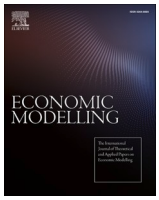
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Exploiting complementarity in applied general-equilibrium models

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Both analytical general-equilibrium models and applied applications generally make compromises to avoid endogenous zeros and discrete regime shifts. In open economy models, assumptions such as national-level product differentiation, a continuum of firms, probabilistic production, and smooth parametric distribution functions ensure interior, closed-form solutions to systems of equations. But resulting complexities restrict models to sparse general-equilibrium structures. I offer an alternative by exploiting non-linear complementarity. While this approach is applicable to many problems where boundary solutions are important, I focus on heterogeneous firms, endogenous exporting and multinational production. There is a discrete set of firm types allowing for any differences in productivity which can be calibrated from observed market shares. The model incorporates large firms with endogenous markups and solves for the set of active firms and their supply modes (no entry, domestic, exporting, multinational). Although focusing on international trade, I argue that the complementarity formulation is valuable for a wide range of economic equilibrium problems, allowing empirically-relevant policy analysis.

1. Introduction

The purpose of this paper is to offer an alternative approach to one nagging issue of general-equilibrium modeling that has generally persisted throughout its history. This is the difficulty of endogenizing corner solutions and regime switches. This has occurred in both analytical models where dealing with inequalities is awkward past very small-dimension problems and in formulating numerical simulation models. In both cases, assumptions are made such that a model consists of a set of equations, preferably with close-form solutions for analytical versions. For counter-factual simulations in numerical models or comparative statics in analytical models, assumptions ensure that positive benchmark values stayed positive or initial zeros are constrained to remained zero.

An early tactic for ruling out switching between active and inactive activities in numerical models is referred to as the Armington assumption in trade applications. Goods in a sector are differentiated by country of origin, and, with CES preferences, every country will always produce and export in all sectors (unless some sector or trade link is constrained to be inactive throughout). Micro-foundations for this approach are weak, but it has remained a workhorse tool to the present. Restrictions ensuring no endogenous zeros persist today in the popular and highly cited Eaton and Kortum (2002) model. This approach relies on probabilistic production for analytical solutions, with all countries producing and exporting at least some small amount in all sectors. There can be no changes in the extensive margin of trade, and indeed under some parameterizations it is equivalent to Armington. But the world trade matrix, exports by country i to country j in industry k, is full of zeros, as much as 80 percent depending on the degree of aggregation.

A major contribution is that of Melitz (2003), who tackled the problem of heterogenous firms and endogenous choices of entry and the decision to export. This is also an analytical model, relying on atomistic firms, a smooth continuous distribution of firm productivities, and probabilistic productivities draws. But it has difficulties dealing with large firms and discrete jumps in variables such as firm numbers or asymmetries among firms in the continuum (e.g., variable versus fixed cost differences in technologies).

I adopt a deterministic formulation as a simple alternative to what I term the probabilistic-continuum approaches.[1] Instead, I formulate general equilibrium as a complementarity problem. This has been relatively rarely exploited in general-equilibrium analytical models. It

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[1] "Probabilistic-continuum approach" is my term, referring to the models just mentioned plus the general use of CES functions. These ensure that solutions do not involve boundary solutions and solving weak inequalities. I suppose that my alternative could be nicknamed "determinist-discreteness approach".

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has been used in perfect-competition applied general-equilibrium models (AGE) however. This was first developed by Mathiesen (1985) and implemented in numerical models by Rutherford (1985, 1995). Deriving its intuition from the Karush-Kuhn-Tucker theorem (KKT: Karush 1939; Kuhn and Tucker (1951)), a model is specified as a system of weak inequalities, each with a complementary non-negative variable. If a weak inequality holds as an equation (e.g., marginal cost greater-than-or-equal to price), then the complementary variable is positive (output). If it holds as a strict inequality, then the complementary variable is zero. The insertion of "slack" variables converts weak inequalities to equations, which can be solved by algorithms such as the Newton method.

I will not attempt a general presentation here, but rather focus on using the tools of complementarity to offer an alternative way to incorporate heterogeneous firms and their mode choices (domestic, exporting, multinational) into AGE models. The models developed do not have analytical solutions, and I believe that this explains their relative scarcity of applications. At least in international economics, there is a high preference for analytical models with closed-form solutions, but their direct translation into numerical simulations for counterfactuals favors the probabilistic-continuum approaches restricted to interior solutions. Here, I develop analytical models which are then solved and simulated numerically.

My view is that analytical tools based on probabilistic production are complex and difficult to incorporate into high-dimension applied general-equilibrium models. Their resulting simple structures cannot address some current policy issues such as trade diversion (China to Vietnam) or foreign firms switching from exporting to building foreign plants (Taiwanese and Korean semi-conductor firms). In addition to ruling out regime, location, and mode switches, a number of other restrictive assumptions are typically (but not universally) made.

First, much of the literature continues to use "large-group" monopolistic competition (MC) in which it is assumed that firms are too small to affect the price index in their industries, leading to constant markups.[2] This removes a difficult endogeneity from the models, but leads to counter-empirical results such as all firms having the same price to marginal cost ratios and an inability to deal with large market-share firms. Second, the pattern of productivities across firms must follow a narrow class of parametric distribution functions so as to permit tractable integration. But the actual values of firm productivities/sizes may depart substantially from any parametric distribution. Third, aggregate profits are disposed of by the assumption that firms must pay for "draws" to learn their productivity. This removes another awkward simultaneity in general equilibrium, allowing total income to be independent of profits despite the fact that aggregate profits are an important source of aggregate GPD.

The alternative way to model heterogeneous firms offered here avoids the problems just mentioned and has the further advantage of a much simpler algebraic formulation. The basic concept is to break the data on the ranking of firms (e.g., sales) into a discrete set of firm "types". Firm types differ in their marginal cost. There is free entry into firm types, but only up to a limited number for each type. This limit per types is the discrete equivalent of the continuous parametric distribution used in the literature which trivially allows only one firm at each point in the continuum. The model will solve for the set of active firm types depending on parameters such as the economy's size or trade costs. Positive aggregate profits are added to total income. The entry limit could result from scarce entrepreneurs, patents, or other types of scarce firm-specific intangible assets that prevent entrants from capturing the productivity of top firms. Entrants cannot just duplicate Nvidia's or TSCM's technology. Bernard et al. (2022) identify additional sources of heterogeneity.

Formulating a heterogeneous-firm model exploiting complementarity, KKT and using the PATH MCP (mixed complementarity problem) solver in GAMS and available in other software has a number of advantages, more or less the opposite of the disadvantages of the traditional continuous approach noted above. First, no integrals or integration are needed. Second, there is no need to impose any parametric distribution function on firm productivity or cost. These can be calculated directly from market-share data once those are divided into firm classes or type. Third, there is no need to assume MC. Nash Cournot (NC) or Nash Bertrand (NB) can allow for large firms and different market shares and thus different markups across firm types, the added endogeneity in general equilibrium being of no consequence. Fourth, there is no need for "draws": the added endogeneity (of income to profits, firm numbers, cutoffs etc.) of positive aggregate profit income is easily incorporated.

In what follows, I first review definition of a complementarity problem, and then describe how double-sided inequalities (upper and lower bounded variables) are formulated in PATH and GAMS. Then I specify a basic general-equilibrium model: a two-sector, one-factor closed-economy model. Paying for productivity draws is unnecessary and the aggregate profits of the active firms are added to the income of the representative consumer. The specific formulation for simulation has five firm types in one increasing-returns sector, free entry up to an upper bound on each firm type. Two simple experiments are introduced to exploit the model properties, comparing results under NC, NB and MC. These sections also show how unobserved marginal costs or productivity can be derived from observed market shares and provide an illustrative application to semi-conductor market-share data.

The second model extends the first to a two-country trade case, with a specification that closely follows that of Melitz (2003). Firms may export, but there is an added fixed cost to entering exporting. Unlike Melitz, I allow for endogenous markups and compare NB to MC (the Melitz original). The third model allows firms to establish a foreign plant to serve that market. Firms incur a fixed cost larger than that for exporting, but do not pay a (unit) trade cost. This is close to the Helpman et al. (2004) model which adds heterogeneous firms to the earlier multinational models of Horstmann and Markusen (1992, 2002).[3]

The second and third models yield simulation results that resemble those in Melitz and Helpman, Melitz and Yeaple. At moderate to high trade costs, the most productive firm type(s) will choose the multinational mode (when allowed in model 3), the middle-productivity firms will choose exporting, the less productive serve only their domestic market and the least productive firms may not enter. The simulations show how the configuration of modes across firm types changes as trade costs fall from a high level to costless trade. Several cases of country or firm asymmetries are included to argue that complementarity allows for the incorporation of many real-world features found in data and needed for policy analysis.

[2] Earlier trade papers with some form of endogenous markup include Horstmann and Markusen (1992), Levinsohn (1993), Bernard et al. (2003), Melitz and Ottaviano (2008), Atkeson and Burstein (2008), and Eaton et al. (2013). A detailed literature review including more recent work is found in Markusen (2023). A paper that does feature positive profit income, the same Bertrand markup rule I use here, and a finite number of firms in a sector is Gaubert and Itskhoki (2021), but it is not formulated as a complementarity problem.

[3] The non-linear complementarity approach is quite different from an integer programming approach and the continuum approach in Melitz and Helpman et al. In KKT, variables are continuous but subject to boundaries. Markusen (2002) applies complementarity to solving for the mode choices of firms active in equilibrium. Helpman et al. (2004) incorporate heterogeneity in sorting firms in the continuum by productivity using zero-profit or equal-profit equations to establish cutoffs.

2. Models with inequality side constraints, complementarity conditions

In this section, I review the complementarity formulation of economic equilibrium models. Suppose we have an equation or set of equations in implicit form $F(Z) = 0$, that we wish to solve. But there is a complication in that there are inequality side constraints $G(Z) \geq 0$. It may be or likely be that a solution to $F(Z) = 0$ violates one (or more) of the inequality constraints. A typical complementarity problem reformulates the set of equations as weak inequalities, $F(Z) \geq 0$. Each of these is matched with a specific weak-inequality side constraint. The complementarity condition is that the product of each weak inequality F times the matched weak inequality G equals zero. A complementarity problem is thus written as

$$Solve \;\; F(Z) \geq 0 \quad s.t. \quad G(Z) \geq 0 \quad F(Z)^T G(Z) = 0 \tag{1}$$

In economics, the archetype situation is that the Z are constrained to be non-negative. In such cases, $G(Z) \geq 0$ reduces to $Z \geq 0$. Later, I introduce a heterogeneous firm model in which there are two-sided bounds on Z. The formulation in (1) looks simple, but it does require the correct direction of the inequalities and the correct association of each member of the set $F(Z)$ with a member of $G(Z)$. These come from the modeler's knowledge of the underlying problem.

We can develop the intuition with a simple supply-demand model, good X with price p_x, an intuition that carries through to much more complex general-equilibrium models. First, there is the optimization condition that price equals marginal cost, $mc(X)$. But a non-negativity restriction on X may mean that the solution is that marginal cost is greater than the demand price p_x at $X = 0$ and the good is not produced. The inequality cannot run the other way; price greater than marginal cost leaves an unexploited profit opportunity.

$$mc(X) - p_x \geq 0 \quad X \geq 0 \quad (mc(X) - p_x)X = 0 \tag{2}$$

If $mc > p$ in equilibrium, X is not produced. There are many examples of this in economics. One is that there are multiple technologies for producing X (electricity), but some are unprofitable under current economic conditions ((2) refers to a specific one of those technologies). Second, the world trade matrix is full of zeros: unprofitable trade links.

The other type of situation that requires a complementarity condition due to a non-negativity constraint is a market clearing equation, supply equal demand, $D(p_x)$. But it may be that the solution involves excess supply in equilibrium, implying that the price is zero.

$$X - D(p_x) \geq 0 \quad p_x \geq 0 \quad (X - D(p_x))p_x = 0 \tag{3}$$

If supply > demand then $p_x = 0$ and X is a free good. This outcome is less common than the optimization condition example, but situations include cases where licenses or permits are in excess supply, a quota is non-binding in equilibrium (X is the supply of licenses or quota level), or a factor of production in excess supply with fixed-coefficient (Leontief) technologies.

