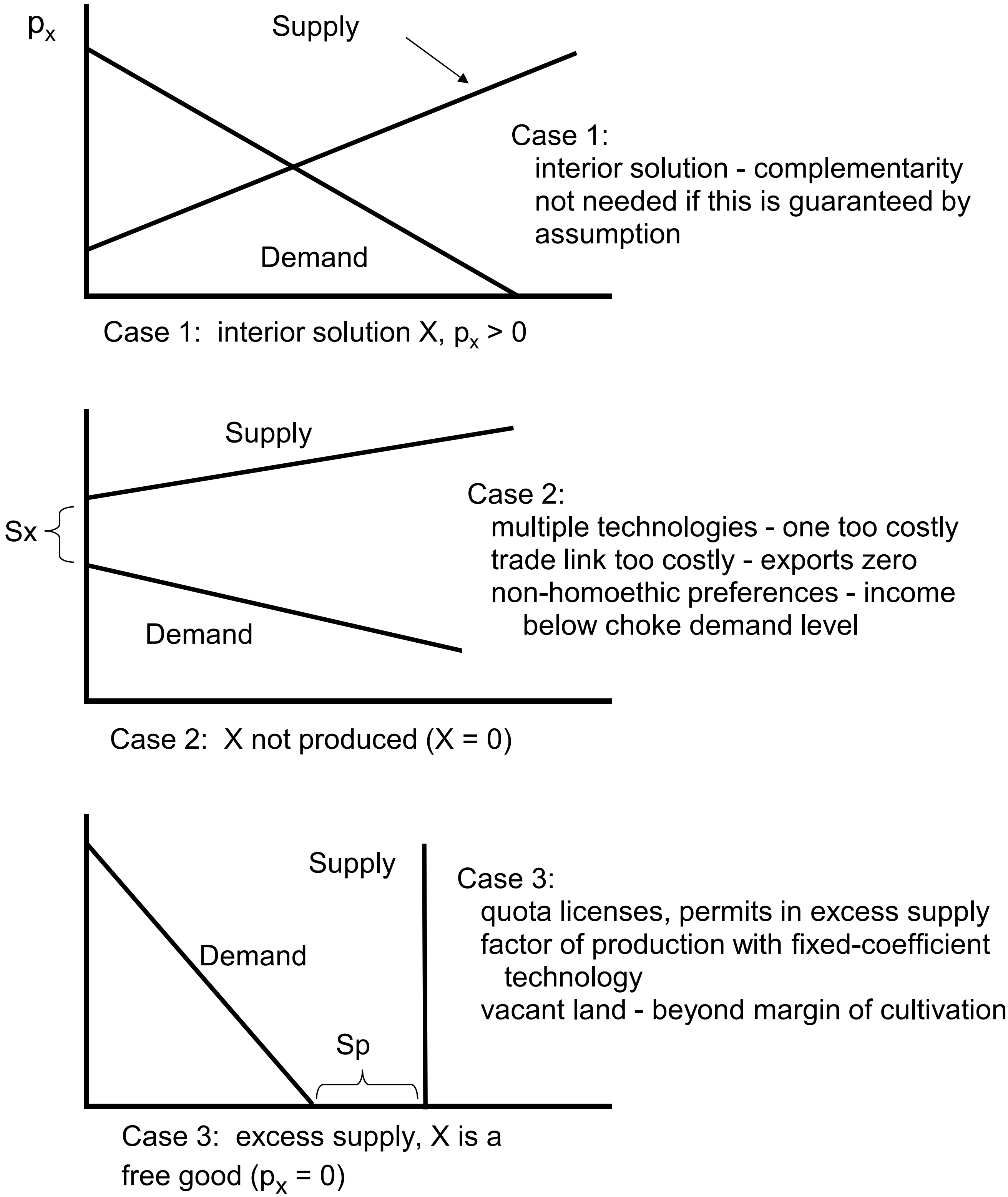
If supply > demand then $p_x = 0$ and X is a free good. This outcome is less common than the optimization condition example, but situations include cases where licenses or permits are in excess supply, a quota is non-binding in equilibrium (X is the suuply of licenses or quota level), or a factor of production in excess supply with fixed-coefficient (Leontief) technologies.

Let me emphasize one last time that the modeler must get the direction of the inequality right and correctly match each inequality to a non-negative variable. A good way to intuit the correct association is to think about the economics of what must be true if a particular weak inequality holds as a strict inequality as we have just done here. The complementary variable to a zero-profit condition is a quantity, the activity level. The complementary variable to a market clearing equation is the price of that good or factor. The complementary variable to an income balance equation is just the income of that agent. Refer back to the specification of our simple model given above, in which each inequality in matched to a variable. As an exercise, make sure you understand the direction of the inequality and why, if the inequality is strict, the matched variable must be zero.

