Endogenizing zeros and incorporating non-unitary income elasticities into applied general-equilibrium models

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Abstract

Traditional applied general-equilibrium (AGE) models have always faced trade-offs between analytical and computational tractability and counter-empirical restrictions. One is the assumption of homothetic preferences, significantly inconsistent with data. Similarly, there is no "choke" income level, below which a certain good is not purchased. The second and related weakness is that there is no choke price above which a good is not purchased and no change in the extensive margin of trade. Here I exploit what I will label a Stone-Geary Modified (SGM) formulation. This produces a model in which there are non-unitary income elasticities, choke income levels for some/all goods, and choke prices. The second approach modifies CRIE (constant relative income elasticity) preferences which are preferred for modeling income elasticities, but don't by themselves permit choke income and prices. While other authors have explored these properties in alternative ways, both my approaches have considerable advantages for simulation models in that they retain CES structures and functional forms so that they can slot right into existing modeling formats. They require only small modifications to off-the-shelf cost and expenditure functions, and goods and factor demand functions via Shepard's lemma.

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Additional references most appreciated. I'd like to hear from you before you write that referee's report.

1. Introduction

High-dimension applied general-equilibrium modeling (AGE) is now 50+ years old, with theoretical and algorithmic roots going back further. A good history of where we were at that point is provided in Shoven and Whalley (1984). It is fair to say that these early efforts had to impose strict limits as to the types of microeconomic features that they could incorporate. An important broadening of model sophistication was Harris' (1985) addition of scale economies and imperfect competition into the more traditional Walrasian, Arrow-Debreu paradigm. A second major development at about the same time was the formulation of an alternative approach based on complementarity developed by Mathiesen (1985), who extended the mathematical programming (optimization) theorems of Karush (1939) and Kuhn and Tucker (1951) to economic equilibrium problems. This alternative was then computationally operationalized by Rutherford (1985, 1995) in his software mps/ge (mathematical programming system for general equilibrium). Brief comments on other historical developments are found in my recent pedagogic article Markusen (2021).

Many limitations remain of course, some due to data restrictions with others due to a lack of insights as to how to incorporate empirically-relevant features into code. The purpose of this paper is to try to make progress on two constraining restrictions that are clearly counter-empirical. The first of these, in no particular order, is the continuing reliance on homothetic preferences and related assumptions on the production side. This assumption implies that consumer budget shares depend only on prices and not on income: all income elasticities of demand are unity. Simple functional forms, CES in particular, allow for solutions to unit cost and expenditure functions depending only on prices, with the further feature that the application of Shepard's lemma to these costs functions give unit demands for goods and factors. These features immensely simplify the construction of high-dimension general-equilibrium models by neutralizing a critical endogeneity between trade, prices and per-capita income.

Data do not support this convenient assumption. Empirical estimates of income elasticities that show significant deviations from unity are presented in Caron, Fally and Markusen (2014, 2020), with further estimation and counter-factual simulations showing important quantitative effects on trade volume and factor prices from incorporating non-homotheticity. The attached "Table II" is from Caron, Fally and Markusen (2014) which lists estimated income elasticities for the GTAP data base, sorted in order from smallest to largest.

In addition to the issue of significant changes in budget shares as incomes rise, the homotheticity assumption does not allow for "choke" income levels below which certain goods or services are not purchased. This is clear counter-empirical since the world trade matrix is full of zeros (export from country i to country j in good k). The general tradition in AGE modeling is to simply hold initially-zero cells fixed at zero when doing counter-factuals. Thus there can be no changes in trade at the extensive margin in simulations. Without in any way demeaning their important contributions, this was embodied in the Armington (1969) assumption and in the popular Eaton-Kortum (2002) formulation. These formulations also imply that any good that has a positive entry in the trade matrix can never go to zero. In the terminology of this paper, there

TABLE II

Estimated Income Elasticity by Sector

GTAP code	Sector name	Income elast.	Std. error	Skill intensity
gro	Cereal grains nec	0.110*	0.133	0.135
pdr	Paddy rice	0.254*	0.199	0.061
pcr	Processed rice	0.352*	0.113	0.130
c_b	Sugar cane, sugar beet	0.433*	0.233	0.091
oap	Animal products nec	0.444*	0.098	0.132
ctl	Bovine cattle, sheep and goats, horses	0.458*	0.137	0.164
vol	Vegetable oils and fats	0.545*	0.063	0.217
sgr	Sugar	0.588*	0.085	0.221
frs	Forestry	0.623*	0.121	0.118
v_f	Vegetables, fruit, nuts	0.640*	0.136	0.095
p_c	Petroleum, coal products	0.664*	0.052	0.313
b_t	Beverages and tobacco products	0.667*	0.079	0.297
tex	Textiles	0.707*	0.064	0.231
ofd	Food products nec	0.777*	0.063	0.268
mil	Dairy products	0.826*	0.077	0.248
ely	Electricity	0.848*	0.073	0.372
nmm	Mineral products nec	0.874	0.097	0.281
crp	Chemical, rubber, plastic products	0.880	0.067	0.356
cns	Construction	0.880	0.061	0.294
wht	Wheat	0.883	0.202	0.117
fsh	Fishing	0.886	0.139	0.124
osd	Oil seeds	0.889	0.194	0.119
ocr	Crops nec	0.893	0.144	0.115
atp	Air transport	0.929	0.070	0.313
wtp	Water transport	0.932	0.100	0.299
ome	Machinery and equipment nec	0.938	0.066	0.372
lum	Wood products	0.970	0.103	0.248
otn	Transport equipment nec	0.981	0.076	0.343
lea	Leather products	0.981	0.066	0.212
otp	Transport nec	0.990	0.074	0.296
fmp	Metal products	0.992	0.077	0.297
cmt	Bovine meat products	1.023	0.078	0.238
osg	Public Administration and services	1.033	0.049	0.503
mvh	Motor vehicles and parts	1.034	0.066	0.341
wtr	Water	1.039	0.087	0.378
ppp	Paper products, publishing	1.044	0.093	0.340
omt	Meat products nec	1.052	0.096	0.233
wap	Wearing apparel	1.057	0.069	0.247
ros	Recreational and other services	1.075	0.067	0.475
ele	Electronic equipment	1.094	0.070	0.358
omf	Manufactures nec	1.095	0.065	0.279
trd	Trade	1.106	0.070	0.308
rmk	Raw milk	1.118	0.145	0.152
cmn	Communication	1.152*	0.078	0.485
obs	Business services nec	1.324*	0.059	0.504
ofi	Financial services nec	1.331*	0.090	0.546
pfb	Plant-based fibers	1.339*	0.193	0.167
isr	Insurance	1.392*	0.104	0.533
wol	Wool, silk-worm cocoons	1.426*	0.177	0.089
gdt	Gas manufacture, distribution	2.221*	0.260	0.362

Notes. Estimates based on the benchmark specification; income elasticities evaluated using median country expenditure shares; bootstrapped standard errors (500 draws); *denotes 5% significance (difference from unity); skill intensity based on total requirements.

is no “choke” income level for produced and traded goods.

The second assumption that defies reality and one closely related to the absence of choke income levels is the absence, by assumption, of choke prices above which a good is not purchased. I’m sure that this is well understood, but a feature of CES preferences, not withstanding their great advantages, is that the demand price for a good goes to infinity as the quantity goes to zero even with elasticities of substitution (much) greater than one. So any good produced anywhere in the world will always be imported be every single country no matter how high the tariff or transport cost. Again, the exceptions are trade cells initially zero constrained to remain zero in running scenarios.

My first alternative proposes and analyzes a simple framework that allows for (a) non-unitary income elasticities of demand, (b) choke income levels below which a good is not purchased, and (c) choke prices above which a good is not purchased. These then allow for rich possibilities in counter-factuals including changes in the extensive marginal of trade (trade from i to j in good k switching on or off), reversals in the direction of trade, and an important role for per capita income in explaining trade flows.

My formulation is a straightforward modification of traditional Stone-Geary preferences (a great review and extensions to the AIDS formulation is in Deaton and Muellbauer (1980)). Where as the Stone-Geary version has traditionally used negative parameters subtracted from consumption purchases and interpreted as minimum consumption requirements (e.g., Markusen, 1986), this version uses positive endowment parameters which could be thought of as (unobserved) fixed endowment goods which are perfect substitutes for the produced and traded goods. This very simple idea implies that goods with large endowment parameters relative to their utility shares have income elasticities great than one, and have maximum choke prices and minimum choke income. Any good with a positive endowment substitute has a choke price. This preference specification has been previously used by Markusen (2013), Simonovska (2015) and Jung, Simonovska and Weinberger (2019). The latter two papers are more focused on price elasticities whereas this current paper is more interested in income elasticities.

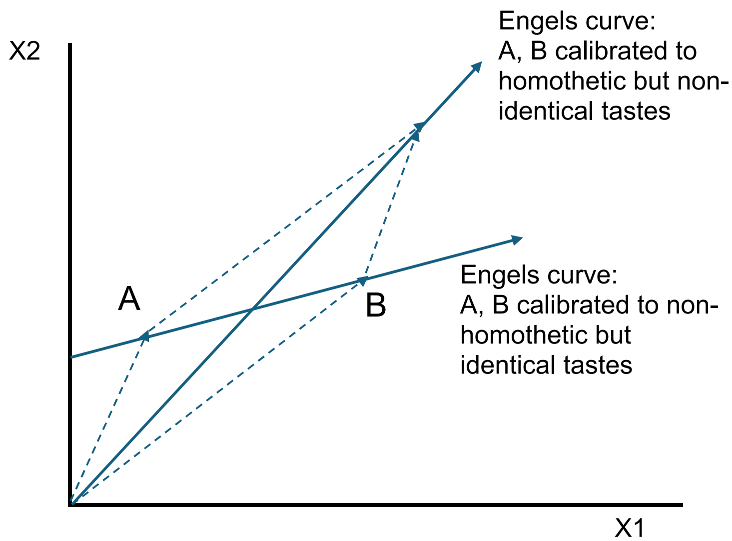
Panel A of Figure 1 tries to provide a motivating example of why non-unitary income elasticities are important for general-equilibrium counter-factuals. Suppose that points A and B are observations on the per-capita income of two households/countries at the same set of prices for goods X_1 and X_2. There are two very different ways of calibrating these data to preferences. One, which I believe is the normal way in general-equilibrium modeling, is to assume that the countries have different, but homothetic preferences. If the countries’ per-capita incomes grows in the same proportion, the Engel’s curve for the sum of their purchases moves out along the steeper line shown. On the other hand, if we assume that the countries have identical but non-homothetic preferences, then equal growth in their incomes moves aggregate consumption out along the flatter income curve. Beginning with the same data, the two options will generate very different counter-factual results.

My second alternative is to use CRIE (constant relative income elasticities) preferences, an idea that has a long but small history: Hanock, (1975), Chao, Kim and Manne (1982). But

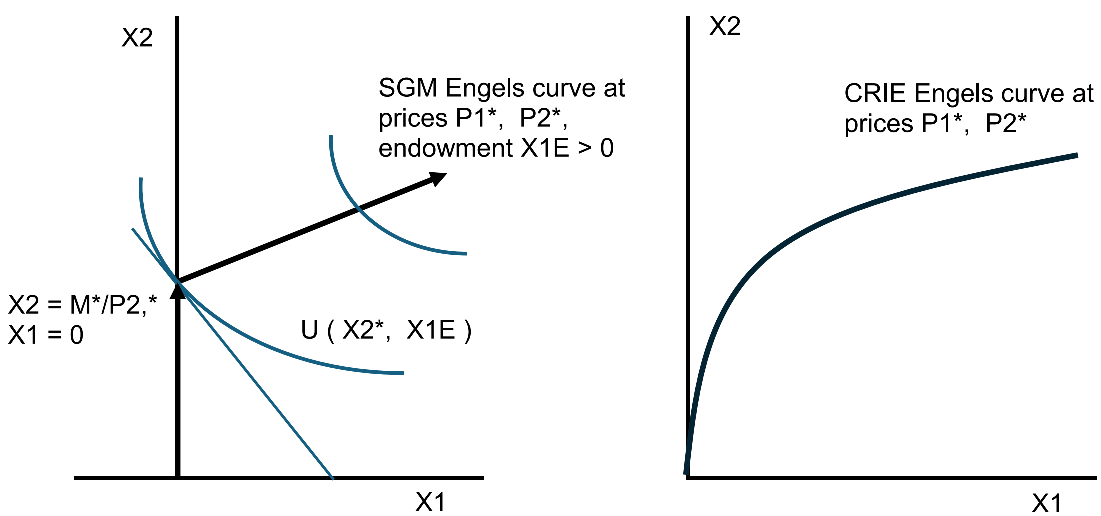
Figure 1: Stone-Geary Modified

Example: endowments X1E >0, X2E = 0
X1 has income elasticity of demand > 1, X2 income elasticity < 1

Panel A: Calibrate (per capita) observations A and B
Homothetic but non-identical tastes
Non-homothetic but identical tastes



Panel B: Engels curves at prices P1*, P2*



they have only been exploited to my knowledge in Fieler (2011) and Caron, Fally and Markusen (2014, 2020). These preferences are preferred for analyzing income elasticities in data where there are large differences in per-capita income across households or countries, but they do allow for choke income and prices. There is some commonality to my SGM version, in that the CRIE version uses endowed sector-specific fixed factors which, combined with purchased goods, map into consumption goods. The two approach can be combined, but I will avoid that here except for making the computer code available for such a case.

I hope that these formulations are of considerable value to researchers working with AGE models, either structural or calibrated. The reason is that both my offerings retain CES preferences with all their advantages in large simulation models. As noted above, these include simple solutions of cost and expenditure functions and then the easy derivation of closed-form goods and factors demand equations via Shepard's lemma. Another advantage of my alternative for empirical/simulation analysis is that all functions are globally regular, making it an attractive choice for simulating large changes in trade costs, technologies, natural disasters etc.