Fig. 1 gives the three possible outcomes of this simple partial-equilibrium, supply-demand problem. The top panel is the classic undergraduate textbook case of an interior solution. If guaranteed, complementarity is not needed: there is no need to associate each equation to a specific variable. The middle panel shows a situation in which the supply curve lies everywhere above the demand curve. This is a case where finding the solution does require complementarity: good X will not be produced in equilibrium. The bottom panel shows the third possibility, where the supply curve lies everywhere to the right of the demand curve (in the case of X being permits or licenses, the supply curve is generally vertical). The complementarity formulation yields a solution in which the price is zero and X is a free good.

How does the modeler know which of these situations will be the solution, especially in a complex general-equilibrium model; e.g., the positions of the supply and demand curves depend on all the other variables in the model? The Karush (1939) and Kuhn and Tucker (1951) theorems introduce non-negative "slack" variables into the problem, which convert the weak inequalities to equations. Denote the non-negative "slack" variables as s_x and $s_p \geq 0$. The addition of two variables requires two added equations, the complementarity conditions.

$$mc(X) - p_x - s_x = 0 \; s_x = \text{unprofitability at } X = 0 \tag{4}$$

$$s_x X = 0 \tag{5}$$

$$X - D(p_x) - s_p = 0 \; s_p = \text{excess supply at } p_x = 0 \tag{6}$$

$$s_p p_x = 0 \tag{7}$$

We now have a model of four equations in four unknowns. This can be solved by an iterative procedure such as the Newton method, and that is done for the modeler in the PATH solver: the modeler just specifies the weak inequalities and matched variables. The values of the slack variables at the solution (equal zero in an interior solution) give the unprofitability of the technology/trade link and the excess supply, and correspond to the measures with the same notation in panels two and three of Fig. 1.[4]

3. Complementarity and KKT with bounded variables

First, I want to introduce how the PATH MCP solver handles a two-sided weak inequality (a variable has both an upper and lower bound). When using the standard technology with constant marginal costs and a fixed cost (at constant factor prices), the zero-profit condition for a firm simplifies to markup revenues equal fixed costs. In a complementarity formulation, the variable associated with the weak inequality is the number of firms in equilibrium. If fixed costs are greater than (potential) markup revenues, then the number of firms is zero.

Here, we are also going to specify an upper bound to the number of firms of a given type, and in PATH, this is handled in the same way. Let i index firm types, differing by marginal costs. Let $FC(i)$ denote the value of fixed costs of firm type i, and $MKR(i)$ the markup revenues of firm type i, calculated regardless of whether or not the firm actually enters. Let $N(i)$ equal the number of firms active at the solution, and let $N.up(i)$ and $N.lo(i)$ be parameters giving the upper and lower bounds on $N(i)$ (this is actual GAMS notation). Firm profits may be positive in equilibrium, this occurring when the number of firms hits the upper bound. The lower bound will be set at $N.lo(i) = 0$ throughout. For most of the paper, the upper bound will be $N.up(i) = 1$. The solution will converge to a single firm type (the most productive) as $N.up(i)$ becomes large, the classic MC Krugman outcome. There are three solutions in general equilibrium, ignoring knife-edged solutions.

$$FC(i) = MKR(i) \quad \Rightarrow \quad N.up(i) \; > \; N(i) \; > \; N.lo(i) \text{ zero profits} \tag{8}$$

$$FC(i) \; > \; MKR(i) \quad \Rightarrow \quad N(i) = N.lo(i) = 0 \text{ losses, no entry} \tag{9}$$

$$FC(i) \; < \; MKR(i) \quad \Rightarrow \quad N(i) = N.up(i) \text{ positive profits} \tag{10}$$

The key to understanding the importance of KKT is that it turns the weak inequalities into a formulation with three equations, by introducing two non-negative "slack" variables into the problem, one for each bound on the variable N(i): denote w(i) for the lower bound, and v (i) for the upper bound slack variables.

$$FC(i) = MKR(i) + w(i) - v(i) \tag{11}$$

[4] KKT are the necessary conditions for a solution to a non-linear programming problem with inequality side constraints. Strictly speaking, a market clearing equation is not a KKT condition. But the logic of KKT was extended to economic equilibrium problems by Mathiesen (1985) and Rutherford (1985, 1995).

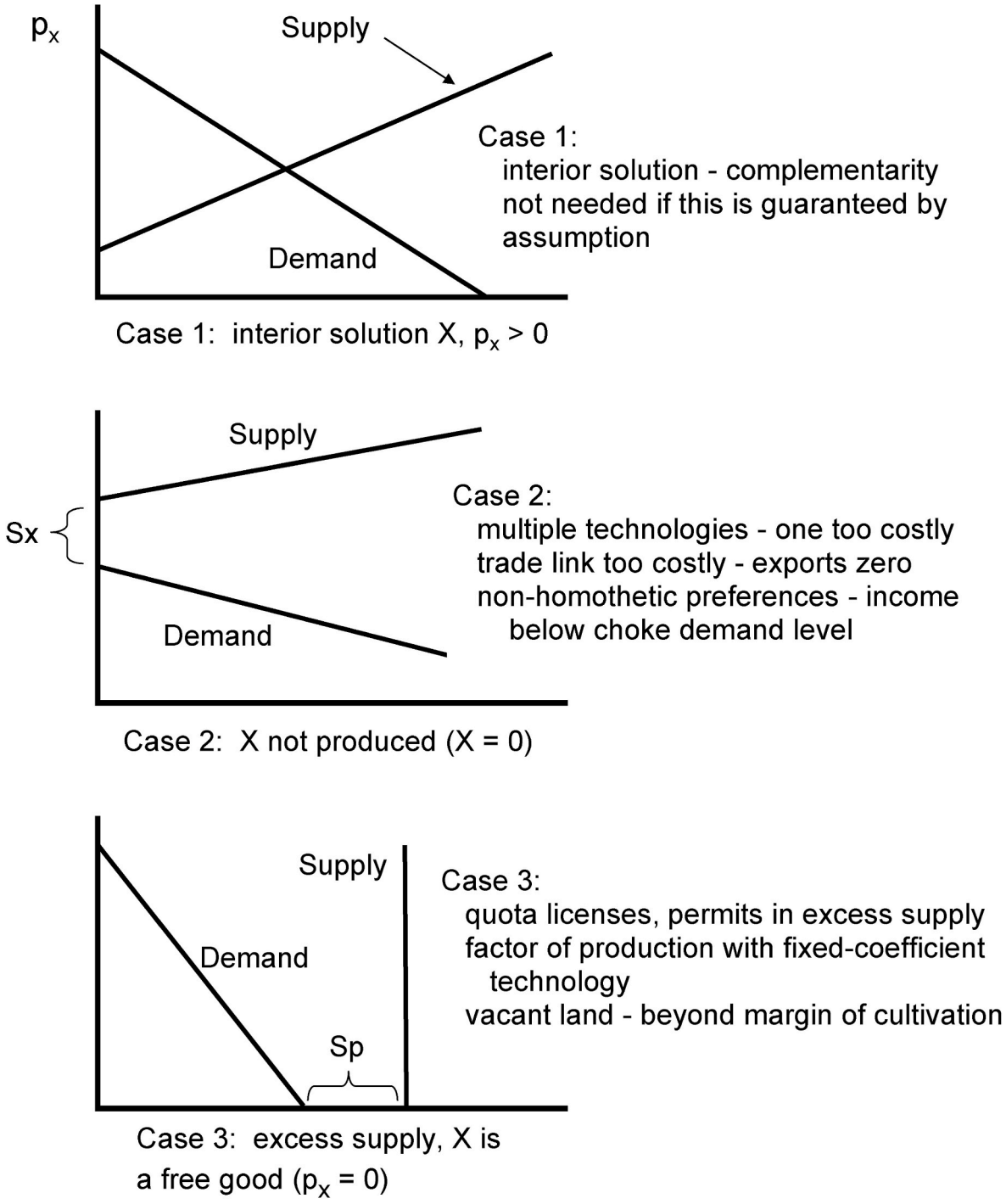


Fig. 1. Three equilibrium outcomes.

$$w(i)(N(i) - N.lo(i)) = 0 \text{ w(i)} = \text{(potential) losses if i enters} \quad (12)$$

$$v(i)(N.up(i) - N(i)) = 0 \text{ v(i)} = \text{profits in equilibrium} \quad (13)$$

Equations (12) and (13) are the complementary conditions. Both *w(i)* and *v(i)* are zero in an interior (zero profit) solution. As just noted, we let *N.lo(i)* = 0 (no entry). *w(i)* is positive if no entry occurs, and its value gives the (negative of) potential profits from entering. *v(i)* is positive if *N (i)* hits its upper bound, and its value gives the positive profits earned in equilibrium. If *w(i)* is positive, then *v(i)* is zero and vice versa. Having converted the weak inequalities to equations, PATH then solves the (full) model by a Newton-type algorithm.

4. Markups, entry and profit income

The representative consumer's welfare is a simple, two-level CES. The upper nest between *X*, our sector of interest, and *Y*, a homogeneous good with constant returns and imperfect competition, is Cobb-Douglas. The lower nest is CES with all *X* goods of all firm types being symmetric but imperfect substitutes. *X(i)* will denote the output of a representative firm of type *i*, with all firms of type *i* having identical technologies. *N(i)* is the number of active type i firms, σ is the elasticity of substitution among the *X* goods. Welfare (*W*) of the representative consumer, and the symmetry of varieties of the *X* goods, gives welfare as follows ($0 < \alpha$, $\beta < 1$).

$$W = X_c^{\beta} \cdot Y^{1-\beta}, X_c = \left[\sum_i N(i) \cdot X(i)^{\alpha}\right]^{1/\alpha}, \sigma = 1 \Big/ (1-\alpha) \; > \; 1 \tag{14}$$

where X_c is often referred to as a composite commodity or sub-utility. Let *s(i)* be the market share of a representative firm of type i in the total output of all *X* firms of all types. $p_x(i)$ denotes the price of all/any *X* firms of type *i*, with these prices varying across firm types. The market share of an individual firm of type *i* is given by the first equation in (15).

$$s_i = \frac{p_x(i) \cdot X(i)}{\Sigma_i p_x(i) \cdot N(i) \cdot X(i)} \quad p_x(i)[1 - 1/\eta(i)] = mc(i) \quad mk(i) \equiv 1 \Big/ \eta(i) \tag{15}$$

The perceived price elasticity of demand for firm type i is given by $\eta(i)$, with the marginal revenue = marginal cost (*mc*) optimization condition given by the second equation in (15). The third equation in (15) is how I will define the markup rate (not revenue) *mk(i)* in this paper, although it is often flipped around to the other side with the markup defined as $p/mc > 1$.

In my earlier paper (Markusen, 2023), I consider and compare three types of imperfect competition. These are MC, NC, and NB. I derive the three perceived elasticities, denoted as η_c or η_b for the small group cases, and therefore their markups in the earlier paper. The Cournot formula is derived under the assumption that the firm makes a best response in quantity holding the quantities of other *X* firms of all types constant. The Bertrand formula holds the price of all other *X* firms constant. Both also hold expenditure on *X* goods constant. In MC, the perceived elasticity of firm demand is just $\eta = \sigma$ with the markup as defined here $mk = 1/\sigma$. The markups for Bertrand and Cournot are

$$mk_b(i) = \frac{1}{\sigma - s(i)(\sigma - 1)} \quad mk_c(i) = s(i) + [1 - s(i)] \cdot \frac{1}{\sigma} \tag{16}$$

In Markusen (2023), I show that the markup for Cournot will be higher than either LGMC or SGB for the same value of *s(i)* strictly between zero and 1. In that sense, we can say that Cournot is "less competitive" than the other two. The limiting values of NB and NC markups converge to $1/\sigma$ as a firm's market share goes to zero.