Figure 1-1 gives the three possible outcomes of this simple partial-equilibrium, supply-demand problem. The top panel is the classic undergraduate textbook case of an interior solution. If we knew this was going to be the case, complementarity is not needed; that is, there is no need to associate each equation to a specific variable. The middle panel shows a situation in which the supply curve lies everywhere above the demand curve. This is a case where finding the solution does require complementarity: good X will not be produced in equilibrium. The bottom panel shows the third possibility, where the supply curve lies everywhere to the right of the demand curve (in the case of X being permits or licenses, the supply curve is generally vertical). The complementarity formulation yields a solution in which the price is zero and X is a free good.

How does the modeler know which of these situations will be the solution, especially in a complex general-equilibrium model; e.g., the positions of the supply and demand curves depend on all the other variables in the model? The Karush (1939) and Kuhn and Tucker (1951) theorems come to the rescue. Their formulations are to introduce non-negative "slack" variables into the problem, which explicitly incorporate the complementarity restriction into the model and convert the weak inequalities to equations. Denote the non-negative "slack" variables as s_x, and $s_p \geq 0$. The addition of two variables requires two added equations, and these are the complementarity conditions. The problem is written as

Figure 1-1: Three equilibrium outcomes of the supply/demand complementarity problem (model M2-3)



$$mc(X) - p_x - s_x = 0 \qquad s_x = \text{unprofitability at } X = 0 \tag{4}$$

$$s_x X = 0 \tag{5}$$

$$X - D(p_x) - s_p = 0 \qquad s_p = \text{excess supply at } p_x = 0 \tag{6}$$

$$s_p p = 0 \tag{7}$$

We now have a model of four equations in four unknowns. This can be solved by an iterative procedure such as the Newton method, and that is done for the modeler in the PATH solver, with the solver adding the slack variables and complementarity equations: the modeler just specifies the weak inequalities and matched variables. There is no need to write a brute force procedure in which different solutions are all tried out: there are only three in Figure 1, but full GE models made have dozens to hundreds. The values of the slack variables at the solution (equal zero in an interior solution) give the unprofitability of the technology/trade link and the excess supply and correspond to the measures with the same notation in panels two and three of Figure 1.[1]

1.6 Solvers Included in GAMS and in Other Modeling Formats

Solvers refer to software program that, as their names suggest, do the heavy lifting of actually solving the numerical models. In GAMS, the solvers are largely independently written and many can be accessed in other software programs such as Python, Julia, R and perhaps others. In this book, we will use three solvers. All three are in complementarity formats and so solve for corner solutions and endogenous zeros.

NLP: non-linear programming: optimize an objective function subject to equality and inequality constraints. The standard NLP solver in GAMS is called CONOPT (as in constrained optimization), the property of the Danish company ARKI Consulting. While very powerful, an NLP requires a single objective function. Most of the models in this book cannot be represented as the solution to a single optimization problem.

MCP: mixed complementarity problem: square system of equations/inequalities and unknowns matching each inequality with a specific variable ("mixed" refers to a mixture of equalities and inequalities, not the best terminology). This will be used heavily in this book. Optimization is embodied at the level of the firm/industry, the household, and in some cases the government. As explained earlier, that is why we construct a general-equilibrium model using

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[1]KKT are the necessary conditions for a solution to a non-linear programming problem with inequality side constraints. Strictly speaking, a market clearing equation is not a KKT condition. But the logic of KKT was extended to economic equilibrium problems by Mathiesen (1985) and Rutherford (1985, 1995). This in turn became the foundation for the PATH MCP solver in GAMS. The values of the slack variables are given in the solution file in GAMS where they are called "marginals".

cost and expenditure functions instead of production and utility functions. PATH is the property of Michael Ferris and colleagues at the University of Wisconsin, Madison.

MPEC: mathematical programming with equilibrium constraints, also called NLPEC. MPEC/NLPEC combines the other two. There is a single objective function to be maximized, but the constraint set is itself a MCP general-equilibrium model. MPEC is a powerful tool for finding the values of optimal taxes, tariffs, public goods provisions, and for optimal redistribution policies to maximize a social welfare function. This solver is the property of GAMS corporation.

The next chapter is an introduction both to GAMS and to the relationship between an NLP and an MCP problem. I feel that economists have had much more schooling in NLP problems, and I want to make sure that users can understand MCP problems. Unfortunately, other software commonly used in age/cge modeling makes strong limiting assumptions to forces interior solutions for all variables. This makes those options unable to compute boundary solutions, endogenize zeros in trade and production activities, and makes changes in the extensive margin of trade difficult or impossible to compute.