There have been many contributions focusing on non-homothetic preferences over the decades, and a few focusing on choke prices or incomes. But I am not aware of any that treat these simultaneously in the same model framework (references welcome!). Linder (1961) is an important early work that focuses on the role of per capita income in explaining trade in non-primary products, although there is no formal model in his work. Some of the earlier work that did formalize the role of per capita income were Markusen (1986), Bergstrand (1990) and Hunter (1991). After an apparent gap, interest reappeared in the new millenium with contributions by Matsuyama (2000), Fieler (2011), Markusen (2013) and Caron, Fally and Markusen (2014, 2020). For large-dimension simulation models, CRIE's usefulness has been limited by the fact that there are no close-form solutions for cost or demand functions, but I have found a work-around that retains all the features of CES.

Some other valuable contributions in this same time period include Simonovska (2015), Fajgelbaum and Khandelwal (2016), and Bertoletti, Federico and Simonovska (2018), Jung, Simonovska and Weinberger (2019) and Matsuyama (2019). These papers generally focus on different issues not closely related to this current paper (e.g., industrial-organization and income-distribution questions) and/or also use alternative preferences (e.g., indirect additivity) which do not lend themselves to multi-sector general-equilibrium models.

The three papers by Simonovska and co-authors just mentioned do feature choke prices. Wales and Woodland (1985) is a significant early contribution which is unfortunately unknown to trade economists. Wales and Woodland uses quadratic preferences which are also not easy to accommodate in large AGE models. Much more recently, Eaton, Kortum and Sotelo (2013) and Anderson and Zhang (2025) endogenize zeros. Like Wales and Woodland, these combine theory and estimation, but are perhaps not particularly easy to slot into large simulations models. Fajgelbaum and Khandelwal and Anderson and Zhang use an AIDS formulation derived from Stone-Geary. While this is useful for estimating parameters to calibrate AGE models initiallly, it is not appropriate for general-equilibrium simulations because (a) some goods will have predicted expenditure shares greater than one and others negative predicted shares and (b) adding up is violated in counter-factuals. I explain this in a short appendix.

2. Stone-Geary-modified preferences, demand, income and price elasticities

First, let's consider the basics in just a partial-equilibrium framework: prices and money income held constant. For each purchased good, there is a perfect substitute endowment good (some of which may be zero). This is the notation for the simple case used to derive demand functions, and price and income elasticities of demand.

Non-negative variables

X_{ip}	amount of good i purchased from the market	(p for purchased)
X_{ic}	amount consumed: purchase plus endowment	(c for consumed)
p_{ic}	implicit consumer price of good i - equal to p_i if $X_{ip} > 0$, less if no i purchased	
e	price index for ces substitute goods	
MT	total "income" including the value of the endowments	
W	welfare (utility)	

Parameters

X_{ie}	perfect substitute endowment good $X_{ie} \geq 0$, can't be traded (e for endowment)
p_i	market price of purchased good i - parameter in partial equilibrium model
β	substitution parameter
α_i	share parameters on the goods
σ	elasticity of substitution among goods
M	money income that can be spent in the market - parameter in the PE model

The modified Stone-Geary (SGM) welfare (utility) function is a simple case, but the quantities of each good are the sum of the purchased amounts plus the perfect substitute endowment good.

$$W = \left[\sum_i \alpha_i (X_{ip} + X_{ie})^{\beta} \right]^{1/\beta} \quad 0 < \alpha_i < 1, \quad \infty < \beta < 1, \quad \sum_i \alpha_i = 1, \quad X_{ie} \geq 0 \qquad (1)$$

Definitional equations for X_i consumed (sum of purchases and endowments), elasticity of substitution, income including the value of endowment.

$$X_{ic} = X_{ip} + X_{ie} \qquad \sigma = \frac{1}{1 - \beta} \qquad MT = M + \sum_i p_{ic} X_{ie} \qquad (2)$$

CES price index giving the cost of one unit of welfare.

$$e = \left[\sum_i \alpha_i^{\sigma} p_{ic}^{(1 - \sigma)} \right]^{\frac{1}{1 - \sigma}} \qquad (3)$$

Demand function for X_{ic}.

$$X_{ic} = X_{ip} + X_{ie} = (\alpha_i/p_{ic})^{\sigma} e^{\sigma - 1} MT \qquad X_{ip} > 0 \iff (\alpha_i/p_{ic})^{\sigma} e^{\sigma - 1} MT > X_{ie} \tag{4}$$

Figure 1 Panel B shows this basic two-good case where $X_{1e} > 0$ and $X_{2e} = 0$. Prices are (p_1^*, p_2^*). X_{1p} is constrained to be non-negative, so the Engel's curve for observed purchases runs up the X_2 axis to the choke point, and then is linear as show holding prices constant. At income and prices M^*, p_1^*, p_2^* the solution is $X_{1p} = 0$, $X_{2p|} = M^*/p_2^*$ and $X_{1c} = X_{1e}$.

The SGM formulation automatically ensures that no endowment is sold. Multiple through by p_{ic} and sum over i.

$$\sum_i p_{ic} X_{ic} = \sum_i (\alpha_i^{\sigma} p_{ic}^{1-\sigma}) e^{\sigma - 1} MT = e^{1-\sigma} e^{\sigma - 1} MT = MT \tag{5}$$

$$\sum_i p_{ic} X_{ic} = \sum_i (p_{ic} X_{ip} + p_{ic} X_{ie}) = M + \sum_i p_{ic} X_{ie} \tag{6}$$

Thus market purchases equal money income so adding up is always (globally) satisfied.

$$\sum_i p_{ic} X_{ip} = \sum_i p_i X_{ip} = M \tag{7}$$

Note that it will be the case that $p_{ic} = p_i \;\; \textit{if} \;\; X_{ip} > 0$ (perfect substitutes X_{ip} and X_{ie} will have the same price if there are market purchases $X_{ip} > 0$). The implicit price p_{ic} will be less than the market price p_i if no market is purchased: $X_{ip} = 0$.

Equation (4) can be used to solve for the "choke" income level, the value of M below which there is no X purchased at given prices, and also for the choke price above which no X is purchased for the income level and other goods' prices. For good X_1, these values are implicitly given by

$$(\alpha_1/p_{1c})^{\sigma} e^{\sigma - 1} (M + \sum_i p_i X_{ie}) \le X_{1e} \quad \Rightarrow \quad X_{1p} = 0 \tag{8}$$

The Cobb-Douglas case with $\sigma = 1$ makes this easier to interpret

$$X_{1p} = 0 \iff \alpha_1 (M + \sum_i p_i X_{ie}) \le p_1 X_{1e} \tag{9}$$

with the choke income level M^* give by

$$M_1^* = Max\left[\frac{1}{\alpha_1}(p_1X_{1e} - \alpha_1\sum_i p_iX_{ie}),\ 0\right] \qquad M_1^* > 0 \iff \frac{p_1X_{1e}}{\sum_i p_iX_{ie}} > \alpha_1 \tag{10}$$

M^* is strictly positive for good X_1 iff its value share of the endowment is great than its utility weight, otherwise there is no choke income level. For a good with $X_{ie} = 0$, there is no choke income level.

From (9) (Cobb-Douglas), the choke price for X_1 given M and the prices of the other goods is given by

$$p_1^* = Min\left[\frac{\alpha_1(M + \sum_{j\neq i} p_jX_{je})}{(1-\alpha_1)X_{1e}},\ +\infty\right] \qquad p_1^* < +\infty \iff X_{1e} > 0 \tag{11}$$

A particular simple case for capturing the intuition occurs when good 1 is the only one with a positive endowment.

$$p_1^* = \frac{\alpha_1 M}{(1-\alpha_1)X_{1e}} \qquad \text{special case with Cobb-Douglas, only } X_{1e} > 0. \tag{12}$$

Bottom left of Figure 1 shows this basic case where $X_{1e} > 0$ and $X_{2e} = 0$ (nothing in the picture relies on Cobb-Douglas). X_{1p} is constrained to be non-negative, so the endowment X_{1e} cannot be sold to obtain X_1. The Engel's curve runs up the X_2 axis to the choke point, and then is linear as show holding prices constant. At income and prices M^*, p_1^*, p_2^* the solution is $X_{1c} = X_{2p|} = M^*/p_2^*$ and $X_{1c} = X_{1e}$.

Now turn to income elasticities of demand. The expenditure on all X_i consumption (purchased plus endowment) as a share of all income (MT = money income plus endowment income) is given by

$$\frac{p_{ic}X_{ic}}{MT} = \frac{\alpha_i^\sigma p_{ic}^{1-\sigma}}{\sum_i \alpha_i^\sigma p_{ic}^{1-\sigma}} = s_i \qquad p_{ic}X_{ic} = p_{ic}(X_{ip} + X_{ie}) = s_i(M + \sum_i p_{ic}X_{ie}) \tag{13}$$

There is one qualifying limitation here: the share defined in this way is the share of total consumption including the endowment quantity $X_{ic} = X_{ip} + X_{ie}$ in total income including the endowment income. That is not the quantity of X and the money income M observed in the data. The amount of purchased, X_{ip}, is given by

$$X_{ip} = \frac{s_iM}{p_{ic}} + \frac{s_i\sum_i p_{ic}X_{ie}}{p_{ic}} - X_{ie} \tag{14}$$

Let $X_{ih} = s_i M / p_{ic}$ denote the purchase of X_i under homothetic demand with no endowments. Note also from (14) that X_{ih} is the asymptotic amount of X_{ip} purchased when M grows arbitrarily large. Alternatively, s_i is the marginal share of added income spent on X_{ip}. Equation (14) then implies that the purchase of a good is less than what it would be under standard homothetic preferences (X_{ih}) when the following holds:

$$X_{ih} = \frac{s_i M}{p_{ic}} > X_{ip} \qquad <=> \qquad \frac{p_{ic} X_{ie}}{\sum_i p_{ic} X_{ie}} > s_i \tag{15}$$

The amount purchased X_{ip} is less than the amount under homothetic preferences (all $X_{ie} = 0$) if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}.

The income elasticity of demand for purchases of X_{ip} with respect to (money) income M is found from (14).

$$\frac{M}{X_{ip}} \frac{dX_{ip}}{dM} = \frac{s_i M}{s_i M + s_i \sum_i p_{ic} X_{ie} - p_{ic} X_{ie}} \tag{16}$$

$$\frac{M}{X_{ip}} \frac{dX_{ip}}{dM} > 1 \qquad <=> \qquad \frac{p_{ic} X_{ie}}{\sum_i p_{ic} X_{ie}} > s_i \tag{17}$$

Note that the right-hand inequalities in (15) and (17) are the same. The income elasticity of demand for X_{ip} is great than one if the share of X_{ie} in endowment value is greater than the marginal share spent on X_{ip}. For example, with two goods and a positive endowment for only X_1, the income elasticity will be greater than one for X_1 and less than one for X_2. But (16) also implies that all the income elasticities converge to one as (money) income becomes large. This property is specific to the Stone-Geary formulation, but for all non-homothetic preferences the income elasticities of demand for high elasticity goods must fall toward one as income grows large. This is required by the budget constraint, and the algebra will be presented later.

The derivation of the own price elasticity of demand for the purchases of good X_{ip} (money income held constant) is more complex since p_{ic} appears a number of times in both the numerator and denominators of (14). The price elasticity of total consumption including the endowment and holding MT constant (or assume zero endowments, same thing) is given by a fairly well-known formula ($MT = M + \sum_i p_{ic} X_{ie}$).

$$X_{ic} = \frac{\alpha_i^{\sigma} p_{ic}^{-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} MT \qquad -\frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = [\sigma - s_i(\sigma - 1)] \quad (MT \text{ held constant}) \tag{18}$$

The step-by-step derivation for the zero-endowment homothetic case is found in my pedagogic article Markusen (2021). But for data purposes, we want the price elasticity of purchases, X_{ip}, and allow for MT to change with the price of the good. A short appendix to the paper shows that price elasticity of purchased X_{ip} allowing MT to change with p_{ic} is given by

$$-\varepsilon_{ip} = -\frac{p_{ic}}{X_{ip}}\frac{dX_{ip}}{dp_{ic}} = [\sigma - s_i(\sigma - 1)] + (1 - s_i)\sigma\frac{X_{ie}}{X_{ip}} \geq [\sigma - s_i(\sigma - 1)] = -\varepsilon_{ih} \tag{19}$$

where ε_{ih}, the term in square brackets, is the formula for the own price elasticity under zero endowments as noted in (18) (*h* for homothetic).

Perhaps to make it a little more intuitive: the Cobb-Douglas case $\sigma = 1$, where from (14) s_i reduces to $s_i = \alpha_i$, (19) is given by

$$-\frac{p_{ic}}{X_{ip}}\frac{dX_{ip}}{dp_{ic}} = 1 + (1 - \alpha_i)\frac{X_{ie}}{X_{ip}} \geq 1 = -\varepsilon_{ih} \qquad \text{Cobb-Douglas: } \sigma = 1, \ s_i = \alpha_i \tag{20}$$

As the price of X_i rises, X_{ip} falls, and so the price elasticity of demand becomes larger (more negative) in both the general and Cobb-Douglas cases . The price elasticity approaches -1 as the price falls toward zero (X_{ip} heads toward infinity) and equals -1 with zero endowment as is well known for Cobb-Douglas. In the general case (19), the formula approaches ε_{ih} as the price falls to zero if (but not only if) $\sigma > 1$, both because X_{ip} heads toward infinity and because s_i head to one.

3. Small open-economy general-equilibrium model

I begin with a very simple, straight-forward case to illustrate the determination of choke prices and income. There are two goods, X_1 and X_2 with prices p_{x1} and p_{x2}, and two factors of production *L* and *K* in inelastic supply, with (endogenous), prices p_l and p_k. One consumer with fixed non-traded endowments X_{1e} and/or X_{2e} with X_{1e} and X_{2e} being perfect substitutes for the produced and trade goods respectively.

The country faces fixed world relative prices for traded goods X_1 and X_2 (no Armington product differentiation). If a good is not purchased (not produced or imported), then its consumer price p_{xic} will be determined by its endowment substitute only and will be less than the world price: the cost inequality for domestic production or imported X_i are strict.

Several other added variables and notation are as follows. Over bars indicate fixed values.