Let CONS be the income of the representative consumer. Firm "entrepreneurs" will be treated much like a consumer: they receive markup revenues as income and demand fixed costs. Added demand for fixed costs constitutes entry, so the variable complementary to the entrepreneur's budget balance equation is the number of firms in equilibrium. The income (from markup revenues) of an entrepreneur of firm type i is denoted *ENTRE(i)*. The markup rate (not markup revenue as in (8)) for a firm of type *i* is denoted *mk(i)*. In this section and the subsequent one, I will use Bertrand as the example but show results for all three cases. I will provide the code for all three versions on request.

Let $\perp$ denote the usual math programming symbol for complementarity: if the weak inequality holds as an equation, the complementary inequality is strict and vice versa. The entry inequalities for a firm of type i under NB are as follows, where market share *s(i)* is given in (15).

$$mk(i) \geq \frac{1}{\sigma - s(i)(\sigma - 1)} \quad \perp mk(i) \geq 0 \tag{17}$$

$$ENTRE(i) \geq mk(i) \cdot p_x(i) \cdot X(i) \quad \perp ENTRE(i) \geq 0 \tag{18}$$

$$ENTRE(i) \geq p_l \cdot fc \geq ENTRE(i) \quad \perp N.up(i) \geq N(i) \geq 0 \tag{19}$$

where p_l is the price of labor, the single factor of production in the model to follow, and fixed cost *fc* is in units of labor, not in value as in (8) (identical for all firm types). Adding the upper bound on the number of firms of a type generates an equation with two slack variables for (19) as described in the previous section. The Bertrand markups formula is the same as Gaubert and Itskhoki (2021).

There is no need to introduce "draws" in order to eliminate aggregate profits from the model. In my version here, all active firm types earn positive profits, except in the case where the least productive active firm type just breaks even. Profits are redistributed to the representative consumer (CONS), so that the budget constraint for the consumer is

$$CONS = p_l \cdot ENDOW + \sum_i N(i)[ENTRE(i) - N(i)\, p_l \cdot fc] \quad \perp CONS \tag{20}$$

where the summation term is the profits of the active firm types ($N(i) > 0$). ENDOW is a parameter giving the economy's endowment of labor L, which we will vary in our experiments.

The model set up computes what the optimal price $p_x(i)$ and output *X(i)* would be for an inactive firm type, but they don't affect the solution: $N(i) = 0$. This has the advantage of showing how unprofitable inactive firm types are at the solution.[5]

The final item for this section is the relationship between marginal costs *mc(i)* across firm types, which are held constant in the counterfactuals to come. This is unobservable and it is unclear how this could be parameterized. Fortunately, CES demand allows for a mapping between observed market shares and unobserved marginal costs. The parameter *N.up(i)* is quantitatively important to the results as well, but that is postponed until section 6. The following shows that there is a convenient mapping in which the observed market shares allow the recovery of marginal costs. We can use the sub-utility function W_x for the *X* sector because all the Y variables cancel out in deriving the equation we need. Setting $N(i) = 1$ for the moment to simplify the notation without loss of generality, the optimization problem is

$$max \;\; W_x = \left[\sum_i X(i)^{\alpha}\right]^{1/\alpha} + \lambda \cdot \left[M - \sum_i p(i) \cdot X(i)\right] \tag{21}$$

where M is income and λ is the Lagrangean multiplier. Take the first-order conditions for types 1 and 2 and form the ratio, cancelling out some algebra including the Lagrangean multiplier λ.

$$\frac{X_1^{\alpha-1}}{X_2^{\alpha-1}} = \frac{p_1}{p_2} \quad \frac{X_1^{-1/\sigma}}{X_2^{-1/\sigma}} = \frac{p_1}{p_2} \quad \frac{X_1}{X_2} = \frac{p_1^{-\sigma}}{p_2^{-\sigma}} \quad \Rightarrow \tag{22}$$

$$\frac{p_1 \cdot X(1)}{p_2 \cdot X(2)} = \frac{s_1}{s_2} \quad \frac{p_1^{1-\sigma}}{p_2^{1-\sigma}} = \frac{s_1^{1/(1-\sigma)}}{s_2^{1/(1-\sigma)}} = \frac{p_1}{p_2} \quad c \cdot s(i)^{1/(1-\sigma)} = p(i) \tag{23}$$

where c is a (endogenous) constant of proportionality. Use the Bertrand conjecture for the relationship between *p* and *mc*.

$$p(i) \cdot [1 - mk(i)] = mc(i) \quad mk(i) = \frac{1}{\sigma - s(i)(\sigma - 1)} \tag{24}$$

Substitute (24) into (23) for *p(i)*. With five firm types, there are six equations in six unknowns, the sixth being the adding-up condition that is complementary to the proportionality variable c. σ is treated as exogenous and would be taken from literature estimates in an empirical application. Ratios of equations in (25) reduce to Melitz equation (6) for MC when $mk(i) = 1/\sigma$.

$$c \cdot s(i)^{1-\sigma} \cdot \left[1 - \frac{1}{\sigma - s(i)(\sigma - 1)}\right] = mc(i) \;\; \perp s(i) \tag{25}$$

$$\sum_i s(i) = 1 \;\; \perp \;\; c \tag{26}$$

Data should give market shares as parameters, so for these six equations the variables are *mc(i)* and *c* ((26) already satisfied). But this could also be solved in the other direction, with marginal costs as the parameters, solving for the market shares and *c* that are consistent with those values (Melitz does this assuming productivities ($1/mc(i)$)). This

[5] An issue arises concerning multiple equilibria: could there be equilibria with less efficient types only active, blockading the entry of the most efficient type? I argue that this can only occur here under strong and restrictive additional assumptions, and I present this in detail in online Appendix 2.

latter is solved as an MCP to show the relationship between marginal costs and market shares in section 6.

5. The single-economy general-equilibrium model

Both the single-economy and two-country general-equilibrium models have two goods, X and Y, and one factor of production L. For variables indexed by i, there are multiple variables and equations. There are five values of i in the simulations to follow in the next section, ordered by marginal costs of the firm type, with $i = 1$ being the lowest cost (most productive) firm type. The variables and parameters of the model for the Bertrand case are listed in Table 1.

Table 2 specifies the equations of the model, each with its complementary variable assignment. Units are chosen such that one unit of Y requires one unit of L, and p_y is arbitrarily used as numeraire so $p_y = p_l = 1$, with p_l then the marginal cost of Y. Fixed-cost parameter fc is the number of units of L required for one N (for all firm types). p_e is the standard CES price index for the composite or sub-utility X_c good, and p_w is the consumer price index (unit expenditure function) for welfare W. Y and X_c have Cobb-Douglas shares 0.5 in welfare.

mc(i) is the marginal cost in units of L of producing an X good of type i. In two cases in section 6, these are chosen to produce figures that show the full range of outcomes, but I check using (25) and (26) that the implied market shares make sense. In two other cases in section 6, market shares are the data and these are used to solve for the marginal costs. These replace some distribution functions and eliminate the need for any integration found in the conventional approach. The values of *mc(i)* and *N.up(i)* across type constitute the discrete equivalent of the continuous parametric distributions in the conventional approach. Throughout, I give *N.up(i)* the same value for all i, but these can be set independently in the code.[6]

6. Calibrating different conjectures and interpreting N.up(i)

Before presenting simulations, a word about calibration when comparing differing formulations is in order. Three versions of the model are calibrated to the same (artificial) data. These are Cournot (NC), Bertrand (NB) and large-group monopolistic competition (MC). We look at the role of the maximum number of firms of a type which we could think of as a density measure. This formulation allows for bunches of firms at particular productivity/cost levels or alternatively a "jagged" firm size distribution which characterizes real data. As mentioned earlier, these *N.up(i)* values could be due to scarce top entrepreneurs, patents or other intangibles which prevent potential entrants from capturing the productivity of top firms.

As AGE modelers know well, if we calibrate the same data to different underlying theoretical formulations, changing one thing such as the markup rule, requires balancing that change by altering at least one other parameter. Referring back to the markup rules in (16), changing from one to the other requires changing the number of firms (the market share s) and/or changing the elasticity of substitution σ. Assume here that there are no upper bounds on firm numbers so that only the most productive firm type is active. I calibrate with the following elasticities for the three conjectures implying a different number of (identical) firms and market shares for the Cournot and Bertrand cases.

$$\begin{array}{llll} \text{NC:} & \sigma = 4 & & \\ \text{NB:} & \sigma = 6.3333 & s = 0.20 & (N = 5) \\ \text{MC:} & \sigma = 5 & s = 0.25 & (N = 4) \end{array} \tag{40}$$

With these assumptions, all three values in (40) yield a markup $1/\eta_c = 1/\eta_b = 1/\eta_{lg} = 0.2$. All three have the same output per firm $X(i)$, the same price $p_x(i)$ and the same welfare level (caveat: preferences are not the same!). In short, the benchmark results of the three cases with no *N.up(i)* constraint, we would see identical outcomes with only type 1 active. Assuming perfect substitutes in the Cournot case is conceptually useful, because then the Cournot example contrasts sharply with MC: in the former, welfare gains from a larger economy will be in the form of increased firm scale (productivity) and lower prices and markups. In MC, gains are in the form of more variety, not firm scale.

Two experiments are performed, with a focus on clarifying the role of *N.up(i)*. In the first, we increase *N.up(i)* uniformly across firm types holding the size (factor endowment) of the economy constant, although the code allows values to be set independently. Methodologically, it confirms that we converge to the single, most-productive firm type only as *N.up(i)* grows large: we are back to the classic identical-firm model. In the second, we raise *N.up(i)* but the economy's size grows in the same proportion. This is Krugman's (1979) parable for adding together

Table 1
Variables and parameters of the single-economy simulation model.

Variables:	
$X(i)$	quantity of one X good (variety) of firm type i
$N(i)$	number of active X sector firms of firm type i
Y	total output of Y
W	welfare
$p_x(i)$	price of one X variety of firm type i
p_y	price of one unit of Y
p_w	price of one unit of welfare W (consumer price index)
p_l	price of one unit of labor L
$CONS$	income
$ENTRE(i)$	entrepreneur revenue
$s(i)$	market share of an individual firm of type i
$mk(i)$	markup on an X good of type i
p_e	price index for composite X goods (sub-utility)
Parameters:	
$ENDOW$	labor endowment
σ	elasticity of substitution among X goods
fc	fixed cost of X in units of labor L
$mc(i)$	marginal cost of X for firm type i in units of labor
$N.up(i)$	maximum number of type i firms that can enter

With 5 firm types, 37 weak inequalities in 37 non-negative variables.

Table 2
Simulation model formulated as a non-linear complementarity problem
All variables restricted to non-negative values.

Pricing inequalities	Complementary variables are quantities	
$p_l \cdot mc(i) \geq p_x(i)[1 - mk(i)]$	$\perp X(i)$	(27)
$ENTRE(i) \geq p_l \cdot fc \geq ENTRE(i)$	$\perp N.up(i) \geq N(i) \geq 0$	(28)
$p_l \geq p_y$	$\perp Y$	(29)
$p_y^{0.5} \cdot p_e^{0.5} \geq p_w$	$\perp W$	(30)
Market-clearing inequalities	**Complementary variables are prices**	
$X(i) \geq p_x^{-\sigma}(i) \cdot p_e^{\sigma-1} \cdot CONS/2$	$\perp p_x(i)$	(31)
$Y \geq CONS/\left(2p_y\right)$	$\perp p_y$	(32)
$W \geq CONS/p_w$	$\perp p_w$	(33)
$ENDOW \geq Y + \sum_i N(i)(mc(i) \cdot X(i) + fc)$	$\perp p_l$	(34)
Income balance	**Complementary variables are incomes**	
$CONS \geq p_l \cdot L + \sum_i [ENTRE(i) - p_l \cdot fc] N_i$	$\perp CONS$	(35)
$ENTRE(i) \geq p_x(i) \cdot mk(i) \cdot X(i)$	$\perp ENTRE(i)$	(36)
Behavioral and definitional inequalities		
$s_i \geq \frac{p_x(i) \cdot X(i)}{\Sigma_i p_x(i) \cdot X(i) \cdot N(i)}$	$\perp s_i$	(37)
$mk(i) \geq [\sigma - s(i)(\sigma - 1)]^{-1}$	$\perp mk(i)$	(38)
$p_e \geq \left[\sum_i N(i) \cdot p_x^{1-\sigma}(i)\right]^{1/(1-\sigma)}$	$\perp p_e$	(39)

[6] A practical issue arises in assigning marginal cost values to firm types that are inactive in benchmark data. This is done automatically in the Melitz tradition: data on active firm types is used to calibrate the parameters of a distribution, and then that calibrated distribution is in turn used to assign values to inactive firm types in the continuum. Some similar procedure could be used here.

