Scattering Theory
Lecture: 7

- Learn about structure of an object (building, person, molecule, atom, proton, ...) by scattering something (photon, electron, ...) off of it — this is "seeing"!

  As we'll see, to lowest order, in far field the scattered amplitude is a Fourier transform of the observed object (Born approximation).

- Want to study dynamics of a packet with $\langle p \rangle = \hbar k_0$ far away from target after "collision"/scattering with target.
  - expand initial state in eigenfnc's of $H = H_0 + V$ ← target

1d

$$t=0 \qquad \psi_k \xrightarrow[x\to-\infty]{} B_k e^{-ikx} + A_k e^{ikx}$$

← initial partial wave.

$$\xrightarrow[x\to\infty]{} C_k e^{ikx} + 0\, e^{-ikx}$$

← initial condition b.c.

← scattered waves

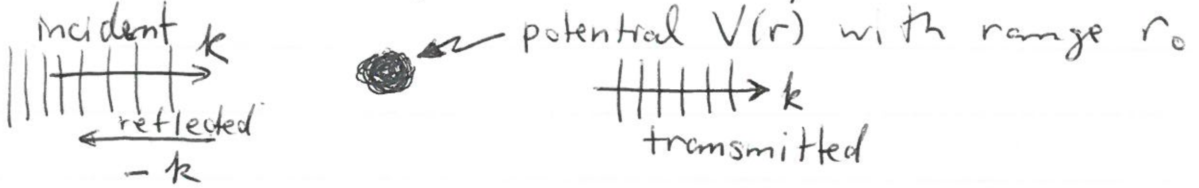
  - evolve in time to t via $e^{-\frac{i}{\hbar}E_k t}$
  - project on reflected and transmitted wavepackets to identify reflection (R) & transmission (T).
  - find that R, T not sensitive to packet shape $\Rightarrow$ (in wide packet, well defined p limit)

$\Rightarrow$ Just take one eigenfnc ψ_{k_0} & determine R, T from ratios of transmitted & <u>reflected currents</u> to <u>incident current</u>: (see e.g. Shankar Ch. 5),

$$\frac{J_R}{J_I} = R = \left|\frac{B_0}{A_0}\right|^2, \quad T = \left|\frac{C_0}{A_0}\right|^2 \frac{k_1}{k_0} = 1 - R = \frac{J_T}{J_R}.$$

"Short-cut":

- consider <u>plane wave</u> (incident) $e^{ik_0 x}$ & unnormalized eigenstate with this b.c. and study current ratios to get R, T.
  $\Rightarrow$ no dynamics for plane wave / infinitely wide wavepacket, steady state current.



$V(r) = V(-r)$

form does not take advantage of parity symmetry

$$\begin{cases} \psi(x) = e^{ikx} + r\, e^{-ikx}, & x \ll -r_0 \\ \quad\;\; = t\, e^{ikx}, & x \gg r_0 \end{cases}$$

$\Rightarrow T = |t|^2, \quad R = |r|^2 \quad ; \quad |r|^2 + |t|^2 = 1$ (unitarity)

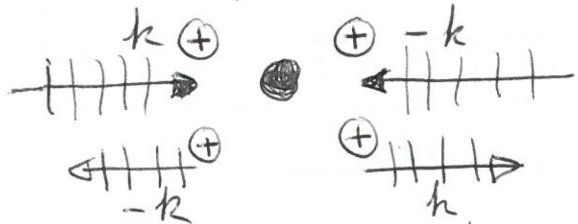
incoming form not symm. but scattered broken up into odd/even components

$$\begin{cases} \psi = e^{ikx} + f_S\, e^{-ikx} - f_A\, e^{-ikx}, & x \ll -r_0 \\ \quad = e^{ikx} + f_S\, e^{ikx} + f_A\, e^{ikx}, & x \gg r_0 \end{cases}$$

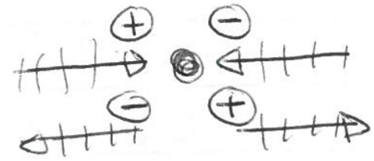
$f_{S,A}$ – symmetric/antisymmetric <u>scattering amplitudes</u>

Equivalently, in terms of <u>scattering matrix</u>:

$\boxed{S_s}$



$\boxed{S_A}$



$x \ll r_0$: $\psi = \frac{1}{2} e^{ikx} + \frac{1}{2} e^{ikx} + \frac{1}{2} e^{-ikx} - \frac{1}{2} e^{-ikx} + f_s e^{-ikx} - f_A e^{-ikx}$