$c_{x1}(p_l, p_k)$ $c_{x2}(p_l, p_k)$	$c_w(p_{x1c}, p_{x2c})$	unit cost functions for X_1, X_2, and W
		partials give unit demands for X_1, X_2, and W by Shepard's lemma
T_{12} T_{21}		activities which export X_1 for X_2, X_2 for X_1 respectively
		the ratio of inputs to output is the world price ratio
tot		parameter which give world price ratio p_{x2}/p_{x1} in the T activities
CONS		consumer income including value of endowments

The model given below consists of 19 weak inequalities with 19 matched non-negative variables. These are divided into three blocks. The first are pricing inequalities derived from optimization conditions, the direction being marginal cost greater-than-or-equal to price, with the complementary variable being the output of that activity. Note that welfare is just treated as a produced good with price p_w, the consumer price index (cost of buying one unit of utility).

Price inequalities (e.g., marginal cost ≥ price) quantities as complementary variables

$$c_{x1}(p_l, p_k) \geq p_{x1} \quad \perp \quad X_1 \tag{21}$$

$$c_{x2}(p_l, p_k) \geq p_{x2} \quad \perp \quad X_2 \tag{22}$$

$$p_{x1} \geq p_{x1c} \quad \perp \quad X_{1p} \tag{23}$$

$$p_{x2} \geq p_{x2c} \quad \perp \quad X_{2p} \tag{24}$$

$$p_{x1e} \geq p_{x1c} \quad \perp \quad X_{1e} \tag{25}$$

$$p_{x2e} \geq p_{x2c} \quad \perp \quad X_{2e} \tag{26}$$

$$p_{x2} \geq p_{x1} tot \quad \perp \quad T_{21} \tag{27}$$

$$p_{x1} tot \geq p_{x2} \quad \perp \quad T_{12} \tag{28}$$

$$c_w(p_{x1c}, p_{x2c}) \geq p_w \quad \perp \quad W \tag{29}$$

The second block are market-clearing inequalities, with their direction being supply greater-than-or-equal to demand. The complementary variables are always prices. It is important for the modeler to emphasize that these good and factor demands on the right-hand side must be consistent with and derived from the cost functions above via Shepard's lemma.

Market clearing (e.g., supply ≥ demand) prices as complementary variables

$$X_1 \geq X_{1p} - T_{21} tot + T_{12} tot \quad \perp \quad p_{x1} \tag{30}$$

$$X_2 \geq X_{2p} + T_{21} - T_{12} \quad \perp \quad p_{x2} \tag{31}$$

$$X_{1p} + X_{1e} \geq \partial c_w / \partial p_{x1c} W \quad \perp \quad p_{x1c} \tag{32}$$

$$X_{2p} + X_{2e} \geq \partial c_w / \partial p_{x2c} W \quad \perp \quad p_{x2c} \tag{33}$$

$$\bar{X}_{1e} \geq X_{1e} \quad \perp \quad p_{x1e} \tag{34}$$

$$\bar{X}_{2e} \geq X_{2e} \quad \perp \quad p_{x1e} \tag{35}$$

$$\bar{L} \geq \partial c_{x1} / \partial p_l X_1 + \partial c_{x2} / \partial p_l X_2 \quad \perp \quad p_l \tag{36}$$

$$\bar{K} \geq \partial c_{x1} / \partial p_k X_1 + \partial c_{x2} / \partial p_k X_2 \quad \perp \quad p_k \tag{37}$$

$$W \geq CONS/p_w \quad \perp \quad p_w \tag{38}$$

The final block is income-balance inequalities. Here there is just a single one for the representative consumer. This is effectively just a definitional equation, since it could be substituted into the market-clearing equation for welfare just above.

Income balance - consumer income is the complementary variable

$$CONS \geq p_l\bar{L} + p_k\bar{K} + p_{x1e}\bar{X}_{1e} + p_{x2e}\bar{X}_{2e} \quad \perp \quad CONS \tag{39}$$

4. A calibrated example with simulations

The following matrix represents a micro-consistent set of initial values for this model. Columns are inputs and output for the sectors of the model including the "production" of welfare and consumer income. Rows are markets with outputs (supplies) positive and inputs (demands) negative. Micro-consistency means that all row and column sums should be zero. There are two features to note. The first is that I have calibrated the model to zero trade initially (no Armington differentiation, so this is possible). Second, X_1 has a large amount of its perfect substitute endowment good, $p_{x1e} = 150$, while $p_{x2e} = 1$. This spread is much greater than the difference in consumption shares, so good X_1 will have an income elasticity of demand greater than one and vice versa for good X_2.

	Production Sectors						Trade		Welfare	Consumer
	X1	X2	X1P	X1E	X2P	X2E	TX21	TX12	W	CONS
Markets										
PX1	100		-100				-0	0		
PX2		100			-100		0	-0		
PL	-25	-75								100
PK	-75	-25								100
PW									351	-351
PX1C			100	150					-250	
PX1E				-150						150
PX2C					100	1			-101	
PX2E						-1				1

These values then allow for the numerical calibration of the cost functions for the two goods and welfare. For simplicity and clarity, I'll just impose Cobb-Douglas functional forms on the goods and welfare. The matrix of values then gives us the numerical versions for producing X_1, X_2 and welfare W.

$$c_{x1}(p_l, p_k) = p_l^{0.25} p_k^{0.75} \qquad c_{x2}(p_l, p_k) = p_l^{0.75} p_k^{0.25} \tag{40}$$

These give unit factor demand equations in (36) and (37) by Shepard's lemma as follows.

$$\frac{\partial c_{x1}(p_l,p_k)}{\partial p_l} = 0.25 p_l^{0.25-1} p_k^{0.75} \qquad \frac{\partial c_{x1}(p_l,p_k)}{\partial p_k} = 0.75 p_l^{0.25} p_k^{0.75-1} \tag{41}$$

$$\frac{\partial c_{x2}(p_l,p_k)}{\partial p_l} = 0.75 p_l^{0.75-1} p_k^{0.25} \qquad \frac{\partial c_{x2}(p_l,p_k)}{\partial p_k} = 0.25 p_l^{0.75} p_k^{0.25-1} \tag{42}$$

The cost (expenditure) function for producing one unit of utility is given by

$$c_w(p_{x1c},p_{x2c}) = p_{x1c}^{\alpha_1} p_{x2c}^{\alpha_2} \qquad \alpha_1 = 250/351 \qquad \alpha_2 = 101/351 \tag{43}$$

Shepard's lemma give the unit commodity demand functions for producing a unit of W in (43) as

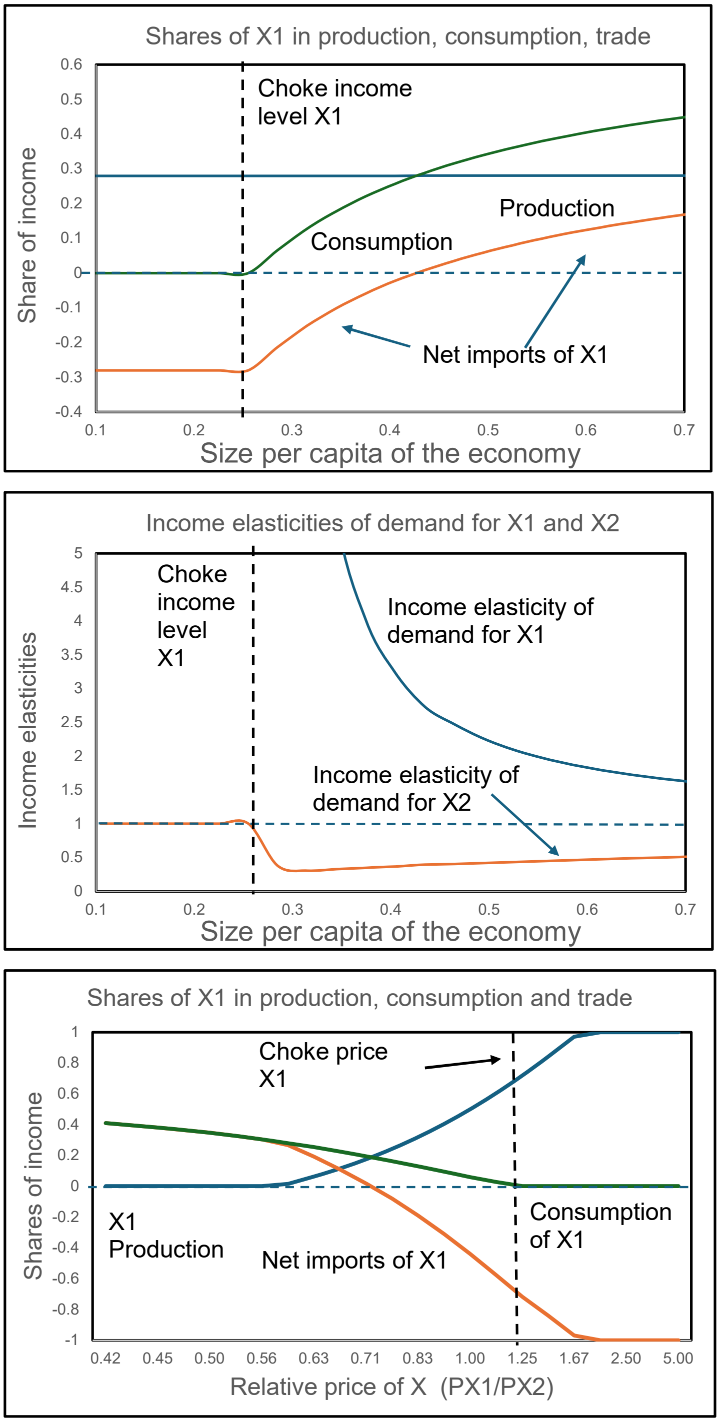
$$\frac{\partial c_w(p_{x1c},p_{x2c})}{\partial p_{x1c}} = \alpha_1 p_{x1c}^{\alpha_1-1} p_{x2c}^{\alpha_2} \qquad \frac{\partial c_w(p_{x1c},p_{x2c})}{\partial p_{x2c}} = \alpha_2 p_{x1c}^{\alpha_1} p_{x2c}^{\alpha_2-1} \tag{44}$$

Figure 2 presents results for this model. Panel A and B loop over the size of the economy, increasing the endowments of labor and capital in the same proportion while holding the levels of the endowment substitutes constant. Panel A focuses on purchased level of good X_{1p} with the high income-elasticity of demand (high endowment substitute), expressing results in shares of income (observed M, labor and capital income). The share of X_1 production is constant due to the small economy assumption. The vertical dashed line is the choke income level for X_{1p} consumption, so in additional to consumption being zero, all production of X_1 is exported. As per-capita income rises, consumption of X_{1p} gradually increases once choke income is passed and trade gradually shifts from exporting all X_1. The curves pass through an autarky point and then the direction of trade switches to importing X_1.

Panel B of Figure 2 graphs the income elasticities of demand for X_{1p} and X_{2p}. When income is below the choke level for X_{1p} the income elasticity for X_{2p} is equal to one. When the choke level is first passed, the income elasticity of demand for X_{1p} is (locally) infinity (at choke income the denominator of (16) is zero) This elasticity falls steeply, and the two income elasticities gradually converge toward one as discussed and shown above.

Panel C of Figure 2 graphs an increase in the relative (world) price of X_1 relative to X_2. At the low price at the left edge, the economy is specialized in the production of X_2. All consumption of X_1 is from imports (which of course equal consumption). As the relative price of X_1 continues to increase that good is produced and eventually the direction of trade reverses. Those results are completely "normal". What is different is that we eventually reach a choke price level in which the consumption of X_1 goes to zero. All production of X_1 is exported.

Figure 2: Small open economy - fixed world prices
SGM preferences - loop over size per capita, relative price of X1



Panel A: increase per-capita income holding world prices constant shares of production, consumption and trade

Panel B: increase per-capita income holding world prices constant income elasticities of demand

Panel C: increase relative price of X1 factor income fixed

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5. CRIE preferences, demand, income and price elasticities

Now we turn to a similar model in many respects, but now preferences are changed to what we have labeled CRIE, constant relative income elasticity. In simple terms, a CRIE function is a CES with different exponents on different goods. I present a very simple version of the model in order to develop the intuition and clearly show its principal properties. The model is a two-final-good, two (conventional, observed) primary factors. It is a small, open economy facing fixed world prices, which can be adjusted as counter-factual. Each produced and traded good is a perfect substitute in trade for same good from the (not-modeled) rest of world.

As noted in the introduction, these preferences have been around for a very long time. CRIE has the advantage that income elasticities do not all converge to one even with large differences in per-capita income. But it has been the view that they are awkward to use in general-equilibrium models since there are no closed form solutions for cost/expenditure and demand functions. In addition, the formulations used in Fieler (2011) and Caron-Fally-Markusen (2014, 2020) do not allow for elasticities of substitution across sectors to be less than one, which seems to be another counter-empirical restriction. That is not the case here.

I have come up with a trick to model CRIE preferences in a way that retains constant-returns, homothetic CES functions so that these preferences slot right into AGE models. There are parameters which we will refer to as sector-specific factors that create the non-homotheticity. They are not observed in the data. The observed goods are combined with the specific factors in fixed supply to produce "utility goods", analogous to the aggregate utility goods produced from different varieties in Armington models or monopolistic competition. CRIE preferences do not allow for choke income and prices. However, the latter can be done by adding the SGM endowments to this formulation. More on that later. I have tried to make this formulation as close as possible to the SMG version, with the specific factors conceptually functioning similar to the endowment goods of SMG.