Table 3
Relationship between market shares and marginal costs using NB.

Experiment 1		Experiment 2	
Increase *N.up(i)* holding size constant		Increase *N.up(i)* and size in proportion	
Assumed *s(i)*	Calibrated *mc(i)*	Calibrated *s(i)*	Assumed *mc(i)*
0.400	1.000	0.447	1.000
0.167	1.263	0.175	1.300
0.153	1.286	0.148	1.350
0.143	1.305	0.125	1.400
0.137	1.319	0.106	1.450

identical economies. I chose to use different calibrations of *mc(i)* and *s(i)* for illustration because when I used the same values, one or the other cases were dominated by a couple of firm types and there was little regime switching. So, the results don't have side-by-side continuity. In both cases, I use the NB assumption. To show it can work either way, in the constant-size case I set market shares and compute marginal costs. In the second case, I start with assumed marginal costs and solve for the market shares using the procedure of section 4. Results are shown in Table 3.

We will look at data from the semi-conductor industry shortly, but I should note that these patterns of market share are qualitatively consistent with actual data for a number of industries I have (casually) looked at: there is generally a pattern where the market shares, starting with the biggest firm, are negatively sloped but convex, often with a long and fairly flat right-hand tail.

Fig. 2 presents results with experiment 1 on the left panel and experiment 2 on the right. In both panels, each of the three cases is run in a loop on *N.up(i)*. The vertical axis indexes the five firm types, T1 being the lowest cost (most productive). As noted, the two panels are not quantitatively comparable as they use different calibrations to see the full range of outcomes. Results in panel A seem intuitive. With all three conjectures, adding firms leads to the successive elimination of less productive types as we move to the right. More firms per capita depresses the price and profits of the firms and so the less productive firms drop out.

Fig. 2 panel B for the Cournot and Bertrand cases shows the

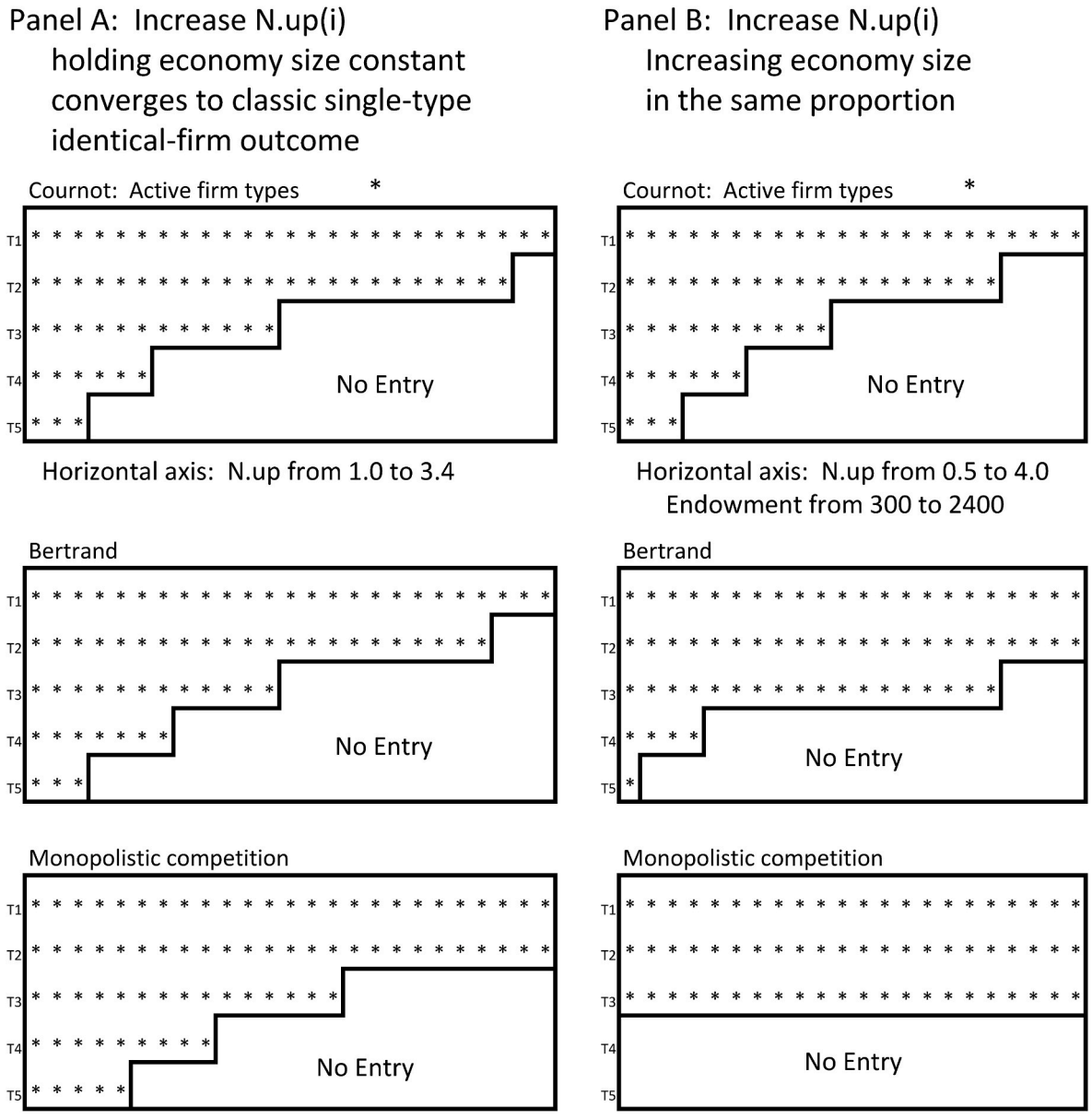


Fig. 2. Role of N.up in determining active firm types
Panel A: Increase N.up(i) holding economy size constant converges to classic single-type identical-firm outcome
Panel B: Increase N.up(i) Increasing economy size in the same proportion.

Table 4
Left (small economy) to right (large economy) in Fig. 2, panel B.

	MC	Bertrand	Cournot
Economy size increases by a factor of:	8.00	8.00	8.00
Firm numbers (variety) increase by a factor of:	8.00	3.43	1.60
Total output of X goods increases by a factor of:	8.00	13.90	18.95
Markup of most productive firm change (percent):	0.00	−48.25	−53.57
Percent change, welfare/capita:	2.97	2.64	2.22

importance of the variable markup assumption. As the economy grows (or identical economies are added), the number of active firm types shrinks, with the most costly firm types exiting one after another. As the economy and *N.up(i)* grow together, there is entry of more firms of the most productive firm types. The market shares of all firms decrease; their markups fall and firm scale increases. This forces down the prices of the *X* goods and leads to losses for the least productive firm types.

However, the MC case in Fig. 2 panel B is a stark comparison and highlights the limitations of this traditional approach. Because markups are fixed at $1/\sigma$ for all firm types, growing the economy and *N.up(i)* in proportion to it just replicates quantities and prices, and active firm numbers for each firm type grow in the same proportion. Fig. 2 suggests caution in making statements such as trade causes sorting and exit: this is not true in the Krugman experiment under MC.

Simulation results behind Fig. 2 panel B, shown in Table 4, emphasize the differences in the composition of the effects contributing to welfare changes. The welfare gains from MC are purely from increased variety: no firm scale effects, no pro-competitive effects, no firm-type-selection effects, and constant markups. Cournot, with the goods being perfect substitutes, is the opposite extreme. There is a small increase in firm numbers, but this has no variety effect on welfare. Instead, the welfare increases due to a strong increase in firm scale (lower average cost), elimination of less productive firms, and a large fall in markups. Bertrand is in between, with added variety, higher firm scale, and lower markups all contributing to welfare.

Overall welfare changes do not show the path. Cournot welfare increases quickly when size is initially small, as firms move down the steep section of their average cost curves. But this effect diminishes when size becomes large. The implication for modeling is that initially small economies benefit greatly from trade under NC or NB, but this effect diminishes relative to MC when economies grow large. Again, the welfare changes are not really comparable insofar as each has different preferences.

The remainder of this section offers a simple example using data for the semi-conductor industry. I take the market shares of just the top five firms for comparison and normalize their shares to sum to one. I choose

Table 5
Observed market shares and marginal costs: semi-conductors
Experiment: Increase *N.up(i)* holding size constant (initially *N.up(i)* = 1).

Data *s(i)*	Calibrated *mc(i)*	Bertrand, $\sigma = 6.333$
0.476	1.000	
0.187	1.314	
0.186	1.315	
0.095	1.523	
0.056	1.694	

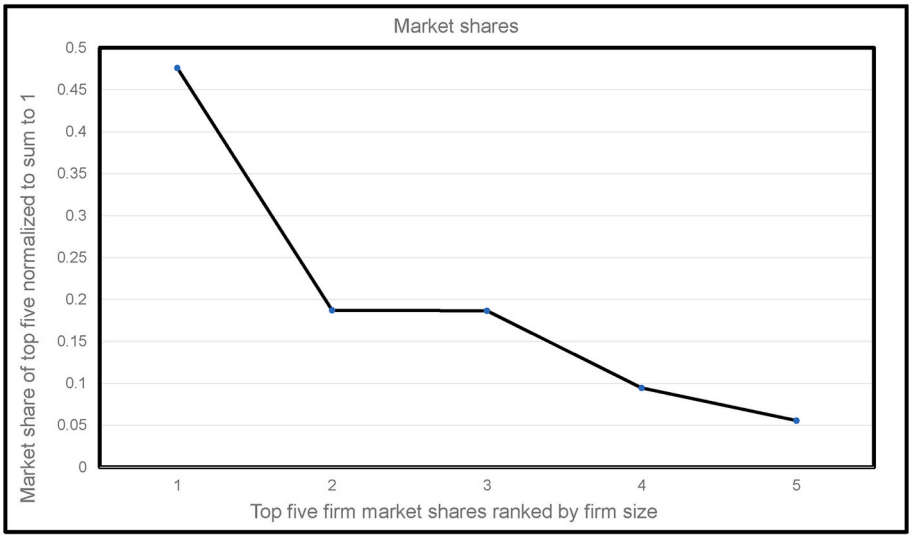


Fig. 3a. Market shares of top 5 semi-conductor firms the five shares normalized to sum to 1 (top 5 is 82 percent of top 10).

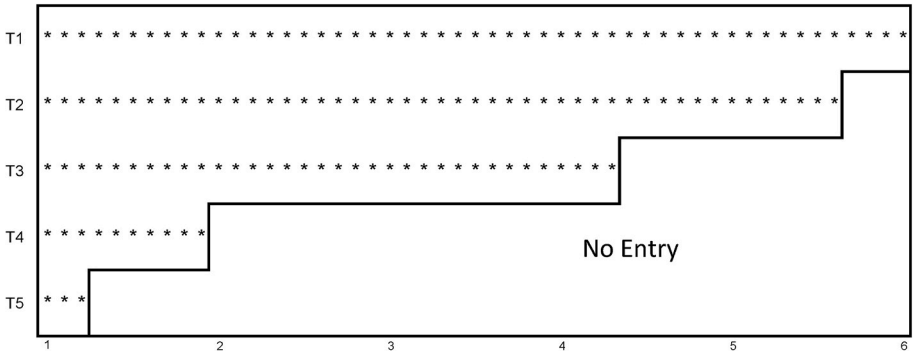


Fig. 3b. Increase N.up(i) holding size constant using market shares and calibrated marginal costs from semi-conductor data.

to use market capitalization as revenue shares are influenced a lot by the types of chips produced. In all the data I have looked at, market shares are negatively sloped but convex as noted above. There are of course many more firms in the market, but their market shares are very small after firm 5 and the right-hand tail of the distribution is very flat.