$x \gg r_0$: $\quad = \frac{1}{2} e^{ikx} + \frac{1}{2} e^{ikx} + \frac{1}{2} e^{-ikx} - \frac{1}{2} e^{-ikx} + f_s e^{ikx} + f_A e^{ikx}$

$x \ll r_0$: $\psi = \frac{1}{2}\left[e^{ikx} + (2f_s+1) e^{-ikx}\right] + \frac{1}{2}\left[e^{ikx} - (2f_A+1) e^{-ikx}\right]$

$x \gg r_0$: $\quad = \frac{1}{2}\left[e^{-ikx} + \underbrace{(2f_s+1)}_{S_s} e^{ikx}\right] + \frac{1}{2}\left[-e^{-ikx} + \underbrace{(2f_A+1)}_{S_A} e^{ikx}\right]$

Uniterity $|S|^2 = 1 \Rightarrow S_{s,A} = e^{i2\delta_{s,A}}$

$$\Rightarrow (2f^*+1)(2f+1) = 1$$

$$4|f|^2 + 4\,\mathrm{Re}\,f + 1 = 1$$

$$\Rightarrow -2f^*f = f + f^* \Rightarrow -2 = \frac{1}{f^*} + \frac{1}{f} = 2\,\mathrm{Re}\left(\frac{1}{f}\right)$$

$$\Rightarrow \boxed{\mathrm{Re}\left(\frac{1}{f}\right) = -1} \Rightarrow \frac{1}{f} = -1 + i\tilde{F}(k)$$

$$\Rightarrow \boxed{f = \frac{1}{i\tilde{F}(k) - 1}}$$

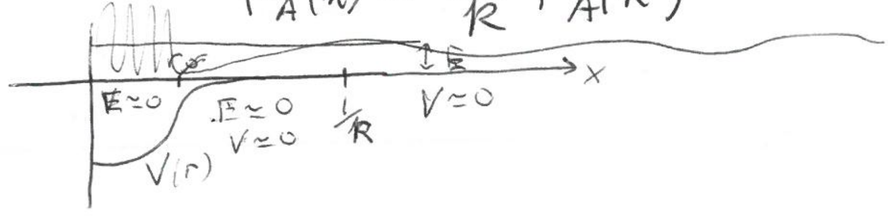
Analysis of S.Eqn:

$\tilde{F}_s(k) = k\,F_s(k^2)$

$\tilde{F}_A(k) = \frac{1}{k} F_A(k^2)$

$$\Rightarrow f_s = \frac{1}{ik\,F_s(k^2) - 1}$$

$$f_A = \frac{1}{ik^{-1} F_A(k^2) - 1}$$



Three-Dimensional Scattering

A. Generalities of partial-wave expansion:

- <u>Scattering amplitude</u>:

Away from potential $r \gg r_0$, free S.Eqn.

$$\Rightarrow \quad \psi_k = e^{\underbrace{i\vec{k}\cdot\vec{r}}_{kz}} + \psi_{sc}(r,\theta,\varphi)$$

$$= e^{ikz} + \sum_{\ell,m}\left(A_\ell j_\ell(kr) + B_\ell n_\ell(kr)\right) Y_{\ell m}(\theta,\varphi)$$

spherical Bessel & Neumann fns.

<u>Use</u>:

$$j_\ell(kr) \xrightarrow[r\to\infty]{kr \gg \ell} \frac{\sin(kr - \frac{\pi}{2}\ell)}{kr} \; ; \quad n_\ell(kr) \xrightarrow[r\to\infty]{} -\frac{\cos(kr-\frac{\pi}{2}\ell)}{kr}$$

$$\underbrace{\xrightarrow[\substack{r\to 0\\ kr \ll \ell}]{} \frac{(kr)^\ell}{(2\ell+1)!!}}_{\text{regular at origin}} \; ; \quad \underbrace{\xrightarrow[\substack{r\to 0\\ kr\ll\ell}]{} -\frac{(2\ell-1)!!}{(kr)^{\ell+1}}}_{\text{irregular at origin.}}$$

$$j_0(\rho) = \frac{\sin\rho}{\rho} \qquad n_0(\rho) = -\frac{\cos\rho}{\rho}$$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho} \qquad n_1(\rho) = -\frac{\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$$

$$\vdots \qquad\qquad \vdots$$

<u>Boundary</u> <u>conditions</u>: $\psi_{sc}(r) \propto$ outgoing wave $\propto \frac{e^{ikr}}{r}$

$$\Rightarrow A_\ell / B_\ell = -i$$

$$\psi \underset{r\to\infty}{=} e^{ikz} + \frac{e^{ikr}}{r}\underbrace{\left(\frac{1}{k}\sum_{\ell,m}(-i)^\ell(-B_\ell)Y_{\ell m}(\theta,\varphi)\right)}_{\equiv f(\theta,\varphi)\text{ - scattering amplitude.}}$$