Notation for quantity variables:

X_1 production of good 1 - observed in data
X_2 production of good 2 - observed in data
X_{1p} domestic purchases of good 1 - observed in data
X_{2p} domestic purchases of good 2 - observed in data

TX_{12} trade activity exports X_1 in exchange for X_2
TX_{21} trade activity exports X_2 in exchange for X_1

$L_1\ K_1$ labor and capital used to produce X_1
$L_2\ K_2$ labor and capital used to produce X_2
$L\ K$ total fixed labor and capital - scaled up/down to change the size of the economy
R_1 specific factor used by the Y_1 sector - fixed supply - not observed in data
R_2 specific factor used by the Y_2 sector - fixed supply - not observed in data

Y_1 utility good produced X_1 and R_1 - Cobb-Douglas - not observed in data
Y_2 utility good produced X_2 and R_2 - Cobb-Douglas - not observed in data
W welfare produced from Y_1 and Y_2 - CES

The following are the key properties of the production, consumption, and trade with the rest of world for the two-sector model.

$$X_1 = L_1^{\beta_{1l}} K_1^{\beta_{1k}} \qquad X_2 = L_2^{\beta_{2l}} K_2^{\beta_{2k}} \qquad L = L_1 + L_2 \quad K = K_1 + K_2 \tag{45}$$

$$X_{1p} = X_1 - TX_{12} + TX_{21} \qquad X_{2p} = X_2 - TX_{21} + TX_{12} \tag{46}$$

$$Y_1 = X_{1p}^{\alpha_{x1}} R_1^{\alpha_{r1}} \qquad Y_2 = X_{2p}^{\alpha_{x2}} R_2^{\alpha_{r2}} \qquad \alpha_{xi} + \alpha_{ri} = 1 \tag{47}$$

$$W = (Y_1^{\rho} + Y_2^{\rho})^{1/\rho} \qquad -\infty \le \rho \le 1 \qquad \sigma = \frac{1}{1 - \rho} \qquad \rho = \frac{\sigma - 1}{\sigma} \tag{48}$$

All production activities have constant returns to scale (CRS), which greatly facilitates computation in large models, especially using Rutherford's mps/ge.

The relationship between α_{x1} and α_{x2} determines which good has the higher income elasticity. Letting $R_1 = R_2 = 1$ for simplicity, W can be written as

$$W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \qquad \gamma_1 = \rho\alpha_{x1} \qquad \gamma_2 = \rho\alpha_{x2} \tag{49}$$

The good with the larger α (larger or less negative) will have the higher income elasticity of demand. If $\rho > 0$ or equivalently $\sigma > 1$, then the good with the *higher* α will have the higher income elasticity of demand. However, if $\rho < 0$ or equivalently $\sigma < 1$, then the good with the *lower* α will have the higher income elasticity; e.g., X_1 has the higher elasticity with a lower α because its exponent is less negative: $\rho\alpha_{x2} < \rho\alpha_{x1} < 0$.

Other notation:

tot	relative price ratio p_{x1}/p_{x2} (although some results presented with the inverse), exogenous parameter in this small open economy case
$c_i(k_1, k_2)$	unit cost functions with arguments k_1 and k_2

Notation for price and income:

p_{x1}	price of production good X_1
p_{x2}	price of production good X_2
p_{x1p}	price of the observed consumption good X_1
p_{x2p}	price of the observed consumption good X_2
p_{r1}	price of the endowment good R_1

p_{r2}	price of the endowment good R_2
p_{y1}	price of consumption good Y_1
p_{y2}	price of consumption good Y_2
p_w	consumer price index (unit expenditure function)
p_l	price of labor
p_k	price of capital
CONS	representative consumer income

CRIE, as its name suggests, has the nice feature that the ratio of income elasticities between two goods is constant, which could aid greatly in calibration. Let p_{x1p} and p_{x2p} denote goods price, held constant here. Maximize utility subject to a budget constraint with income M and Lagrangean multiplier λ.

$$\max\ W = (X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho} \quad s.t. \quad M = p_{x1}X_{1p} + p_{x2}X_{2p} \tag{50}$$

$$\delta\gamma_1 X_{1p}^{\gamma_1 - 1} - \lambda p_{x1p} = 0 \quad \delta\gamma_2 X_{2p}^{\gamma_2 - 1} - \lambda p_{x2p} = 0 \quad \delta = (1/\rho)(X_{1p}^{\gamma_1} + X_{2p}^{\gamma_2})^{1/\rho - 1}$$

An increase in M is proxied here by a change (fall) in λ. Hold prices constant (normalize them equal to 1). Forming the ratio, λ and δ cancel out, giving

$$\frac{dX_{1p}}{dX_{2p}} = \frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 2}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 2}} \qquad \frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{\gamma_2(1-\gamma_2)X_{2p}^{\gamma_2 - 1}}{\gamma_1(1-\gamma_1)X_{1p}^{\gamma_1 - 1}}$$

$$\frac{\gamma_2 X_{2p}^{\gamma_2 - 1}}{\gamma_1 X_{1p}^{\gamma_1 - 1}} = 1 \qquad \text{from the first-order conditions (xx)}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} = \frac{1-\gamma_2}{1-\gamma_1} \tag{53}$$

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \quad \textit{if}\ \gamma_i > 0 \ \ \textit{and}\ \ \gamma_1 > \gamma_2 \ \ \textit{or}\ \ \gamma_i < 0 \ \ \textit{and}\ \gamma_1 < \gamma_2$$

or alternatively

$$\frac{dX_{1p}/X_{1p}}{dX_{2p}/X_{2p}} > 1 \quad \textit{if}\ \rho > 0 \ \ \textit{and}\ \ a_{x1} > a_{x2} \ \ \textit{or}\ \ \rho < 0 \ \ \textit{and}\ \ a_{x1} < a_{x2} \tag{54}$$

This relationship is verified in the simulations, both for $\sigma > 1$ ($0 < \rho < 1$) and $\sigma < 1$ ($\rho < 0$).

Next, consider the effect on demands of an increase in income M holding the prices of the two goods constant.

$$M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \qquad 1 = p_{x1p}\frac{dX_{1p}}{dM} + p_{x2p}\frac{dX_{2p}}{dM} \tag{55}$$

$$1 = \left[\frac{p_{x1p}X_{1p}}{M}\right]\frac{dX_{1p}/X_{1p}}{dM/M} + \left[\frac{p_{x2p}X_{2p}}{M}\right]\frac{dX_{2p}/X_{2p}}{dM/M} \tag{56}$$

Let s_{xi} denote the expenditure share on good i and let ηi denote the income elasticity of demand for good i. Equation (xx) reduces to

$$\sum_i s_{xi}\eta_i = 1 \qquad i = 1, 2 \tag{57}$$

Note that, although the ratio of income elasticities is constant as noted above, they cannot be constant as M increases (unless all income elasticities are one - homothetic preferences). As the expenditure share of the good with the high η increases with income, the η's cannot remain constant or (57) is violated. Both income elasticities must fall, maintaining their ratio, as income grows. Note that this coming exclusively from the budget constraint: it has nothing to do with the particular specification of preferences. Equation (57) is verified in the simulations.

Now consider the effect of increasing the price of good 1 holding income and the price of good 2 constant. While the algebra that follows here is not specific to any type of preferences, it is useful to help interpret simulation results to follow.

$$M = p_{x1p}X_{1p} + p_{x2p}X_{2p} \qquad \frac{dM}{dp_{x1p}} = X_{1p} + p_{x1p}\frac{dX_{1p}}{dp_{x1p}} + p_{x2p}\frac{dX_{2p}}{dp_{x1p}} = 0 \tag{58}$$

$$1 + \frac{p_{x1p}}{X_{1p}}\frac{dX_{1p}}{dp_{x1p}} + \left[\frac{p_{x2p}X_{2p}}{p_{x1p}X_{1p}}\right]\frac{p_{x1p}}{X_{2p}}\frac{dX_{2p}}{dp_{x1p}} = \frac{1}{X_{1p}}\frac{dM}{dp_{x1p}} = 0$$

$$\varepsilon_{11} + \frac{s_{x2}}{s_{x1}}\varepsilon_{21} = -1 \qquad \varepsilon_{ij} \text{ - elasticity of demand for } X_{ip} \text{ wrt } p_{xjp} \tag{60}$$

$$\sum_i s_{xi}\varepsilon_{i1} = -s_{x1} \le 0 \qquad M \text{ constant or economy specialized in producing } X_2 \tag{61}$$

In general, we expect the own-price elasticity to be negative. However, the cross-price elasticity could be negative as well: with a low elasticity of substitution between goods, the effect of the price increase holding income constant could lead to a fall in demand for both goods and indeed this must be the case with a zero elasticity of substitution.

Suppose that the economy is specialized in X_2 so that $M = p_{x2p}\bar{X}_2$, where the over bar is the end of the production frontier. The (60) and (61) remain valid in that a change in the price of X_1 has no effect on M.

Suppose instead that the economy is specialized in X_1 so that $M = p_{x1p}\bar{X}_1$. Now the right-hand side of (xx) is no longer zero.

$$M = p_{x1p}\bar{X}_1, \quad \frac{dM}{dp_{x1p}} = \bar{X}_1 \qquad \frac{1}{X_{1p}}\frac{dM}{dp_{x1p}} = \frac{\bar{X}_1}{X_{1p}} = \frac{p_{x1p}\bar{X}_1}{p_{x1p}X_{1p}} = \frac{1}{s_{x1}} \tag{62}$$

$$\varepsilon_{11} + \frac{s_{x2}}{s_{x1}}\varepsilon_{21} = -1 + \frac{1}{s_{x1}} \tag{63}$$

$$\sum_i s_{xi}\varepsilon_{i1} = -s_{x1} + 1 \geq 0 \qquad \text{economy specialized in producing } X_1 \tag{64}$$

As seems intuitive, the response of demands to an increase in the price of good 1 are larger - less negative or even turning positive - if the economy is specialized in good 1 that if it is specialized in good 2. The effect of the increase in the price of good 1on income when the economy is specialized in good 1 is positive while it is negative if specialized in good 2. Note that in this case, the price elasticities of both goods can be positive. This will be confirmed in the simulations.

The bottom right-hand diagram in Figure 1 gives a (hand drawn) picture of the Engels curve for the CRIE case. As noted, these preferences by themselves do not allow for choke income and prices, but they have significant advantages in that the income elasticities do not converge to one, particularly relevant for goods with low income elasticities. It is hard to swallow these goods having unity income elasticities at high levels of income.

The next section gives the inequalities and complementary variables for the small-economy general-equilibrium model in the same format as the SGM model in (21)-(39). I have tried to construct the model in a manner that stay close to the SGM formulation and indeed, as I'll comment later, the two can be combined with both endowment goods and specific fixed factors.

5. CRIE small open-economy general-equilibrium model

Price inequalities		quantities as complementary variables	
$c_{x1}(p_l, p_k) \geq p_{x1}$	$\perp$	X_1	(65)
$c_{x2}(p_l, p_k) \geq p_{x2}$	$\perp$	X_2	(66)
$p_{x1} \geq p_{x1p}$	$\perp$	X_{1p}	(67)
$p_{x2} \geq p_{x2p}$	$\perp$	X_{2p}	(68)
$p_{x2} \geq p_{x1} tot$	$\perp$	T_{21}	(69)
$p_{x1} tot \geq p_{x2}$	$\perp$	T_{12}	(70)
$c_{y1}(p_{x1p}, p_{r1}) \geq p_{y1}$	$\perp$	Y_1	(71)
$c_{y2}(p_{x2p}, p_{r2}) \geq p_{y2}$	$\perp$	Y_1	(72)
$c_w(p_{y1}, p_{y2}) \geq p_w$	$\perp$	W_1	(73)

Market clearing		prices as complementary variables	
$X_1 \geq X_{1p} - T_{21} tot + T_{12} tot$	$\perp$	p_{x1}	(74)
$X_2 \geq X_{2p} + T_{21} - T_{12}$	$\perp$	p_{x2}	(75)
$X_{1p} \geq \partial c_{y1}/\partial p_{x1p} Y_1$	$\perp$	p_{x1p}	(76)
$X_{2p} \geq \partial c_{y2}/\partial p_{x2p} Y_2$	$\perp$	p_{x2p}	(77)
$L \geq \partial c_{x1}/\partial p_l X_1 + \partial c_{x2}/\partial p_l X_2$	$\perp$	p_l	(78)
$K \geq \partial c_{x1}/\partial p_k X_1 + \partial c_{x2}/\partial p_k X_2$	$\perp$	p_k	(79)
$R_1 \geq \partial c_{y1}/\partial p_{r1} Y_1$	$\perp$	p_{r1}	(80)
$R_2 \geq \partial c_{y2}/\partial p_{r2} Y_2$	$\perp$	p_{r2}	(81)
$Y_1 \geq \partial c_w/\partial p_{y1} W$	$\perp$	p_{y1}	(82)
$Y_2 \geq \partial c_w/\partial p_{y2} W$	$\perp$	p_{y2}	(83)
$W \geq CONS/p_w$	$\perp$	p_w	(84)

Income balance		consumer income as complementary variable	
$CONS \geq p_l L + p_k K + p_{r1} R_1 + p_{r2} R_{r2}$	$\perp$	$CONS$	(85)

The initial calibration of the model is as follows, with a micro-consistent set of numbers requiring row and column sums equal to zero. As in the SGM case, I calibrated it to zero trade initially. The share of the specific factor in Y_1 is lower than in Y_2 so that, with an elasticity of substitution greater than one, Y_1 is the sector with the higher income elasticity of demand.

	Production Sectors						Trade		Welfare	Consumer
	X1	X1P	X2	X2P	Y1	Y2	TX21	TX12	W	CONS
Markets										
PX1	100	-100					-0	0		
PX1P		100			-100					
PX2			100	-100			0	-0		
PX2P				100		-100				
PR1					-25					25
PR2						-75				75
PY1					125				-125	
PY2						175			-175	
PW									300	-300
PL	-25		-75							100
PK	-75		-25							100

Figure (3) presents a simulation of a case with the elasticity of substitution σ between Y_1 and Y_2 in (48) equal to 2.5. Panels A and B correspond to the results shown in panels A and B of Figure 2, looping over the endowment size of the economy. Endowments of the specific factors are held fixe as L and K increase, similar to holding the endowment goods constant in SGM.