Fig. 3a graphs the (actual) market sizes *s(i)* for the top five firms normalized to sum to one, showing the convex relationship alluded to earlier. The right-hand numbers in Table 5 are then the computed marginal costs *mc(i)* using the procedure outlined in equations (25) and (26) for NB. Fig. 3b computes the counter-factual of increasing the density of firms holding the market size constant. The results look qualitatively very much like those in Fig. 2 panel A, although I needed to have a larger range of *N.up(i)* in order to see the full range of configurations appearing. From this actual data and eye-balling other industries, I feel that the experiments above and values given in Table 3 bear a reasonable relationship to data in many industries.

7. Two-country trade model

Consider next a two-country trade model, where each country is identical to the closed-economy version presented in the previous sections and therefore identical to one another. There is a fixed cost to exporting in addition to an iceberg trade cost. The experiment considered is changes in (symmetric) trade costs between the two countries.

For the differentiated-goods Bertrand case used in this section I will assume that firms can price exports and domestic sales independently, which has two consequences. First, firms will charge different markups on domestic sales and export sales in the presence of trade costs: markets are segmented but arbitrage constraints don't bind in any case. This works through the market share variables in the markup equations. Second, within a country, imported varieties and domestic varieties will sell for different prices. Trade costs are modeled as an iceberg cost rather than as a tariff. *tc* is the gross trade cost (one plus the rate). If the home marginal cost is *mc*, the cost of an exported unit is *mc*tc*. But the trade cost shrinks the amount received rather than creating an income stream. If the amount exported is *X*, the amount received is *X/tc*. The revenue received by the exporter equals the expenditure by the importer. I continue to use the five-firm-type formulation (set i).

The two countries are denoted with subscripts *h* and *f* as in home and foreign. In addition to firm types indexed by marginal costs, for each of these types there are two sub-types (call them modes): domestic sales only and domestic plus exports. Subscript d will denote a domestic-sales-only firm, and subscript *x* will denote a firm that also exports. There cannot exist a firm that only exports with symmetric countries. For a firm of a given type, domestic sales will be the same whether or not it also exports, which economizes a bit on notation. Output of a firm of type i is given by X_{jk} *(i)* which is production in country j for sale in country k. Domestic-sales-only firms will have X_{hf} *(i)* or X_{fh} *(i)* equal to zero.

There is free entry into what are now five firm types each with two modes. With a fixed exporting cost, more productive firms will export and less productive but active types will only sell domestically as in the Melitz tradition. The difficulty is that in a different sense the two firm modes are the same firms and the limit on each firm type must be on the sum of the exporting and strictly domestic modes of each type. *N.up(i)* denotes the maximum number of a type that can enter (ignoring country subscripts), which will be set equal to one for all firm types unless otherwise noted. Denoting the number of exporting firms and the number of domestic-sales-only firms as $N_x(i)$ and $N_d(i)$, the constraints for the five types are now weak inequalities.

$$N.up(i) \geq N_x(i) + N_d(i) \quad \perp \lambda(i) \ (N.up(i) \text{ is a parameter}, N(i) \text{ are variables}) \tag{41}$$

This added weak inequality requires a complementary variable, denoted $\lambda(i)$ in (41), which must appear somewhere else in the model. The approach I am implementing is to introduce what is in effect a tax on fixed costs (one for each country and firm type in units of labor) that raises the fixed costs for both domestic and exporting firms if the unrestricted number of firms violates (41). Let $\lambda(i)$ be the shadow tax on fixed costs for a particular country, with $\lambda(i)$ complementary to (41). Let fc_d and fc_x denote the fixed costs of a domestic and exporting firm (same for all firm types) respectively, with subscripts *d* and *x* on entrepreneur's markup income for the two modes. In the code, the equations complementary to the number of firms of each type (fixed costs greater-than-or-equal-to entrepreneur markup revenues) are then given by

$$p_l \cdot fc_d + p_l \cdot \lambda(i) \geq ENTRE_d(i) \quad \perp N_d(i) \tag{42}$$

$$p_l \cdot fc_x + p_l \cdot \lambda(i) \geq ENTRE_x(i) \quad \perp N_x(i) \tag{43}$$

Consistent and valid general-equilibrium solutions require that this $\lambda(i)$ be accounted for somewhere else in the model, otherwise there will be a residual imbalance which invalidates any "solution". So, I have modeled it as a virtual tax, with the tax revenue being returned to the representative consumer. Or we can just say that profits are the property of the representative consumer. Firm (retained) profits under this scheme are zero: $\lambda(i)$ and profits are both zero if the number of firms is less than *N.up(i)* in equilibrium, and $\lambda(i)$ is endogenously set to make (42) and (43) equations when the number of firms *hits N.up(i)*. With tax revenue returned to (profits assigned to the consumer), the income balance constraint for country h is given as follows.

$$CONS_h = p_{lh} \cdot ENDOWH + \sum_i N_{hd}(i) \cdot p_{lh} \cdot \lambda_h(i) + \sum_i N_{hx}(i) \cdot p_{lh} \cdot \lambda_h(i) \tag{44}$$

Table 6 gives the variables of the model. Trade costs on *X* are the same in both directions and there is no trade cost for *Y* without loss of generality. Again, this model is a straightforward extension of the single-economy model of the previous section, differing primarily in number of dimensions. These are due to the ability of the firms to discriminate on markups and outputs to the two markets, added firm types by country and modes, markups, and entrepreneurs. The full GAMS model is given at the end of the paper.[7]

Table 7 gives the full specification of the model. This specification of a general-equilibrium model follows the format of Mathiesen (1985) and Rutherford (1995), which treats general equilibrium as a sequence of complementarity problems. For those who have seen this formulation before, an important characteristic of my approach is that the inequality set in Table 7 adds in heterogeneous firms in a simple and straightforward way. The only additions required to a simple single-firm-type model are the set dimension of *i*, the constraint equation on the sum of mode *d* and *x* firms, and the virtual tax revenue added to the consumer income definition. Put differently, this methodology allows heterogeneous firms to be added to large-dimension general-equilibrium models with minimal complexity.

Pushing the ideal of simplicity a little further, I note that the model in Tables 6 and 7 converts to a single firm-type model by simply setting the dimension of set *i* to a singleton. This can also be done by removing *N.up (i)* constraints. The model converts to MC by fixing the markup variables to $1/\sigma$ (GAMS then drops the complementary equations from the model).

Fig. 4 presents the results for a simulation, looping over trade costs *tc*. A value high enough to induce autarky is on the left side of the horizontal axis, and free trade is on the right edge.[8] The small-group Bertrand case is used, and countries are identical so only one is

[7] I report only identical country results here, but the model can handle most any type of asymmetries such as country size or marginal costs (comparative advantage) as noted earlier, and also tariffs on X or Y. A couple of experiments with asymmetric countries are included in the final section.

[8] A very small trade cost $tc = 1.0001$ is needed to prevent model degeneracy, infinitely many solutions, all of which have the same *net* trade flows.

Table 6
Variables and parameters of the two-country Bertrand model.

Quantity variables complementary to pricing inequalities	
$X_{hh}(i)$	X good produced in h and sold in h, firm type i
$X_{hf}(i)$	X good produced in h and sold in f, firm type i
$X_{ff}(i)$	X good produced in f and sold in f, firm type i
$X_{fh}(i)$	X good produced in f and sold in h, firm type i
$N_{hd}(i)$	number of country h domestic X sector firms of firm type i
$N_{hx}(i)$	number of country h exporting X sector firms of firm type i
$N_{fd}(i)$	number of country f domestic X sector firms of firm type i
$N_{fx}(i)$	number of country f exporting X sector firms of firm type i
Y_h, Y_f	total output of Y in countries h and f
W_h, W_f	welfare of countries h and f
Price variables complementary to market clearing inequalities	
$p_{xhh}(i)$	price of X good produced in h and sold in h, firm type i
$p_{xhf}(i)$	price of X good produced in h and sold in f, firm type i
$p_{xff}(i)$	price of X good produced in f and sold in f, firm type i
$p_{xfh}(i)$	price of X good produced in f and sold in h, firm type i
p_y	world price of Y (complementary to trade balance condition)
p_{wh}, p_{wf}	price of one unit of welfare W (consumer price index) in h and f
p_{lh}, p_{lf}	price of one unit of labor L in countries h and f
Income variables complementary to income balance inequalities	
$CONS_h$	consumer income in h (includes profit income, virtual tax revenue)
$CONS_f$	consumer income in f (includes profit income, virtual tax revenue)
$ENTRE_{hd}(i)$	entrepreneur markup revenue country h domestic firm type i
$ENTRE_{hx}(i)$	entrepreneur markup revenue country h exporting firm type i
$ENTRE_{fd}(i)$	entrepreneur markup revenue country f domestic firm type i
$ENTRE_{fx}(i)$	entrepreneur markup revenue country f exporting firm type i
Variables complementary to definitional inequalities	
p_{eh}, p_{ef}	price index for X goods in countries h and f
$mk_{xhh}(i)$	markup on X good produced in h and sold in h, firm type i
$mk_{xhf}(i)$	markup on X good produced in h and sold in f, firm type i
$mk_{xff}(i)$	markup on X good produced in f and sold in f, firm type i
$mk_{xfh}(i)$	markup on X good produced in f and sold in h, firm type i
Auxiliary variables (virtual taxes) to implement firm number constraints *N.up(i)*	
$\lambda_h(i)$	virtual tax on entrepreneur income country h, firm type i
$\lambda_f(i)$	virtual tax on entrepreneur income country f, firm type i
Parameters:	
$ENDOW_h$	labor endowment of country h
$ENDOW_f$	labor endowment of country f
σ	elasticity of substitution among X goods
fcd, fcx	fixed cost of X in units of labor L, domestic and exporting firms
mc(i)	marginal cost of X for firm type i in units of labor
tc	gross trade cost (1 + trade cost rate)
N.up(i)	maximum number of type i firms, set = 1 for all types, both countries

With 5 firm types, 145 weak inequalities in 145 non-negative variables.

Table 7
Inequalities of the two-country Bertrand model.

Pricing inequalities, quantities complementary variables	
$p_{lh}\cdot mc(i) \geq p_{xhh}(i)(1 - mk_{xhh}(i))$	$\perp X_{hh}(i)$
$p_{lh}\cdot mc(i)\cdot tc \geq p_{xhf}(i)(1 - mk_{xhf}(i))$	$\perp X_{hf}(i)$
$p_{lf}\cdot mc(i) \geq p_{xff}(i)(1 - mk_{xff}(i))$	$\perp X_{ff}(i)$
$p_{lf}\cdot mc(i)\cdot tc \geq p_{xfh}(i)(1 - mk_{xfh}(i))$	$\perp X_{fh}(i)$
$p_{lh}\cdot fcd + p_{lh}\cdot\lambda_h(i) \geq ENTRE_{hd}(i)$	$\perp N_{hd}(i)$
$p_{lh}\cdot fcx + p_{lh}\cdot\lambda_h(i) \geq ENTRE_{hx}(i)$	$\perp N_{hx}(i)$
$p_{lf}\cdot fcd + p_{lf}\cdot\lambda_f(i) \geq ENTRE_{fd}(i)$	$\perp N_{fd}(i)$
$p_{lf}\cdot fcx + p_{lf}\cdot\lambda_f(i) \geq ENTRE_{fx}(i)$	$\perp N_{fx}(i)$
$p_{lh} \geq p_y$	$\perp Y_h$
$p_{lf} \geq p_y$	$\perp Y_f$
$p_{eh}^{0.5}\cdot p_y^{0.5} \geq p_{wh}$	$\perp W_h$
$p_{ef}^{0.5}\cdot p_y^{0.5} \geq p_{wf}$	$\perp W_f$
Market clearing inequalities, prices complementary variables	
$X_{hh}(i) \geq p_{xhh}^{-\sigma}(i)\cdot p_{eh}^{\sigma-1}\cdot CONS_h/2$	$\perp p_{xhh}(i)$
$X_{hf}(i)/tc \geq p_{xhf}^{-\sigma}(i)\cdot p_{ef}^{\sigma-1}\cdot CONS_f/2$	$\perp p_{xhf}(i)$
$X_{ff}(i) \geq p_{xff}^{-\sigma}(i)\cdot p_{ef}^{\sigma-1}\cdot CONS_f/2$	$\perp p_{xff}(i)$
$X_{fh}(i)/tc \geq p_{xfh}^{-\sigma}(i)\cdot p_{eh}^{\sigma-1}\cdot CONS_h/2$	$\perp p_{xfh}(i)$
$Y_h + Y_f \geq CONS_h/(2p_y) - CONS_f/(2p_y)$	$\perp p_y$
$W_h \geq CONS_h/p_{wh}$	$\perp p_{wh}$
$W_f \geq CONS_f/p_{wf}$	$\perp p_{wf}$
$ENDOW_h = Y_h + \sum_i[(N_{hd}(i) + N_{hx}(i))\cdot X_{hh}(i) + N_{hx}(i)\cdot X_{hf}(i)]mc(i) + \sum_i[N_{hd}(i)\cdot fc_d + N_{hx}(i)\cdot fc_x]$	$\perp p_{lh}$
$ENDOW_f = Y_f + \sum_i[(N_{fd}(i) + N_{fx}(i))\cdot X_{ff}(i) + N_{fx}(i)\cdot X_{fh}(i)]mc(i) + \sum_i[N_{fd}(i)\cdot fc_d + N_{fx}(i)\cdot fc_x]$	$\perp p_{lf}$
Income balance inequalities, income complementary variables	
$CONS_h = p_{lh}\cdot ENDOW_h + \sum_i[N_{hd}(i) + N_{hx}(i)]p_{lh}\cdot\lambda_h(i)$	$\perp CONS_h$
$CONS_f = p_{lf}\cdot ENDOW_f + \sum_i[N_{fd}(i) + N_{fx}(i))]\cdot p_{lf}\cdot\lambda_f(i)$	$\perp CONS_f$
$ENTRE_{hd}(i) \geq p_{xhh}(i)\cdot mk_{xhh}(i)\cdot X_{hh}(i)$	$\perp ENTRE_{hd}(i)$
$ENTRE_{hx}(i) \geq p_{xhh}(i)\cdot mk_{xhh}(i)\cdot X_{hh}(i) + p_{xhf}(i)\cdot mk_{xhf}(i)\cdot X_{hf}(i)/tc$	$\perp ENTRE_{hx}(i)$
$ENTRE_{fd}(i) \geq p_{xff}(i)\cdot mk_{xff}(i)\cdot X_{ff}(i)$	$\perp ENTRE_{fd}(i)$
$ENTRE_{fx}(i) \geq p_{xff}(i)\cdot mk_{xff}(i)\cdot X_{ff}(i) + p_{xfh}(i)\cdot mk_{xfh}(i)\cdot X_{fh}(i)/tc$	$\perp ENTRE_{fx}(i)$
Definitional inequalities, Bertrand markup inequalities, firm-number constraints	
$p_{eh} = \left[\sum_i (N_{hd}(i) + N_{hx}(i))\cdot p_{xhh}(i)^{1-\sigma} + N_{fx}(i)\cdot p_{xfh}(i)^{1-\sigma}\right]^{1/(1-\sigma)}$	$\perp p_{eh}$
$p_{ef} = \left[\sum_i (N_{fd}(i) + N_{fx}(i))\cdot p_{xff}^{1-\sigma}(i) + N_{hx}(i)\cdot p_{xhf}^{1-\sigma}(i)\right]^{1/(1-\sigma)}$	$\perp p_{ef}$
$mk_{xhh}(i) \geq [\sigma - s_{hh}(i)(\sigma-1)]^{-1}$	$\perp mk_{xhh}(i)$
$mk_{xhf}(i) \geq [\sigma - s_{hf}(i)(\sigma-1)]^{-1}$	$\perp mk_{xhf}(i)$
$mk_{xff}(i) \geq [\sigma - s_{ff}(i)(\sigma-1)]^{-1}$	$\perp mk_{xff}(i)$
$mk_{xfh}(i) \geq [\sigma - s_{fh}(i)(\sigma-1)]^{-1}$	$\perp mk_{xfh}(i)$
(in the code, market share equations and substituted directly into markup equations)	
$s_{hh}(i) = p_{xhh}(i)\cdot X_{hh}(i)\left[\sum_i (N_{hd}(i) + N_{hx}(i))\cdot p_{xhh}(i)\cdot X_{hh}(i) + N_{fx}(i)\cdot p_{xfh}(i)\cdot X_{fh}(i)/tc\right]^{-1}$	
$s_{fh}(i) = p_{xfh}(i)\cdot X_{fh}(i)/tc\left[\sum_i (N_{hd}(i) + N_{hx}(i))\cdot p_{xhh}(i)\cdot X_{hh}(i) + N_{fx}(i)\cdot p_{xfh}(i)\cdot X_{fh}(i)/tc\right]^{-1}$	
$s_{ff}(i) = p_{xff}(i)\cdot X_{ff}(i)\left[\sum_i (N_{fd}(i) + N_{fx}(i))\cdot p_{xff}(i)\cdot X_{ff}(i) + N_{hx}(i)\cdot p_{xhf}(i)\cdot\frac{X_{hf}(i)}{tc}\right]^{-1}$	
$s_{hf}(i) = p_{xhf}(i)\cdot X_{hf}(i)/tc\left[\sum_i (N_{fd}(i) + N_{fx}(i))\cdot p_{xff}(i)\cdot X_{ff}(i) + N_{hx}(i)\cdot p_{xhf}(i)\cdot\frac{X_{hf}(i)}{tc}\right]^{-1}$	
$NUP(i) \geq N_{hd}(i) + N_{hx}(i)$	$\perp\lambda_h(i)$
$NUP(i) \geq N_{fd}(i) + N_{fx}(i)$	$\perp\lambda_f(i)$

shown. The fixed costs of being an exporting firm are set at 75 percent higher than for a domestic-sales-only firm. *N.up(i) = 1* for all firm types for both countries, with type T1 in the top row of Fig. 4. Results on active firm types in Fig. 4 duplicate the finding we have come to know from Melitz (2003) and successors. As trade costs fall, the most productive type T1 begins exporting, while types 2 and 3 enter exporting successively as costs fall. This increases supply from the other country, which in turn reduces profits and leads to exit of less productive firms.

Fig. 5a shows the output per firm for the five firm types over the range of trade costs. Output jumps up when a firm begins exporting: domestic sales actually fall due to competition with imports from the other symmetric country, but this is more than offset by increased exports. At the same time, the increased imports from the other country reduce the markups and profits of the less productive firms on their home sales which cannot afford to export, and so first type T5 exits and then T4 as well as shown in Figs. 4 and 5aa.

Fig. 5b plots the markups for the two most productive firm types and shows both the domestic and export markups. For positive trade costs, the domestic markup exceeds the export markup when exports are positive. This is working through the market share variables in the domestic and export markup equations: with the countries symmetric, a firm's market share in the foreign market is less than its domestic share until trade costs go to zero. In alternative terminology, this can be called "partial pass through": the firm absorbs part of the trade cost through a lower markup. Second, the convergence of domestic and export markups in which the domestic markup falls and the export markup rises is also a reflection of the market share changes. In particular, the domestic markup falls due to the import competition reducing the firm's domestic market share. Third, the more productive firm has the higher markup, again a reflection of its larger market share. I believe that this is consistent with empirical evidence. Note that none of these results would be true under large-group monopolistic competition with factory-gate pricing where domestic and export markups, as well as across types, are the same.

Figs. 6a and 6ba and b shows some results for MC. Markups are fixed at 0.20 and the elasticity of substitution is changed from 6.333 to 5.0 as in (40) above in order to calibrate to the same benchmark data. Otherwise, this model version is exactly the same. Fixed markups do not lead to results substantially different from the traditional MC approach with

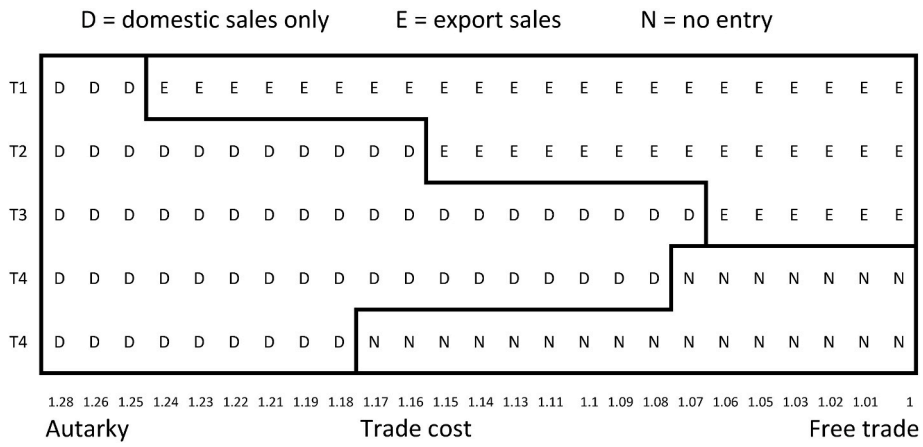


Fig. 4. Sequence of equilibria, autarky to free trade small-group Bertrand competition.

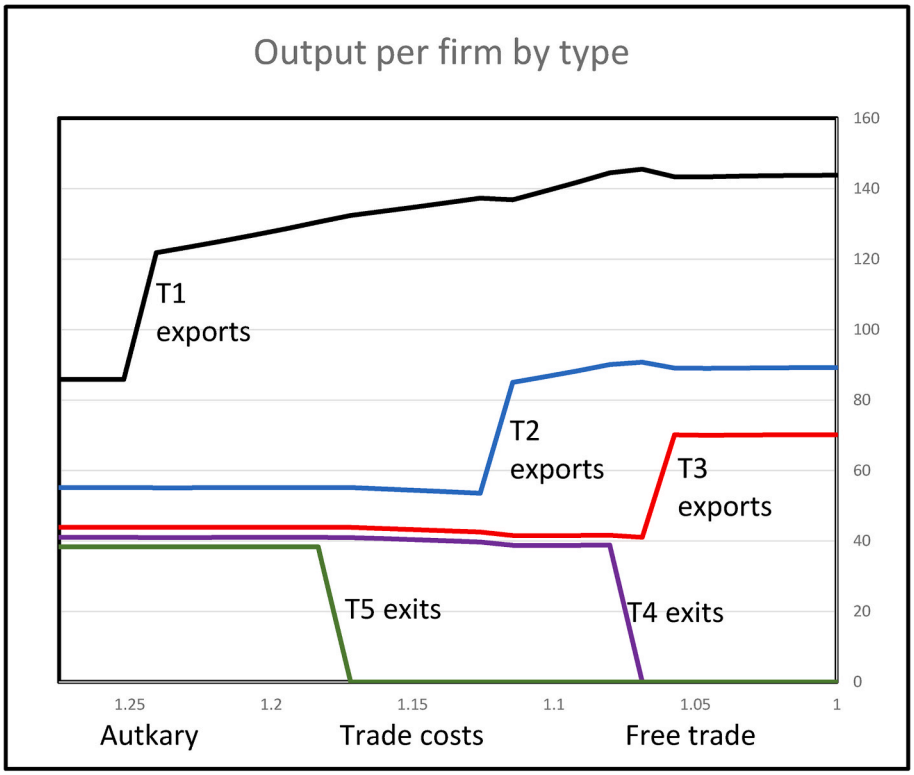


Fig. 5a. Output per firm by type small-group Bertrand competition.

respect to firm exit and/or entry into exporting. Fig. 6a is quantitatively a little different from Fig. 4, but qualitatively similar. I have left the scale on the horizontal axis the same for comparison purposes. Types T2 and T3 begin exporting with falling trade costs, and T5 and T4 exit.

Fig. 6b completes the analysis by showing the pattern of firm output for the MC case. Fig. 6b looks qualitatively similar to Fig. 4. Again, the point is that the traditional results about firm entry to exporting and firm exit under MC continue to hold up in my alternative formulation.