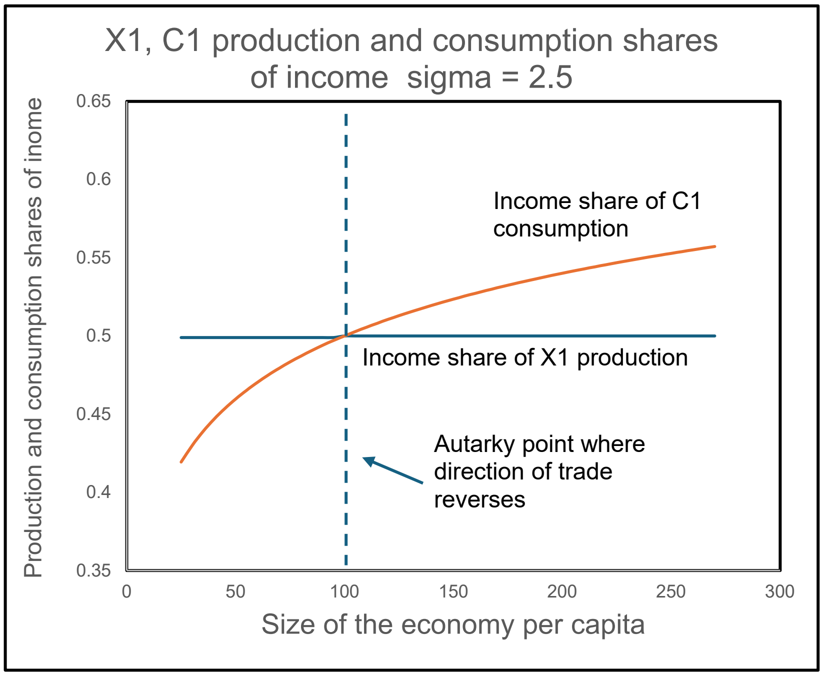
Panel A of Figure (3) shows how the income shares of X_1 consumption and production change with per-capita size. The production share of X_1 is constant by virtue of the small-country assumption. The dashed line is the autarky size and also the calibration point as just noted. At a small size, X_1 is exported while at large size it is imported. The vertical distance between the production and consumption lines is the share of imports or exports. Panel B of Figure (3) give the income elasticities of demand for the same per-capita income experiment. As noted a couple of times, a feature of CRIE is that these income elasticity differences persist over a much larger range of income and the good with the low income elasticity of demand never rises (and of course doesn't approach one) contrary to SGM.

It is certainly not a surprise that a high-income country specializes in consuming the income elastic good. I should note that the result here that this maps directly into the direction and balance is trade is not general however. Fieler (2011) and especially Caron-Fally-Markusen (2014, 2020) show that in models with much richer production structures, high income countries are relatively specialized in the production of (skilled-labor-intensive) goods that are also the high income-elasticity goods, so those countries may export the high income elasticity goods.

Figure (4) shows the effect of increasing the terms-of-trade, the relative price of good X_1. I don't want to spend much time on this because the results are very similar to what we would get from a case with homothetic preferences. Panel A shows that the income share of production has bounds where there is no X_1 produced on the left (specializing in X_2) to specializing in X_1 and the right: this of course depends on domestic and foreign goods being perfect substitutes. The income share of X_1 consumption declines with its rising relative price due to the elasticity of

Figure 3: Small open economy - fixed world prices

CRIE preferences - loop over size (income) per capita



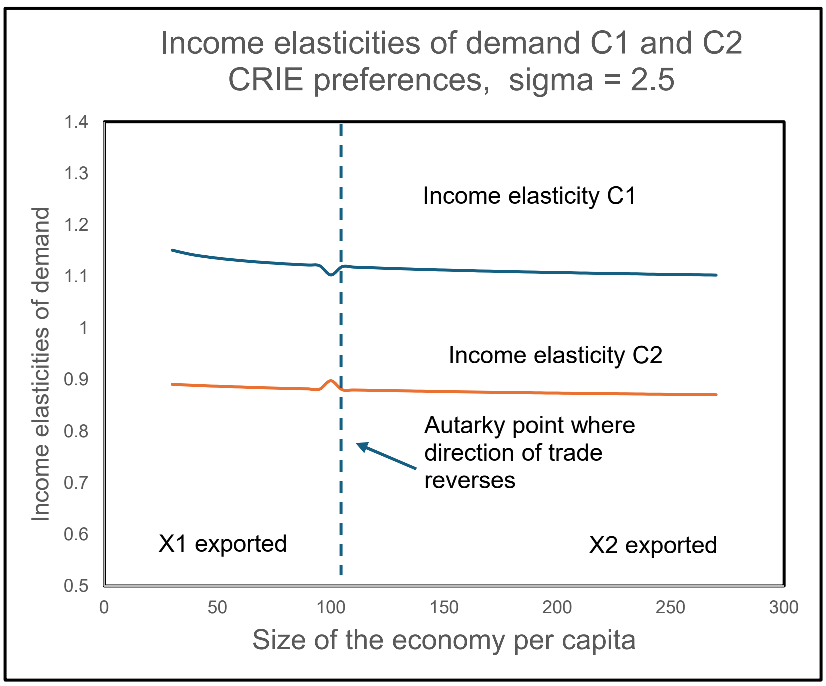


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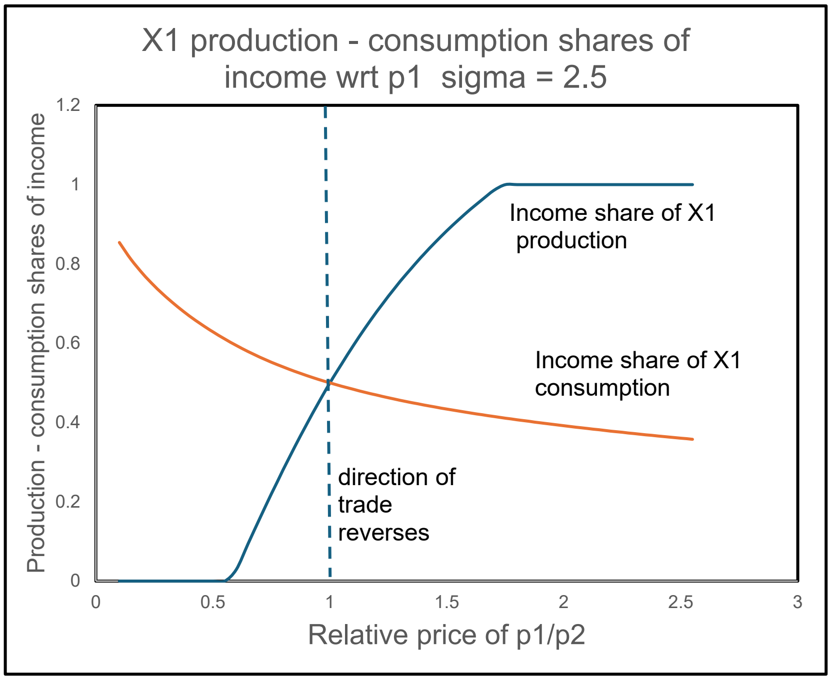
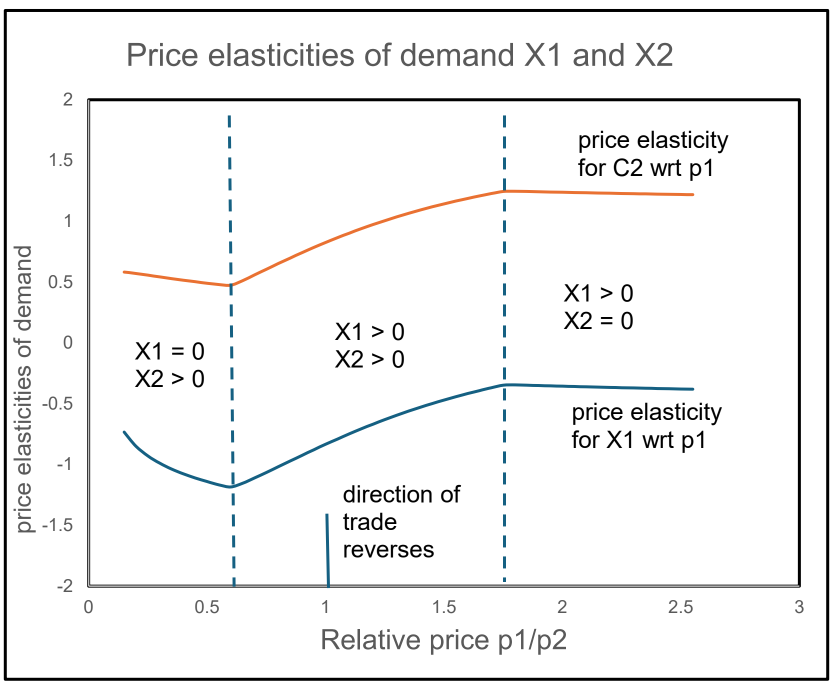
Figure 4: Small open economy - fixed world prices
CRIE preferences - loop over relative price of X1





c:\jim\OPTand SIM\CRIE\CRIE-open-p-25.gms c:\jim\OPTand SIM\CRIE\CRIE-open.xlsx SIGMA = 2.5

substitute being greater than one. Again, these results would be the case in any Heckscher-Ohlin type of model.

Panel B shows the price elasticities of demand for X_1 and X_2. The cross-price elasticity of demand for good X_2 is always positive, indicating that the goods are gross substitutes. The non-monotonic curve for px1 reflects the discussion about price elasticities presented above; specifically the dependence of the own-price elasticity on the income effect which in turn depends on the direction of trade. When the economy specializes in X_1 on the right-hand side of the figure, the strong positive income effect of a further increase in p_{x1p} leads to a significantly smaller reduction in demand than one the left-hand side.

My CRIE formulation allows for elasticities of substitution between sectors to be less than one, which I believe is supported by empirical evidence (references very welcome - memory loss on this). This does require a re-calibration if X_1 is to remain the good with the high income elasticity as discussed above, specifically now a larger Cobb-Douglas share of the specific factor R_1 in X_1. The results don't differ much from those I just presented for σ = 2.5, so I won't show them here. The code and figures for a case with σ = 0.5 are included with my supplementary materials. The only qualitative different with the low elasticity of substitution occurs in the experiment of increasing the price of X_1. Now the share spent on X_1 increases with the price, but this would also be true with homothetic preferences, so I don't think that further comment is warranted.

6. Strengths, Weaknesses and Challenges

As noted early on, a number of other authors have considered adding non-homothetic preferences to general-equilibrium trade theory and empirics. Many or most of these are significantly more technically complex than this present paper and proposal. Many have analytical solutions for their theory. This is a great strength, but the restrictive assumptions needed for analytical solutions have a cost for modelers seeking to formulate empirically-relevant models calibrated to large, multi-sector, multi-country, multi-factor and multi-household-type data sets. I could include myself in this catagory in that the CRIE preferences used in Caron, Fally and Markusen (2014, 2020) do not yield analytical solutions.

I am arguing that the simplicity of my formulation is a strength and an attractive feature for AGE modelers. In particular, it retains all the valuable features of CES preferences which generate solutions for cost and expenditure functions depending only on prices. This in turn generates straightforward and simple derivations of factor and goods-demand equations via Shepard's lemma. These features then allow a model to be specified in a complementarity format where pricing (optimization) inequalities have quantities as complementary variables and market clearing inequalities have prices as complementary variables. The final step is that these characteristic allow computation of solutions with choke income and prices levels.

However, I am the first to acknowledge that there are two significant problems to implementation on general-equilibrium data sets. The first of these is calibration: we have to solve for the level of the unobserved endowment goods in the SGM version in order to get the income elasticities. Imagine that we have a set of estimates of income elasticities of demand such as those from Caron, Fallly and Markusen (2014, 2020) shown earlier. We could make the assumption that these apply to every country regardless of their per-capita income (the theory says these elasticities converge to one as income becomes large). But prices differ substantially in the benchmark data. Perhaps running a simulation which eliminates all trade costs could form a basis for setting the endowment parameters by country by good? But the implied income elasticities may then not be consistent with price elasticities if there are estimates of those. Again, we did make progress on this in the two Caron, Fally and Markusen papers. Nevertheless, I accept that this is a significant barrier to implementation.

The second difficulty lies in assessing the choke prices and incomes for goods not traded or not produced initially. How unprofitable are these trade links for latent goods at the benchmark? This is another difficult problem but one which I think AGE modelers can crack. I should point out that we have always had this problem. The standard procedure is to simply hold latent (zero) trade links in the benchmark constant at zero. In effect, there is an unbounded unobserved and implicit trade barrier on these links. This comes at a huge cost in that there cannot be an expansion of trade at the extensive margin following a liberalization. Conversely, an initially-active link cannot go to zero given standard CES preferences. The other approach is the Eaton-Kortum formulation in which (as I understand it) there are no zeros initially, just extremely small but positive trade flows on all links. So again, there is no change in trade at the extensive margin. When trade costs fall just a little, there will be immediate positive or negative (but still non-zero) responses on all links.

The CRIE formulation with its "fictitious factors" not only has theoretical advantages (the non-convergence of income elasticities to one), but is probably a better candidate for calibration to existing data sets. Given a set of income elasticities as in Caron, Fally and Markusen shown earlier, these allow for relatively easy parameterization of expenditure and demand functions. Further, the CRIE preferences permit the independent calibration of price and income elasticities: in the notation of the paper, the ρ parameter can be combined with the differences in the α_i across sectors to acheive some independence in the two elasticities. The drawback of CRIE, given the goals of the paper, is that the basic formulation does not allow for choke income and prices. SGM and CRIE can be combined (just add the endowment goods to CRIE), but this then revisits the calibration issues.

Available from the author directly

(1) notes and code for a SGM model with Armington product differentiation
(2) notes and code for a SGM Armington model in which the endowment goods are produced domestically, but inefficiently, so choke prices occur when the foreign substitute become to pricey.
(3) notes and code for a model combining CRIE with fictitious factors with a SGM model with substitute endowment goods.
(4) .gms and excel files for all models, a couple which have both mcp and mps/ge formats.