It is interesting to note that the welfare gains under MC (not shown) in Figs. 6a and 6ba and b are not due to increased product variety, but are due to the transfer of production from less efficient to more efficient firms and higher outputs per firm by the more efficient types. As the text to the right of Fig. 5b notes, there are six varieties consumed in each country at both the left-hand edge and the right-hand side. The fact that the number (mass in a continuum) of firms/varieties may increase or decrease as competitive pressures increase is one issue identified and explored systematically in Matsuyama and Ushchev (2023). Welfare continues to increase on the flat sections of Fig. 5b as trade costs fall even without changes in variety or firm scale. This is due to a more even consumption of domestic and foreign varieties: in the terminology and results of Arkolakis et al. (2012), the share of expenditure on domestic varieties continues to fall with falling trade costs.

8. Extension 1: adding a horizontal multinational mode

The open economy model can be altered or extended to include an endogenous choice between exporting and foreign affiliate production by almost trivial changes to the formulation. I published a number of papers in the 1980s and 1990s with endogenous multinationals (many with Ignatius Horstmann, Anthony Venables or Keith Maskus), most of which are integrated and extended in Carr et al. (2001) and in my MIT Press book (Markusen (2002)). But this was prior to the heterogeneous

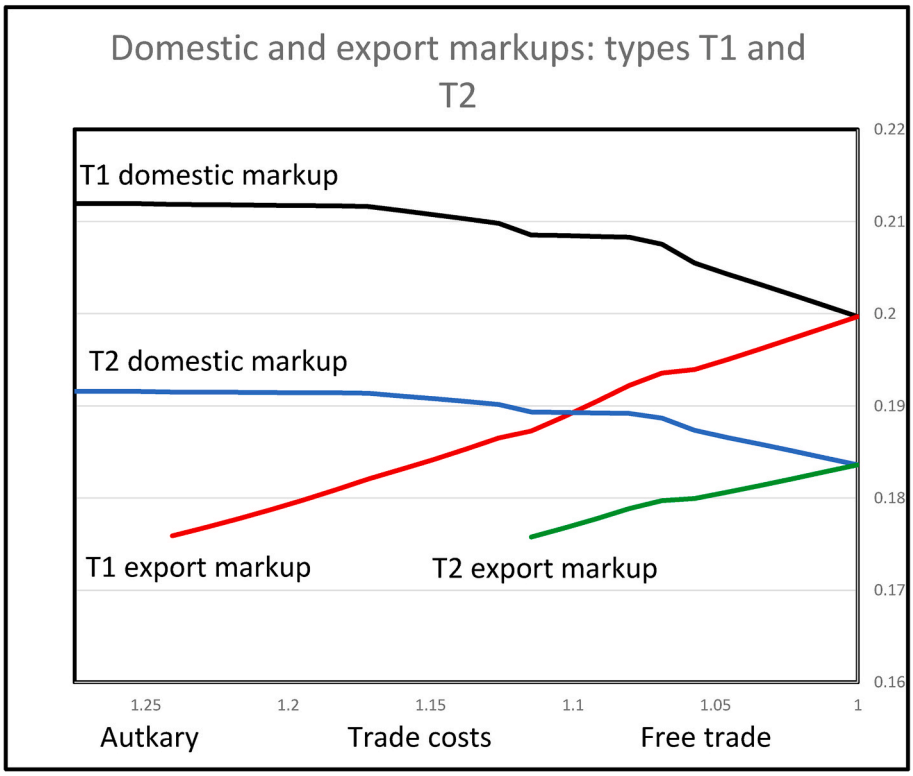


Fig. 5b. Domestic and export markups two most productive firm types: SGB.

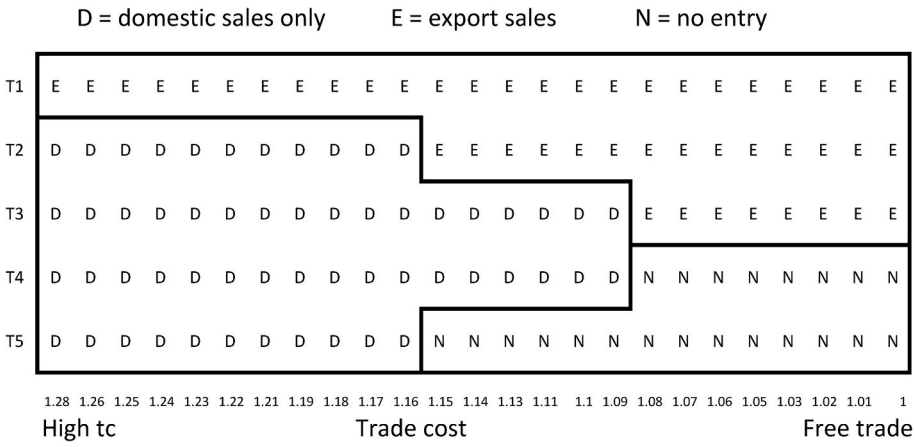


Fig. 6a. Sequence of equilibria, autarky to free trade large -group monopolistic competition.

firm revolution. The "horizontal" approach was extended to heterogeneous firms by Helpman et al. (2004), using the continuum approach of Melitz (2003), solving for "cutoff conditions" to determine domestic versus exporting versus foreign production firm types.[9]

The model developed above using the fixed cost of exporting formulation of Melitz can be extended to a model with a foreign production option by simply adding an additional option as to the size of fixed costs and marginal costs for the multinational type and where they are incurred. I assume that the fixed cost of a foreign plant is greater than the fixed cost of exporting and that the foreign affiliate fixed cost is incurred in units of foreign labor. The marginal costs of foreign affiliate production are also in units of foreign (local) labor. The advantage of the foreign production mode is that it does not incur the unit trade costs of exporting. The multinational mode will be chosen if the added fixed cost of foreign production above that for exporting is less than the saving on trade costs. No firm will choose both foreign production and exporting in this formulation: trade and foreign production are substitutes.

Including three modes - domestic, exporting, and multinationals - requires adding the third mode choice (MNEs) to the existing model developed above. This implies an increase in model dimensionality, but really no other complications. To simplify a little, the blocks of four inequalities in Table 7 become blocks of six. For example, there are now six markup equations instead of four with the added markups of firms choosing the multinational mode for their domestic and foreign sales. I

[9] By "horizontal" multinationals, we mean that foreign affiliates serve the host-country market, but exports back to the home country are not considered, neither are they in Helpman, Melitz, Yeaple. "Vertical" affiliates which produce abroad and export back to the parent country are included in my Knowledge Capital Model (Markusen (2002)). Vertical firms only arise if countries are of different sizes or have different factor prices.

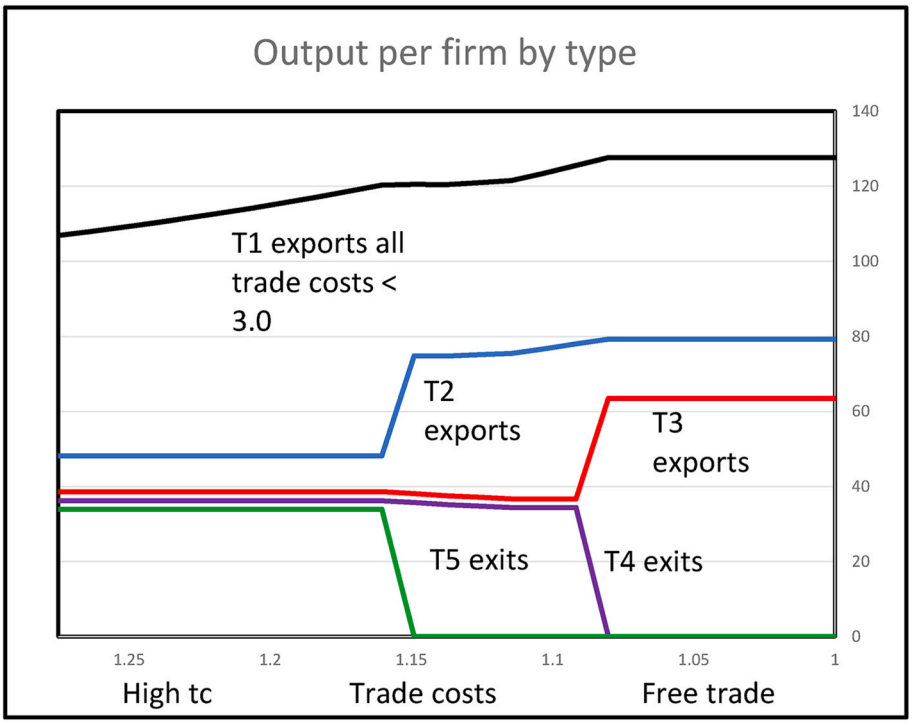


Fig. 6b. Output per firms by type large-group monopolistic competition.

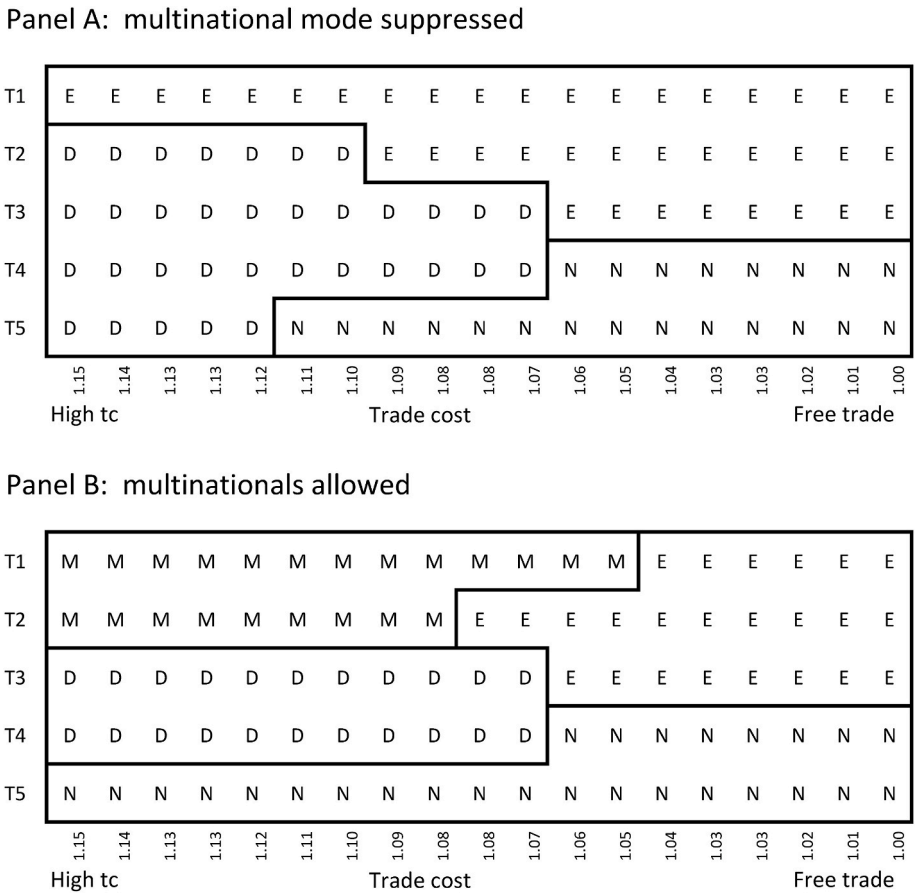


Fig. 7. Adding horizontal multinationals: sequence of equilibria, autarky to free trade.

will not present the equivalent of Tables 6 and 7 here, but the GAMS code is found in the online appendix. I use a parameter *NUP(i)* in the code, allowing users to specify levels other than one within or across firm types.

$$NUP(i) \geq N_{hd}(i) + N_{hx}(i) + N_{hm}(i) \quad \perp \lambda_h(i) \tag{45}$$

$$NUP(i) \geq N_{fd}(i) + N_{fx}(i) + N_{fm}(i) \quad \perp \lambda_f(i) \tag{46}$$

where the subscript m denotes the multinational mode. Redistributed profits for country h imply

$$CONS_h = p_{lh} \cdot ENDOW_h + \sum_i (N_{hd}(i) + N_{hx}(i) + N_{hm})\, p_{lh} \cdot \lambda_h(i) \tag{47}$$

Results of a simulation are shown in Fig. 7. I have adjusted only one parameter (fixed cost of exporting) a small amount from Fig. 4 in order to show the effect of the multinational option more clearly. So the upper Panel A of Fig. 7 gives the equivalent of Fig. 4 (multinational mode suppressed) for these slightly different parameters.