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Appendix 1: Algebra for the (partial equilibrium) own price elasticity of demand for X_{1p}

$$X_{ic} = \frac{\alpha_i^{\sigma} p_{ic}^{-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} MT \qquad MT = M + \sum_i p_{ic} X_{ie} \tag{A1}$$

$$\frac{p_{ic}}{X_{ic}} \frac{dX_{ic}}{dp_{ic}} = -[\sigma - s_i(\sigma - 1)] + \frac{\alpha_i^{\sigma} p_{ic}^{1-\sigma}}{\sum_i \alpha_i^{\sigma} p_{ic}^{1-\sigma}} \frac{\partial MT}{\partial p_{ic}} \frac{1}{X_{ic}} \qquad \frac{\partial MT}{\partial p_{ic}} = X_{ie} \tag{A2}$$

where the terms in square brackets is derived in detail in Markusen (2023). (A2) simplified to

$$\frac{p_{ic}}{X_{ic}} \frac{dX_{ic}}{dp_{ic}} = -[\sigma - s_i(\sigma - 1)] + s_i \frac{X_{ie}}{X_{ic}} \tag{A3}$$

We want the elasticity of purchased (observed) X_{ip} however, not the elasticity of total consumption including the endowment. Multiple both sides through by X_{ic}/X_{ip} and replace dX_{ic} = dX_{ip} since these are identical ($dX_{ie} = 0$).

$$\frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = -[\sigma - s_i(\sigma - 1)] \frac{X_{ic}}{X_{ip}} + s_i \frac{X_{ie}}{X_{ip}} \qquad \frac{X_{ic}}{X_{ip}} = \frac{X_{ip} + X_{ie}}{X_{ip}} = 1 + \frac{X_{ie}}{X_{ip}} \tag{A4}$$

$$\frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = -[\sigma - s_i(\sigma - 1)] \left[1 + \frac{X_{ie}}{X_{ip}}\right] + s_i \frac{X_{ie}}{X_{ip}} \tag{A5}$$

$$\frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = -[\sigma - s_i(\sigma - 1)] - (1 - s_i)\sigma \frac{X_{ie}}{X_{ip}} \tag{A6}$$

$$\varepsilon_{ip} = \frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = \varepsilon_{ih} - (1 - s_i)\sigma \frac{X_{ie}}{X_{ip}} \tag{A7}$$

which is more negative than ε_{ih} (larger in absolute value).

The Cobb-Douglas special case is given by

$$\varepsilon_{ip} = \frac{p_{ic}}{X_{ip}} \frac{dX_{ip}}{dp_{ic}} = -\left[1 + (1 - \alpha_i) \frac{X_{ie}}{X_{ip}}\right] \leq -1 \quad \text{Cobb-Douglas: } \sigma = 1, \;\; s_i = \alpha_i \tag{A8}$$

I checked my derivation by deriving the Cobb-Douglas case with an alternative procedure using (15).

$$X_{ip} = \frac{s_i M}{p_{ic}} + \frac{s_i \sum_i p_{ic} X_{ie}}{p_{ic}} - X_{ie} \tag{A9}$$

$$\frac{dX_{ip}}{dp_{ic}} = -\frac{\alpha_i M}{p_{ic}^2} - \frac{\alpha_i \sum_i p_{ic} X_{ie}}{p_{ic}^2} + \frac{p_i \alpha_i X_{ie}}{p_{ic}^2} \tag{A10}$$

$$p_{ic}\frac{dX_{ip}}{dp_{ic}} = -\frac{\alpha_i M}{p_{ic}} - \frac{\alpha_i \sum_i p_{ic} X_{ie}}{p_{ic}} + \alpha_i X_{ie} \tag{A11}$$

$$= -\left[\frac{\alpha_i M}{p_{ic}} + \frac{\alpha_i \sum_i p_{ic} X_{ie}}{p_{ic}} - X_{ie}\right] - (1-\alpha_i)X_{ie} = -X_{ip} - (1-\alpha_i)X_{ie} \tag{A12}$$

$$-\frac{p_{ic}}{X_{ip}}\frac{dX_{ip}}{dp_{ic}} = 1 + (1-\alpha_i)\frac{X_{ie}}{X_{ip}} \geq 1 \qquad \text{(Cobb-Douglas)} \tag{A13}$$

Appendix 2: Income constraint implies restrictions on income elasticities of demand, AIDS

$$M = \sum_i p_i X_i \qquad dM = \sum_i p_i \frac{dX_i}{dM} dM \qquad \text{(holding prices constant)} \tag{A14}$$

$$1 = \sum_i \left[\frac{p_i X_i}{M}\right]\left[\frac{M}{X_i}\frac{dX_i}{dM}\right] = \sum_i s_i \eta_i \tag{A15}$$

where s_i is the expenditure share on X_i and η_i is the income elasticity of demand for X_i. Consider a two-good case with X_1 the income elastic good $\eta_1 > 1$, $\eta_2 < 1$. As income increases, s_1 increases and s_2 decreases: $ds_1 = -ds_2 > 0$. For the summation in (A15) to hold, η_1 must decrease (eventually at least) as income grows large, although η_2 may increases or decrease depending on the functional form. Intuitively, as the share s_1 spent on X_1 heads to one with a very large income, then the income elasticity of X_1 must approach 1: (almost) all additional income is spent on X_1.

$$s_1 \Rightarrow 1 \quad \textit{implies} \quad \eta_1 \Rightarrow 1 \tag{A16}$$

In the modified Stone-Geary formulation, all income elasticities will converge to one at should be clear from (16). The denominator converges to $s_i M$ as income grows large and so all income elasticities converge to 1.

Now consider a basic AIDS (almost ideal demand system) based on the analysis and exposition in Deaton and Muellbauer (1980, p.75 equation (4.6)). The AIDS is derived from or at least motived by the Stone-Geary utility function, and has been used by a number of authors for estimating parameters such as price and income elasticities. I think that is entirely fine, but I am arguing that it should not be used for counter-factuals because it is only locally regular (specifically doesn't satisfy adding up in counter-factuals.

Let X_i be a good and M be income. Ignore all prices, set them all = 1. The w_i are budget (expenditure) shares and the basic AIDS equation is written as

$$w_i = \frac{X_i}{M} = \alpha_i + \beta_i \ln M \qquad \text{(A17)}$$

Deaton and Muellbauer state that adding up is satisfies if $\sum \alpha_i = 1 \;\; and \;\; \sum \beta_i = 0$ but this is only true locally, with two fatal problems. The first, if the β_i are constant, as always assumed, is obvious. As M grows to a large value, goods with positive βs (income elasticities greater than one) will have expenditures shares greater than one and goods with negative βs will have shares less than zero. The second flaw is that even if calibration satisfied the adding-up condition initially, it is immediately violated when income changes. Differentiate (A17).

$$\frac{dw_i}{w_i} = \frac{dX_i}{X_i} - \frac{dM}{M} = \beta_i \frac{1}{M} dM \qquad \text{(A18)}$$

$$\frac{dX_i}{X_i} \Big/ \frac{dM}{M} - 1 = \beta_i \qquad \frac{dX_i}{X_i} \Big/ \frac{dM}{M} = 1 + \beta_i \equiv \eta_i \quad \text{income elasticity} \qquad \text{(A19)}$$

$\beta > 0$ is a good with an income elasticity greater than one and goods with $\beta < 0$ have income elasticities less than one. Budget constraint adding up condition

$$\sum_i w_i \eta_i = 1 \qquad \text{requires} \qquad \sum_i \left[\alpha_i + \beta_i \ln X_i\right](1 + \beta_i) = 1 \qquad \text{(A20)}$$

Even if this is calibrated to be satisfied initially, it cannot hold once income changes. Consider this for the two-good case. Assume that $\eta_1 > 1$ ($\beta_1 > 0$) and $\eta_2 < 1$ ($\beta_2 < 0$). The βs and therefore the ηs are treated as constants. As income increases, the share spent on X_1 increases, call that $dw_1 > 0$ and the share spent on good X_2 decreases by the same amount, $dw_2 = -\, dw_1$. Then the change in the left-hand side of the adding-up requirement is

$$\eta_1 dw_1 - \eta_2 dw_1 = (\eta_1 - \eta_2) dw_1 > 0 \qquad \text{(A21)}$$

As income increases, adding up is immediately violated. The only way AIDS satisfies "regularity" is if all beta = 0 (or the betas somehow vary endogenously) so we are back to homothetic preferences with shares depending only on prices. Thus my assertion that this approach is not useful for performing valid counter-factuals, but may be useful for empirical estimation of the initial values of the α_i and β_i.

Appendix 3: Computer code in GAMS:

I have attached the programs for the first SGM model (no Armington differentiation). Two versions that produce identical results follow. The first is in MPC format (mixed complementarity problem), in which all weak inequalities are written out, and the model statement gives the complementary variable associated with each inequality. The MCP format is awkward to use in big models (and coding error prone), so the second version is in Rutherford's MPS/GE format (mathematical programming system for general equilibrium). This will be much more user friendly for experiences GAMS and GTAP data users, and might convince others to give it a try. Translated into set or indexed versions, MPS/GE can compactly code very large-dimension models with many countries, sectors, and factors.