The introduction of the multinational option in Fig. 7 Panel B leads to regime changes for high or moderate trade costs. At high trade costs, the two lowest-cost (most productive) firm types switch to foreign production from exporting and domestic-only production for types 1 and 2 respectively. The effect of this is to make these foreign firms more competitive (no trade costs) in each other's domestic markets, so type 5 never enters. At middle-level costs, type 2 switches to exporting. At slightly lower costs (further to the right), type 3 starts exporting and type

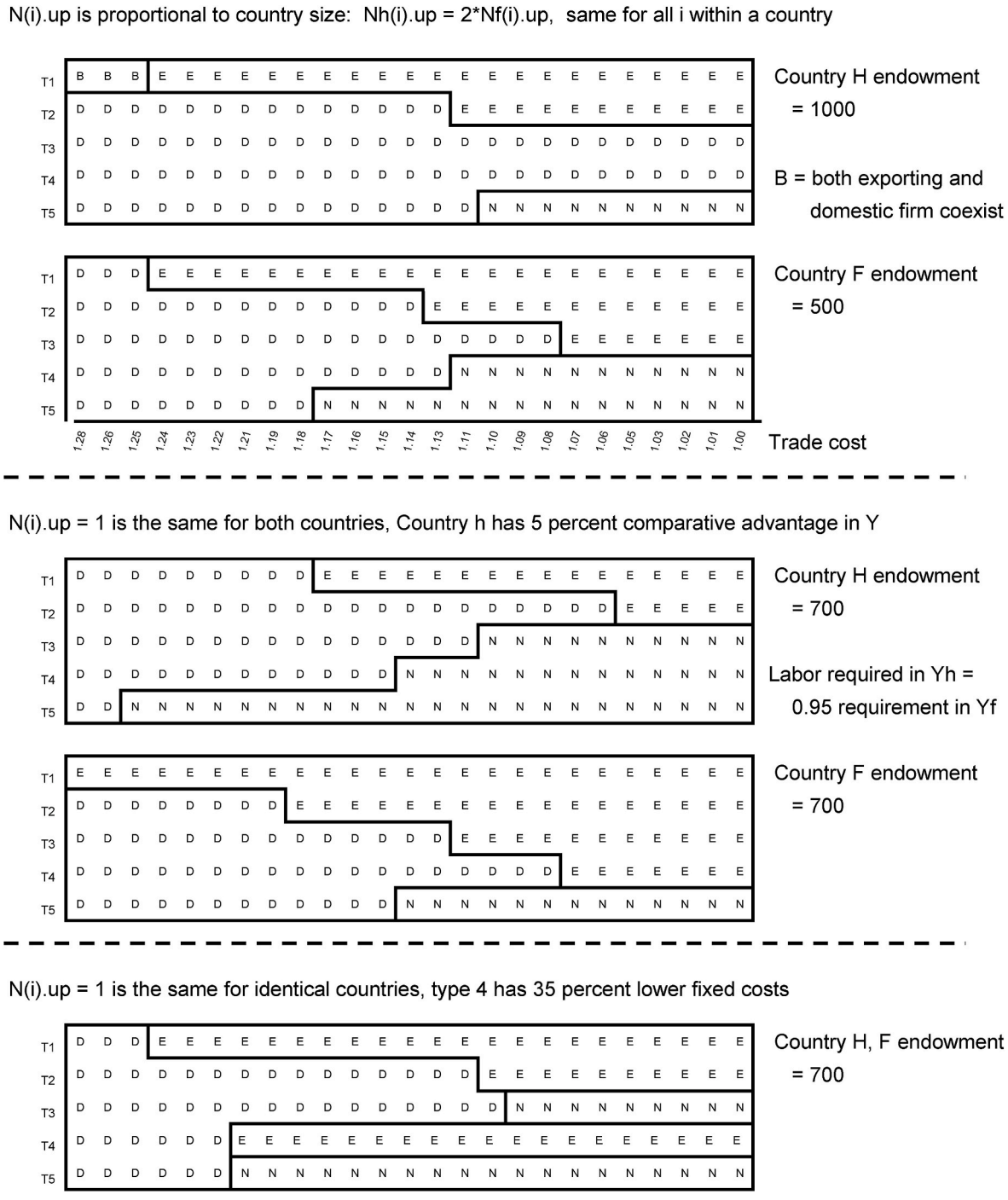


Fig. 8. Examples of asymmetries in countries or firm types.

4 exits due to foreign-firm competition in its domestic market. Finally, when trade costs fall further, type 1 switches to exporting. From that point on, panel B is the same as panel A.

There are two columns in Fig. 7 Panel B that illustrate the focus of Helpman et al. (2004). These are two values of trade costs (1.08 and 1.07) in which all four modes are active over the firm types. The most productive type chooses the multinational mode; the second type chooses exporting. The third and fourth most productive types choose domestic sales only and the least productive type doesn't enter.

Both panels have the same parameter values. These figures have a slightly higher value of FCX than Fig. 2 in order to show a rich set of outcomes with multinationals, but otherwise the code is the same as Fig. 2. Columns 1.08 and 1.07, Panel B, have the full set of four modes active and correspond to the analysis of Helpman et al. (2004).

9. Extension 2: examples of asymmetric countries and technologies

A principal motivation or rather "selling point" for this paper is that the complementarity approach to general-equilibrium modeling is fully scalable to complex and high-dimension models. It expands to multiple sectors, factors of production, countries and accommodates intermediate use matrices, etc. Fig. 8 presents three examples of how asymmetries are easily examined, requiring only a couple of keystroke changes to parameters. This flexibility is valuable for empirically-relevant policy counter-factuals which are difficult in the probabilistic-continuum models.

The top two matrices of Fig. 8 give the results corresponding to Fig. 4 but with the countries of different sizes (exporting allowed but no MNEs). Country h has twice the labor endowment of country f, and I have let country h have twice the *N.up(i)* values of country f. At very high trade costs, there are a couple of points where firm type 1 in country h has a (fractional) firm that is exporting and one that sells only domestically, cells denoted "B". Skipping to the right-hand area with low to zero trade costs, country h will have some middle-productivity firms that are only domestic due to the fixed cost of entering exporting. Country f has only exporting and no-entry firm types. However, country h is the net exporter of *X* good (net importer of *Y*), consistent with the classic presumption of home-market advantage.

It is interesting to note, however, that if both countries have the same value of *N.up(i)* (not shown), then country h turns out to be the net importer of *X*, net exporter of *Y*. Home-market advantage for the larger country depends on it having proportionately larger numbers of potential firms.

The middle two panels of Fig. 8 consider a second example of asymmetry. Both countries have the same labor endowment or size, but country h has an absolute/comparative advantage in good *Y* production requiring 0.95 units of labor to produce one unit of *Y* instead of 1.0 units in country f. Both countries have the same technologies for *X* production for a firm type. Examining the right-hand area of the two panels with low trade costs, I think that the results are quite intuitive. Country f has more firms/firm types producing *X* and is the net exporter of *X*. As an application, think of the US as having a comparative advantage in services and agriculture, the *Y* sector in our example, though equally productive in *X*, manufacturing. The US will be a net importer of *X*, for a reason that is not caused by being a low productivity producer of manufactured goods.

The bottom panel of Fig. 8 is motivating the idea that the complementarity approach offers a richness of possibilities which I believe are not feasible in the probabilistic-continuum approaches. Specifically, in the Melitz model, small output firms are necessarily branded inefficient firms. But I imagine that we can all think of many counter examples where smaller market-share firms are highly profitable (Porsche). The last matrix in Fig. 8 assumes that the ranking of the five firms by marginal cost is the same and thus so is their ranking by market shares. But firm type T4 has a lower fixed cost for domestic production and exporting: T4 is small but profitable. For the other parameters chosen, a fall in trade costs leads firm T4 to start exporting and T3 to exit rather than the other way around. There are certainly alternative reasons why some smaller firms may nevertheless be highly profitable, such as targeting niche markets, or high quality but high marginal cost products. But my point is that the complementarity framework allows for rich differences across individual firms as well as countries.

10. Summary

The general objective of this paper is to illustrate the advantages of formulating general-equilibrium models as non-linear complementarity problems. My view is that this allows modelers a much easier option for incorporating corner solutions, regime shifts, capacity constraints and endogenous zeros than other analytic approaches. Common assumptions in the latter are designed to force interior solutions such as the Armington assumption or Eaton and Kortum (2002). They also have difficulty with large firms and discrete regime shifts.

There are several advantages but also disadvantages to my offered alternative. The latter first. It is difficult in my approach to find analytical solutions to the model, and analytical methods clearly remain a desirable property of economic theory. My first defense is that, with a high level of complexity and dimensionality, analytically solvable models come with costs, requiring multiple simplifications that often eliminate many of the most interesting parts of a problem and/or clearly employ counter-empirical assumptions. Assuming that firms are atomistic yet produce with increasing returns is rather absurd. Second, when it comes to performing counter-factual experiments on models calibrated to empirical estimates, a numerical version of the underlying theory is used in any case. There seems to be growing recognition that numerical models are a valid theory tool as well as an empirical one.

The advantages of my alternative were laid out in the introduction, so just a quick recap here. First, no integrals/integration are required. Second, there is no need to impose some parametric continuous distribution of firm productivities/costs. Third, endogenous markup rules such as Nash Bertrand or Nash Cournot are easily incorporated.[10] Fourth, the addition of endogenous markups eliminates one of the nagging counter-empirical implications of the traditional approach, equal price to marginal costs for all firms, and replaces it with endogenous markups increasing in firm size/productivity. Fifth, the need to eliminate aggregate profit income bypasses an awkward endogeneity by firms paying for "draws" to find out their productivity is not needed. Sixth, I argue that my approach is completely scalable to high-dimension multi-country, multi-factor, multi-sector models, and many types of country and firm asymmetries. Finally, I imagine that working with real data requires aggregating firm-size distributions into discrete size classes in any case. All of these features are important for realistic policy analysis involving large firm entry/exit, mode switches and trade diversion. While I don't extend the analysis to protectionist policies here, it is easy to add tariffs for example in order to examine how policies can induce the switch from foreign firms exporting to producing via a subsidiary inside the protecting country.

Future work should involve applying this approach to real data, with available sources (as I understand it) allowing the distribution of firm sales in an industry to be divided into discrete sets such as quintiles. First attempts could be on smaller models which focus on particular key industries such as autos or semi-conductors, with data on both sales market share and market capitalization, so as to not rule out small but highly productive firms. A difficulty is that data which are rich in firm

[10] Again, I acknowledge that there are a number of other papers incorporating endogenous markups, and a number of these are reviewed in detail in my pedagogic paper (Markusen (2023)). A practical advantage of my formulation, using traditional CES preferences, is that it slots directly into traditional AGE models.

detail are generally disjoint from detailed general-equilibrium data.

For generalizations to the theory, the knowledge-capital model (Markusen (2002)) incorporates vertical multinationals and could be extended in a heterogeneous firm context, allowing the analysis of the effects of policy on supply chains. More generally, there is a great need to allow for endogenous zeros in the world trade matrix in large competitive AGE models, such as allowing for minimum choke income levels and maximum choke prices.

Declaration of competing interest

No competing interests.

Acknowledgements

Special thanks to Michael Ferris at the University of Wisconsin, Madison, Computer Science. Ferris and colleagues are the authors of PATH, the non-linear complementarity or MCP (mixed complementarity problem) solver that can be accessed in GAMS, Julia, R and Matlab. Also, to Thomas Rutherford to led me to break out from the constraints and limitations of analytical models. Thanks to Danielle Parks for capable research assistance. Additional computer code and all detailed numerical results are available from the author. A detailed literature review, derivations of endogenous markups and alternative calibration subtleties are found in my recent pedagogic paper (Markusen 2023) in the open-access *Journal of Global Economic Analysis*.

Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.econmod.2026.107816.

Data availability

No data was used for the research described in the article.

Exploiting complementarity in applied general-equilibrium models (Original data) ()

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