Then the code for the basic CRIE models follow in mcp and mps/ge formats. Again, no Armington differentiation to keep things simple. As noted at the end of the concluding section, more version are available from myself.

```
$TITLE: SG-model 1 text version  Stone-Geary Modified model  GAMS MCP format
*  small open economy,  homogeneous goods,  no Armington differentiation

$ontext
Small open economy model with positive endowment of a good which is a
perfect substitute for good X1, denoted X1E, small endowment of X2E

  The benchmark accounting matrix:
             Production Sectors                  Trade      Welfare Consumers

  Markets|    X1    X2   X1P  X1E    X2P   X2E   TX21   TX12    W     CONS1
  ------------------------------------------------------------------------
    PX1   |  100        -100                       -0      0
    PX2   |       100              -100             0     -0

    PL    |  -25  -75                                                100
    PK    |  -75  -25                                                100

    PW    |                                                  351   -351
    PX1C  |              100   150                          -250
    PX1E  |                   -150                                    150
    PX2C  |                           100     1              -101
    PX2E  |                                  -1                         1
  ------------------------------------------------------------------------
$offtext


PARAMETERS
   SCALEX1  scales up or down the endowment X1E /1/
   SCALEX2  scales up or down the endowment X2E /1/
   ENDOW    scales up or down the endowment of L and K/1/
   TOT      world price PX1 over PX2 /1/;

NONNEGATIVE VARIABLES

    X1         Output of sector X1
    X2         Output of sector X2
    X1P        Output of X1P (produces PX1C from PX1)
    X1E        Output of X1E (produces PX1C from endowment PX1E)
    X2P        Output of X2P (produces PX2C from PX2)
    X2E        Output of X2E (produces PX2C from endowment PX2E)
    W          Produces welfare from PW1 and PX2
    TX21       Export X2 in eX1change for X1
    TX12       Export X1 in eX1change for X2

    PX1        Price of commoditX2 X1
    PX2        Price of commoditX2 X2  (used as numeraire here)
    PL         Price of primarX2 factor L
    PK         Price of primarX2 factor K
    PX1C       Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
    PX2C       Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
    PW         Price of welfare
    PX1E       Price of the endowment good - perfect substitute for X1 cannot be traded
    PX2E       Price of the endowment good - perfect substitute for X2 cannot be traded

    CONS       Consumer;

EQUATIONS

    COSTX1      Marginal cost - price X1
    COSTX2      Marginal cost - price X2
    COSTX1P     Cost of producing X1 consumption from X1 production
    COSTX1E     Cost of producing X1 consumption from X1E endowment
    COSTX2P     Cost of producing X2 consumption from X2 production
    COSTX2E     Cost of producing X2 consumption from X2E endowment
    COSTW       Cost of producing a unit of welfare
    COSTTX21    Cost of exporting X2 in exchange for X1
    COSTTX12    Cost of exporting X1 in exchange for X2

    MKTPX1      Supply-demand for X1
    MKTPX2      Supply-demand for X2
    MKTPL       Supply-demand for labor
    MKTPK       Supply-demand for capital
    MKTPX1C     Market for X1 consumption of good 1
    MKTPX2C     Market for X2 consumption of good 2
    MKTPW       Supply demand for welfare
    MKTPX1E     Supply-demand for endowment good 1
    MKTPX2E     Supply demand for endowment good 2

    INC         Consumer income balance;
```

```
COSTX1..   PL**0.25*PK**(0.75) =G= PX1;
COSTX2..   PL**0.75*PK**(0.25) =G= PX2;
COSTX1P..  PX1 =G= PX1C;
COSTX1E..  PX1E =G= PX1C;
COSTX2P..  PX2 =G= PX2C;
COSTX2E..  PX2E =G= PX2C;

COSTW..    PX1C**(250/351)*PX2C**(101/351) =G= PW;

COSTTX21.. PX2*TOT =G= PX1;
COSTTX12.. PX1*1.001 =G= PX2*TOT;

MKTPX1..   X1 =G= X1P + TX12 - TX21/TOT;
MKTPX2..   X2 =G= X2P + TX21 - TX12*TOT/1.001;

MKTPL..    100*ENDOW*PL =G= 0.25*(PL**0.25*PK**(0.75))*100*X1 +
                          0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..    100*ENDOW*PK =G= 0.75*(PL**0.25*PK**(0.75))*100*X1 +
                          0.25*(PL**0.75*PK**(0.25))*100*X2;

MKTPX1C..  100*(X1P + X1E)*PX1C =G= (250/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;
MKTPX2C..  100*(X2P + X2E)*PX2C =G= (101/351)*(PX1C**(250/351)*PX2C**(101/351))*W*351;

MKTPW..    W*351*(PX1C**(250/351)*PX2C**(101/351)) =G= CONS*351;

MKTPX1E..  1.5*SCALEX1 =G= X1E;
MKTPX2E..  (1/100)*SCALEX2 =G= X2E;

INC..      CONS*351 =G= PL*100*ENDOW + PK*100*ENDOW + PX1E*150*SCALEX1 + PX2E*1*SCALEX2;

MODEL SG1 /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX1E.X1E, COSTX2P.X2P, COSTX2E.X2E,
           COSTW.W, COSTTX21.TX21, COSTTX12.TX12,
           MKTPX1.PX1, MKTPX2.PX2, MKTPL.PL, MKTPK.PK, MKTPX1C.PX1C, MKTPX2C.PX2C,
           MKTPW.PW, MKTPX1E.PX1E, MKTPX2E.PX2E, INC.CONS/;

PL.L = 1; PK.L = 1; PX1C.L = 1; PX2C.L = 1; PX1.L = 1; PX2.L = 1; PX1E.L = 1; PX2E.L = 1;
PW.L = 1; X1.L = 1; X1P.L = 1; X1E.L = 1.5; X2.L = 1; X2P.L = 1; X2E.L = 1/100; W.L = 1;
PX2.FX = 1; CONS.L = 1;

OPTION ITERLIM = 0;
SOLVE SG1 USING MCP;

OPTION ITERLIM = 1000;
SOLVE SG1 USING MCP;


SETS I /I1*I25/;

PARAMETERS RESULTS1(I, *) economX2 size, RESULTS2(I, *) terms of trade,
           X10, X20, M, M0, PX10;

X10 = 0; X20 = 0; M0 = PL.L*ENDOW +PK.L*ENDOW;
SCALEX1 = 1; SCALEX2 = 1; TOT = 1/1.25;

SOLVE SG1 USING MCP;

LOOP(I,

ENDOW = 0.03*ORD(I) + 0.045;

SOLVE SG1 USING MCP;

M = PL.L*ENDOW +PK.L*ENDOW;

RESULTS1(I, "ENDOW") = ENDOW;
RESULTS1(I, "X1") = MAX(EPS, PX1.L*X1.L/M);
RESULTS1(I, "TX21") = PX1.L*TX21.L/TOT/M;
RESULTS1(I, "TX12") = PX1.L*TX12.L*1.001/M;
RESULTS1(I, "SHX1") = PX1.L*X1P.L/M;
RESULTS1(I, "INCELASX1")$(X10 GT 0) = ((X1P.L-X10)/X10)/((M-M0)/M0);
RESULTS1(I, "INCELASX2")$(X20 GT 0) = ((X2P.L-X20)/X20)/((M-M0)/M0);

X10 = X1P.L; X20 = X2P.L; M0 = PL.L*ENDOW +PK.L*ENDOW;

);
DISPLAY RESULTS1;
```

```
$TITLE: SG-model 1 text version  Stone-Geary Modified model  GAMS  MPS/GE format
*  small open economy,  homogeneous goods,  no Armington differentation
*  MPS/GE format for GTAP users

$ontext
Small open economX2 model with positive endowment of a good which is a
perfect substitute for good X1, denoted X1E. Small endowment of X2E

  The benchmark accounting matrix:
             Production Sectors                  Trade      Welfare Consumers

  Markets|    X1    X2   X1P  X1E    X2P   X2E   TX21   TX12    W     CONS1
  ------------------------------------------------------------------------
    PX1   |  100        -100                     -0      0
    PX2   |       100              -00             0     -0

    PL    |  -25  -75                                                100
    PK    |  -75  -25                                                100

    PW    |                                                    351   -351
    PX1C  |              100  150                             -250
    PX1E  |                  -150                                    150
    PX2C  |                          100     1                -101
    PX2E  |                                 -1                         1
  ------------------------------------------------------------------------
$offtext

PARAMETERS
   SCALEX1  scales up or down the endowment X1E /1/
   SCALEX2  scales up or down the endowment X1E /1/
   ENDOW   scales up or down the endowment of L and K/1/
   TOT     world price PX1 over PX2 /1/;

$ontext

$MODEL:SOESG

$SECTORS:
    X1       ! Output of sector X1
    X2       ! Output of sector X2
    X1P      ! Output of X1P (produces PX1C from PX1)
    X1E      ! Output of X1E (produces PX1C from endowment PX1E)
    X2P      ! Output of X2P (produces PX2C from PX2)
    X2E      ! Output of X2E (produces PX2C from endowment PX2E)
    W        ! Produces welfare from PW1 and PX2
    TX21     ! EX1port X2 in eX1change for X1
    TX12     ! EX1port X1 in eX1change for X2

$COMMODITIES:
    PX1      ! Price of commoditX2 X1
    PX2      ! Price of commoditX2 X2  (used as numeraire here)
    PL       ! Price of primarX2 factor L
    PK       ! Price of primarX2 factor K
    PX1C     ! Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
    PX2C     ! Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
    PW       ! Price of welfare
    PX1E     ! Price of the endowment good - perfect substitute for X1 cannot be traded
    PX2E     ! Price of the endowment good - perfect substitute for X2 cannot be traded

$CONSUMERS:
    CONS   ! Consumer

$PROD:X1  s:1
    O:PX1     Q:100
    I:PL      Q: 25
    I:PK      Q: 75

$PROD:X2   s:1
    O:PX2     Q:100
    I:PL      Q: 75
    I:PK      Q: 25

$PROD:TX21
    O:PX1     Q:(100*TOT)
    I:PX2     Q:100

$PROD:TX12
    O:PX2     Q:100
    I:PX1     Q:(100.1*TOT*1.)
```

```
$PROD:X1P
    O:PX1C   Q:100
    I:PX1    Q:100

$PROD:X1E
    O:PX1C   Q:150
    I:PX1E   Q:150

$PROD:X2P
    O:PX2C   Q:100
    I:PX2    Q:100

$PROD:X2E
    O:PX2C   Q: 1
    I:PX2E   Q: 1

$PROD:W  s:1.0
   O:PW     Q:351
   I:PX1C   Q:250
   I:PX2C   Q:101

$DEMAND:CONS
   D:PW     Q:351
   E:PL     Q:(100*ENDOW)
   E:PK     Q:(100*ENDOW)
   E:PX1E   Q:(150*SCALEX1)
   E:PX2E   Q:(1*SCALEX2)

$OFFTEXT
$SYSINCLUDE mpsgeset SOESG

PX2.FX = 1;

*       Benchmark replication
TX21.L = 0; TX12.L = 0;
OPTION ITERLIM = 0;
$INCLUDE SOESG.GEN
SOLVE SOESG USING MCP;

OPTION ITERLIM = 1000;
$INCLUDE SOESG.GEN
SOLVE SOESG USING MCP;


SETS I /I1*I25/;

PARAMETERS RESULTS1(I, *) economX2 size, RESULTS2(I, *) terms of trade,
           X10, X20, M, M0, PX10;

X10 = 0; X20 = 0; M0 = PL.L*ENDOW*100 +PK.L*ENDOW*100;
SCALEX1 = 1; SCALEX2 = 1; TOT = 1.25;
$INCLUDE SOESG.GEN
SOLVE SOESG USING MCP;

LOOP(I,

ENDOW = 0.03*ORD(I) + 0.045;
SCALEX2 = 1;

$INCLUDE SOESG.GEN
SOLVE SOESG USING MCP;

M = PL.L*ENDOW*100 +PK.L*ENDOW*100;

RESULTS1(I, "ENDOW") = ENDOW;
RESULTS1(I, "X1") = MAX(EPS, PX1.L*X1.L*100/M);
RESULTS1(I, "TX21") = PX1.L*TX21.L*125/M;
RESULTS1(I, "TX12") = PX1.L*TX12.L*125*1.*1.001/M;
RESULTS1(I, "SHX1") = PX1.L*X1P.L*100/M;
RESULTS1(I, "INCELASX1")$(X10 GT 0) = ((X1P.L-X10)/X10)/((M-M0)/M0);
RESULTS1(I, "INCELASX2")$(X20 GT 0) = ((X2P.L-X20)/X20)/((M-M0)/M0);

X10 = X1P.L; X20 = X2P.L; M0 = PL.L*ENDOW*100 +PK.L*ENDOW*100;

);
DISPLAY RESULTS1;

*$exit
```

```
ENDOW = 0.325; PX10 = 2;

LOOP(I,

TOT = 0.1*ORD(I) + 0.1;
SCALEX2 = 1;

$INCLUDE SOESG.GEN
SOLVE SOESG USING MCP;

M = PL.L*ENDOW*100 +PK.L*ENDOW*100;

RESULTS2(I, "TOT") = TOT;
RESULTS2(I, "X1") = MAX(EPS, PX1.L*X1.L*100/M);
RESULTS2(I, "TX21") = PX1.L*TX21.L*100*TOT/M;
RESULTS2(I, "TX12") = PX1.L*TX12.L*100*TOT*1.001/M;
RESULTS2(I, "SHX1") = PX1.L*X1P.L*100/M;
RESULTS2(I, "PELASX1")$(X10 GT 0) = ((X1P.L-X10)/X10)/((PX1.L-PX10)/PX10);

X10 = X1P.L; X20 = X2P.L; M0 = PL.L*ENDOW +PK.L*ENDOW; PX10 = PX1.L;

);
DISPLAY RESULTS1, RESULTS2;
```

```
$TITLE: CRIE IN MPS/GE USING FIXED FACTOR TRICK  GAMS  MCP format,   SIGMA = 2.5
*       c:\jim\optandsim\crie\CRIE-open-m-25.gms
*       THIS VERSON CALCULATES INCOME ELASTICITIES OF DEMAND

$onText
How to do CRIE+SG in mps/ge
   CRIE converted from standard CES by fixed factors R1 and R2

borrows a Markusen/Rutherford trick from the early 1990s - we used a fixed
factor to create concavity in the foreign offer curve LOE model back when
Two produced goods, X1 and X2, observed in the data
then think of two CRS utility goods Y1 and Y2 produced respectively
with X1 and a fixed factor R1, X2 and a fixed factor R2 so Y1 and Y2 have CRS

Y1 and Y2 are Cobb-Douglas in this model which is equivalent to the usual
CRIE function where the exponents on X1 and X2 are different but constants

utility is then a CRS CES function of Y1 and Y2
the different shares of X1/R1 and X2/R2 determine which good is more income elastic

    W  =  (Y1**rho + Y2**rho)**(1/rho)     rho = (sigma - 1)/sigma

    Y1  = ((X1)**alpha1x)*(R1**alpha1r)      alpha1x + alpha1r = 1
    Y2  = ((X2)**alpha2x)*(R2**alpha2r)      alpha2x + alpha2r = 1
        X1 = (L1**beta1l)*(K1**beta1k)       beta1l + beta1k = 1
        X2 = (L2**beta2l)*(K2**beta2k)       beta2l + beta2k = 1
        LBAR  =  L1 + L2   (LBAR = 100*SIZE)
        KBAR  =  K1 + K2   (KBAR = 100*SIZE)
        R1 and R2 specific factors in fixed supply

Income elasticities are calculated as the response of produced (and observed) goods
X1 and X2 to increases in the labor supply (observed income)

      The benchmark accounting matrix:
                  Production Sectors     Utility goods  Welfare  Consumers
----------------------------------------------------------------------
Markets
          X1      X1P     X2    X2P      Y1      Y2        W       CONS
  PX1     100    -100
  PXIC            100                  -100
  PX2                    100   -100
  PX2C                           100            -100
  PR1                                   -25                         25
  PR2                                            -75                75
  PY1                                   100              -125
  PY2                                            175     -175
  PW                                                      300      -300
  PL      -25             -75                                       100
  PK      -75             -25                                       100
----------------------------------------------------------------------
$offtext

Parameters
 SIGMA   elasticity of substitution between Y1 and Y2 /2.5/,
 SIZE    multiplier on endowment of L and K primary factors /1/,
 TOT     relative price p2 over p1 /1/;

NONNEGATIVE VARIABLES

   X1    produces X1 from L, K
   X2    produces X2 from L, K
   X1P   Output of X1P (produces PX1C from PX1)
   X2P   Output of X2P (produces PX2C from PX2)
   Y1    produces consumption good Y1 from X1 and R1
   Y2    produces consumption good Y2 from X2 and R2
   TX21  Export X2 in exchange for X1
   TX12  Export X1 in exchange for X2
   W     Produces welfare from PW1 and PX2

   PX1   price of X1
   PX2   price of X2
   PX1P  Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
   PX2P  Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
   PR1   price of endowment of R1
   PR2   price of endowment of R2
   PY1   price of consumption good Y1
   PY2   price of consumption good Y2
   PW    consumer price index
   PL    price of labor
   PK    price of capital
```

```
 CONS  !representative consumer's income

EQUATIONS

   COSTX1     Marginal cost - price X1
   COSTX2     Marginal cost - price X2
   COSTX1P    Cost of producing X1 consumption from X1 production
   COSTX2P    Cost of producing X2 consumption from X2 production
   COSTY1     Cost of producing X1 consumption from X1E endowment
   COSTY2     Cost of producing X2 consumption from X2E endowment
   COSTTX21   Cost of exporting X2 in exchange for X1
   COSTTX12   Cost of exporting X1 in exchange for X2
   COSTW      Cost of producing a unit of welfare

   MKTPX1     Supply-demand for X1
   MKTPX2     Supply-demand for X2
   MKTPX1P    Supply-demand for X1
   MKTPX2P    Supply-demand for X2
   MKTPR1
   MKTPR2
   MKTY1
   MKTY2
   MKTPW      Supply demand for welfare
   MKTPL      Supply-demand for labor
   MKTPK      Supply-demand for capital

   INC        Consumer income balance;

COSTX1..   PL**0.25*PK**0.75 =G= PX1;
COSTX2..   PL**0.75*PK**0.25 =G= PX2;
COSTX1P..  PX1 =G= PX1P;
COSTX2P..  PX2 =G= PX2P;

COSTY1..   PR1**(25/125)*PX1P**(100/125) =G= PY1;
COSTY2..   PR2**(75/175)*PX2P**(100/175) =G= PY2;

COSTTX21.. PX2*TOT =G= PX1;
COSTTX12.. PX1*1.001 =G= PX2*TOT;

COSTW..    ((5/12)*PY1**(1-SIGMA) + (7/12)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= PW;

MKTPX1..   X1 =G= X1P + TX12 - TX21/TOT;
MKTPX2..   X2 =G= X2P + TX21 - TX12*TOT/1.001;

MKTPX1P..  PX1P*25*X1P =G= (25/125)*PR1**(25/125)*PX1P**(100/125)*125*Y1;
MKTPX2P..  PX2P*75*X2P =G= (75/175)*PR2**(75/175)*PX2P**(100/175)*175*Y2;

MKTPR1..   PR1*25 =G=  (25/125)*PR1**(25/125)*PX1P**(100/125)*125*Y1;
MKTPR2..   PR2*75 =G=  (75/175)*PR2**(75/175)*PX2P**(100/175)*175*Y2;

MKTY1..    125*Y1 =G= (5/12)*PY1**(-SIGMA)*((5/12)*PY1**(1-SIGMA) +
(7/12)*PY2**(1-SIGMA))**(SIGMA/(1-SIGMA))*300*W;
MKTY2..    175*Y2 =G= (7/12)*PY2**(-SIGMA)*((5/12)*PY1**(1-SIGMA) +
(7/12)*PY2**(1-SIGMA))**(SIGMA/(1-SIGMA))*300*W;

MKTPW..    300*W*((5/12)*PY1**(1-SIGMA) + (7/12)*PY2**(1-SIGMA))**(1/(1-SIGMA)) =G= CONS;

MKTPL..    100*SIZE*PL =G= 0.25*(PL**0.25*PK**(0.75))*100*X1 +
                          0.75*(PL**0.75*PK**(0.25))*100*X2;
MKTPK..    100*SIZE*PK =G= 0.75*(PL**0.25*PK**(0.75))*100*X1 +
                          0.25*(PL**0.75*PK**(0.25))*100*X2;

INC..      CONS =G= PL*100*SIZE + PK*100*SIZE + PR1*25 + PR2*75;

MODEL CRIEMCP /COSTX1.X1, COSTX2.X2, COSTX1P.X1P, COSTX2P.X2P, COSTY1.Y1, COSTY2.Y2,
           COSTTX21.TX21, COSTTX12.TX12, COSTW.W,
           MKTPX1.PX1, MKTPX2.PX2,  MKTPX1P.PX1P, MKTPX2P.PX2P, MKTPR1.PR1, MKTPR2.PR2,
           MKTY1.PY1, MKTY2.PY2, MKTPW.PW, MKTPL.PL, MKTPK.PK,INC.CONS/;


PL.L = 1; PK.L = 1; PR1.L = 1; PR2.L = 1; PX1P.L = 1; PX2P.L = 1; PX1.L = 1; PX2.L = 1;
PW.L = 1; X1.L = 1; PY1.L = 1; PY2.L = 1;
X1.L = 1; X1P.L = 1;  X2.L = 1; X2P.L = 1; Y1.L = 1; Y2.L = 1; W.L = 1;
PX2.FX = 1; CONS.L = 300;

OPTION ITERLIM = 0;
SOLVE CRIEMCP USING MCP;

OPTION ITERLIM = 1000;
SOLVE CRIEMCP USING MCP;

SIZE = 1.3;
```

```
TOT = 1.2;
SOLVE CRIEMCP USING MCP;

*$EXIT

SETS I /I1*I50/;
PARAMETERS
 RESULTS1(I, *) L = SIZE  SIGMA = 2.5
 RESULTS2(I, *) INC and SHARES
 SIZE0, X10, X20, INCOME, INC0
 INCELAS1, INCELAS2, CONS0, Y10, Y20;

SIZE0 = 0.15; X10 = .1; X20 = .1; INC0 = 1;
INCELAS1 = 1; INCELAS2 = 1; CONS0 = .1;

TOT = 1.;

LOOP (I,

SIZE =  0.2 + 0.05*ORD(I);
SIGMA = 2.5;

SOLVE CRIEMCP USING MCP;

INCOME = PL.L*100*SIZE + PK.L*100*SIZE + PR1.L*25 + PR2.L*75;

INCELAS1 = (((X1P.L - X10)/X10)/((INCOME-INC0)/INC0))$(X10 GT 0);
INCELAS2 = (((X2P.L - X20)/X20)/((INCOME-INC0)/INC0))$(X20 GT 0);

RESULTS1(I, "L") = SIZE*100;
RESULTS1(I, "X1") = X1.L;
RESULTS1(I, "X2") = X2.L;
RESULTS1(I, "T12") = TX12.L;
RESULTS1(I, "T21") = TX21.L;
RESULTS1(I, "C1") = X1P.L;
RESULTS1(I, "C2") = X2P.L;
RESULTS1(I, "C1/C2") = (X1P.L/X2P.L)$(X2.L GT 0);
RESULTS1(I, "X1/X2") =  X1.L/X2.L;
RESULTS1(I, "INELAS1") = INCELAS1$(ORD(I) GT 1);
RESULTS1(I, "INELAS2") = INCELAS2$(ORD(I) GT 1);
RESULTS1(I, "RATIOINELAS") = (INCELAS1/INCELAS2)$(ORD(I) GT 1);

RESULTS2(I, "INCOME") = INCOME;
RESULTS2(I, "X1") = MAX(EPS, PX1.L*X1.L*100/INCOME);
RESULTS2(I, "TX21") = PX1.L*TX21.L*100*TOT/INCOME;
RESULTS2(I, "TX12") = PX1.L*TX12.L*100*TOT*1.*1.001/INCOME;
RESULTS2(I, "SHX1") = PX1.L*X1P.L*100/INCOME;

SIZE0 = SIZE;  X10 = X1P.L; X20 = X2P.L;
Y10 = Y1.L; Y20 = Y2.L; CONS0 = CONS.L; INC0 =  PL.L*100*SIZE + PK.L*100*SIZE + PR1.L*25 + PR2.L*75;
);

DISPLAY RESULTS1, RESULTS2;



$exit

Execute_Unload 'CRIE-open.gdx' RESULTS1
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS1 rng=SHEET1!A3:M55'

Execute_Unload 'CRIE-open.gdx' RESULTS2
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS2 rng=SHEET1!A56:F109'
```

```
$TITLE: CRIE IN MPS/GE USING FIXED FACTOR TRICK  GAMS  MPS/GE format   SIGMA = 2.5
*       c:\jim\optandsim\crie\CRIE-open-m-25.gms
*       THIS VERSON CALCULATES INCOME ELASTICITIES OF DEMAND

$onText
How to do CRIE+SG in mps/ge
   CRIE converted from standard CES by fixed factors R1 and R2

borrows a Markusen/Rutherford trick from the early 1990s - we used a fixed
factor to create concavity in the foreign offer curve LOE model back when
Two produced goods, X1 and X2, observed in the data
then think of two CRS utility goods Y1 and Y2 produced respectively
with X1 and a fixed factor R1, X2 and a fixed factor R2 so Y1 and Y2 have CRS

Y1 and Y2 are Cobb-Douglas in this model which is equivalent to the usual
CRIE function where the exponents on X1 and X2 are different but constants

utility is then a CRS CES function of Y1 and Y2
the different shares of X1/R1 and X2/R2 determine which good is more income elastic

    W  =  (Y1**rho + Y2**rho)**(1/rho)     rho = (sigma - 1)/sigma

    Y1  = ((X1)**alpha1x)*(R1**alpha1r)      alpha1x + alpha1r = 1
    Y2  = ((X2)**alpha2x)*(R2**alpha2r)      alpha2x + alpha2r = 1
        X1 = (L1**beta1l)*(K1**beta1k)       beta1l + beta1k = 1
        X2 = (L2**beta2l)*(K2**beta2k)       beta2l + beta2k = 1
        LBAR  =  L1 + L2   (LBAR = 100*SIZE)
        KBAR  =  K1 + K2   (KBAR = 100*SIZE)
        R1 and R2 specific factors in fixed supply

Income elasticities are calculated as the response of produced (and observed) goods
X1 and X2 to increases in the labor supply (observed income)


       The benchmark accounting matrix:
                  Production Sectors     Utility goods  Welfare  Consumers
---------------------------------------------------------------------
Markets
          X1      X1P     X2    X2P      Y1      Y2        W       CONS
  PX1     100    -100
  PXIC            100                  -100
  PX2                    100   -100
  PX2C                           100           -100
  PR1                                   -25                         25
  PR2                                           -75                 75
  PY1                                   100            -125
  PY2                                           175    -175
  PW                                                    300       -300
  PL      -25            -75                                       100
  PK      -75            -25                                       100
---------------------------------------------------------------------
$offText

Parameters
 SIGMA   elasticity of substitution between Y1 and Y2,
 SIZE    multiplier on endowment of L and K primary factors,
 TOT     relative price p2 over p1 /1/;

$ONTEXT
$MODEL: CRIEMPS

$SECTORS:
 X1   ! produces X1 from L, K
 X2   ! produces X2 from L, K
 X1P  ! Output of X1P (produces PX1C from PX1)
 X2P  ! Output of X2P (produces PX2C from PX2)
 Y1   ! produces consumption good Y1 from X1 and R1
 Y2   ! produces consumption good Y2 from X2 and R2
 TX21 ! Export X2 in exchange for X1
 TX12 ! Export X1 in exchange for X2
 W    ! Produces welfare from PW1 and PX2

$COMMODITIES:
 PX1  ! price of X1
 PX2  ! price of X2
 PX1C ! Price of the X1 and X1E perfect substitutes  X1C = X1 + X1E
 PX2C ! Price of the X2 and X2E perfect substitutes  X2C = X2 + X2E
 PR1  ! price of endowment of R1
 PR2  ! price of endowment of R2
 PY1  ! price of consumption good Y1
 PY2  ! price of consumption good Y2
 PW   ! consumer price index
```

```
 PL   ! price of labor
 PK   ! price of capital

$CONSUMERS:
 CONS  !representative consumer's income

$PROD:X1   s:1
 O:PX1   Q:100
 I:PL    Q:25
 I:PK    Q:75

$PROD:X1P
 O:PX1C  Q:100
 I:PX1   Q:100

$PROD:X2   s:1
 O:PX2   Q:100
 I:PL    Q:75
 I:PK    Q:25

$PROD:X2P
 O:PX2C  Q:100
 I:PX2   Q:100

$PROD:TX21
 O:PX1   Q:(100*TOT)
 I:PX2   Q:100

$PROD:TX12
 O:PX2   Q:100
 I:PX1   Q:(100.1*TOT)

$PROD:Y1     S:1.0
 O: PY1  Q:125
 I: PX1C Q:100
 I: PR1  Q:1   P:25

$PROD:Y2     S:1.0
 O: PY2  Q:175
 I: PX2C Q:100
 I: PR2  Q:1   P:75

$PROD:W     S:SIGMA
 O: PW   Q:300
 I: PY1  Q:125
 I: PY2  Q:175

$DEMAND: CONS
 D:PW    Q:300
 E:PL    Q:(100*SIZE)
 E:PK    Q:(100*SIZE)
 E:PR1   Q:1
 E:PR2   Q:1

$OFFTEXT
$SYSINCLUDE mpsgeset CRIEMPS

PX2.FX = 1;
SIGMA = 2.5;  SIZE = 1;
TOT = 1; TX21.L = 0; TX12.L = 0; X1.L = 1; X1P.L = 1;


*       Benchmark replication

OPTION ITERLIM = 0;
$INCLUDE CRIEMPS.GEN
SOLVE CRIEMPS USING MCP;

OPTION ITERLIM = 1000;
$INCLUDE CRIEMPS.GEN
SOLVE CRIEMPS USING MCP;

*$EXIT

SETS I /I1*I50/;
PARAMETERS
 RESULTS1(I, *) L = SIZE  SIGMA = 2.5
 RESULTS2(I, *) INC and SHARES
 SIZE0, X10, X20, INC, INC0
 INCELAS1, INCELAS2, CONS0, Y10, Y20;

SIZE0 = 0.15; X10 = .1; X20 = .1; INC0 = 1;
```

```
INCELAS1 = 1; INCELAS2 = 1; CONS0 = .1;

TOT = 1;

LOOP (I,

SIZE =  0.2 + 0.05*ORD(I);
SIGMA = 2.5;

$INCLUDE CRIEMPS.GEN
SOLVE CRIEMPS USING MCP;

INC = PL.L*SIZE*100 + PK.L*SIZE*100+ PR1.L + PR2.L;

INCELAS1 = (((X1P.L - X10)/X10)/((INC-INC0)/INC0))$(X10 GT 0);
INCELAS2 = (((X2P.L - X20)/X20)/((INC-INC0)/INC0))$(X20 GT 0);

RESULTS1(I, "L") = SIZE*100;
RESULTS1(I, "X1") = X1.L;
RESULTS1(I, "X2") = X2.L;
RESULTS1(I, "T12") = TX12.L;
RESULTS1(I, "T21") = TX21.L;
RESULTS1(I, "C1") = X1P.L;
RESULTS1(I, "C2") = X2P.L;
RESULTS1(I, "C1/C2") = (X1P.L/X2P.L)$(X2.L GT 0);
RESULTS1(I, "X1/X2") =  X1.L/X2.L;
RESULTS1(I, "INELAS1") = INCELAS1$(ORD(I) GT 1);
RESULTS1(I, "INELAS2") = INCELAS2$(ORD(I) GT 1);
RESULTS1(I, "RATIOINELAS") = (INCELAS1/INCELAS2)$(ORD(I) GT 1);

RESULTS2(I, "INCOME") = INC;
RESULTS2(I, "X1") = MAX(EPS, PX1.L*X1.L*100/INC);
RESULTS2(I, "TX21") = PX1.L*TX21.L*100*TOT/INC;
RESULTS2(I, "TX12") = - PX1.L*TX12.L*100*TOT*1.*1.001/INC;
RESULTS2(I, "SHX1") = PX1.L*X1P.L*100/INC;

SIZE0 = SIZE;  X10 = X1P.L; X20 = X2P.L;
Y10 = Y1.L; Y20 = Y2.L; CONS0 = CONS.L; INC0 = PL.L*SIZE*100 + PK.L*SIZE*100+ PR1.L + PR2.L;

);

DISPLAY RESULTS1, RESULTS2;



$exit

Execute_Unload 'CRIE-open.gdx' RESULTS1
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS1 rng=SHEET1!A3:M55'

Execute_Unload 'CRIE-open.gdx' RESULTS2
execute 'gdxxrw.exe CRIE-open.gdx par=RESULTS2 rng=SHEET1!A56:F109'
```