- <u>particle current</u> $\vec{j} = \vec{j}_{inc} + \vec{j}_{sc}$.

$$\vec{j} = \frac{1}{2}\left[\psi^*\left(-\frac{i\hbar}{m}\nabla\psi\right) + \psi\left(\frac{i\hbar}{m}\nabla\psi^*\right)\right]$$

$$= \frac{\hbar}{2mi}\left[e^{-ikz}\nabla e^{ikz} - e^{ikz}\nabla e^{-ikz}\right]$$

$$+ \frac{\hbar}{2mi}\left[\psi_{sc}^*\nabla\psi_{sc} - \psi_{sc}\nabla\psi_{sc}^*\right]$$

$\left(\vec{\nabla} = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\varphi}\frac{1}{r\sin\theta}\partial_\varphi\right)$

$$\vec{j} \simeq \underbrace{\frac{\hbar\vec{k}}{m}}_{j_{inc}} + \underbrace{\frac{\hbar k}{m}\frac{\hat{r}}{r^2}|f(\theta,\varphi)|^2}_{j_{sc}} + \text{fast-oscillating terms for } r\to\infty$$

look away from $\vec{k}$, i.e. $\theta \neq 0$

- <u>Differential cross section</u>: then $\vec{j} \simeq \vec{j}_{sc}$

Probability flow into $d\Omega$ at rate:

$$R(d\Omega) = \vec{j}_{sc}\cdot d\vec{A} = \vec{j}_{sc}\cdot\hat{r}\,r^2 d\Omega$$

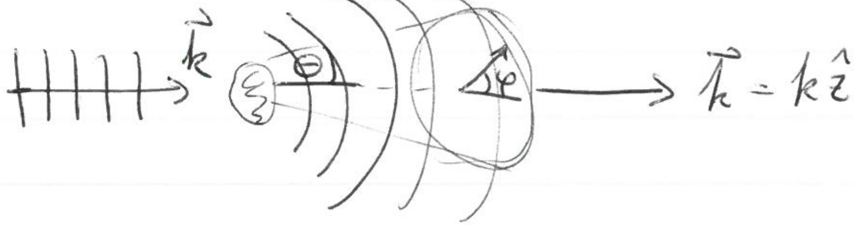
$$R(d\Omega) = \frac{\hbar k}{m}|f|^2 d\Omega$$

$$\Rightarrow \frac{d\sigma}{d\Omega}d\Omega = \frac{R(d\Omega)}{j_{inc}} = |f|^2 d\Omega$$

$$\Rightarrow \underline{\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2}$$

- <u>Spherical potential $V(\vec{r}) = V(r)$</u> (rot. inv. $\vec{L}$ conserved)

$\Rightarrow f(\theta)$ — φ independent (about $\vec{k}$)



$\rightarrow V(r) = 0$:

$$\psi_k^0 = \underset{\text{plane-wave}}{e^{ikz}} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} (i)^\ell (2\ell+1) \underset{\text{spherical waves expansion.}}{j_\ell(kr)} P_\ell(\cos\theta)$$

$$\underset{\substack{kr \gg \ell \\ r\to\infty}}{\simeq} \sum_{\ell=0}^{\infty} e^{i\frac{\pi}{2}\ell}(2\ell+1)\left[\frac{e^{i(kr-\frac{\pi}{2}\ell)}}{2ikr} - \frac{e^{-i(kr-\frac{\pi}{2}\ell)}}{2ikr}\right] P_\ell(\cos\theta)$$

$$\psi_k^0 \underset{r\to\infty}{=} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik}\left[\underbrace{\underset{\text{outgoing}}{\frac{e^{ikr}}{r}} - \underset{\text{incoming}}{\frac{e^{-ikr}}{r}}}_{\text{equal fluxes by unitarity}} \underset{\text{phase difference due to centrif. barrier}}{(-1)^\ell}\right] P_\ell(\cos\theta)$$

$$\psi_k^{0(\ell)} \underset{r\to\infty}{\sim} \sin\left(kr - \tfrac{\pi}{2}\ell\right)/r \quad P_\ell(\cos\theta)$$

→ $V(r) \neq 0$: just a phase shift for $r \to \infty$

$$\psi_k^{(\ell)} \sim A_\ell \frac{\sin(kr - \frac{\pi}{2}\ell + \delta_\ell(k))}{r} P_\ell(\cos\theta)$$

$$\psi_k(\vec{r}) \underset{r\to\infty}{=} \sum_{\ell=0}^{\infty} A_\ell \frac{e^{i(kr-\frac{\pi}{2}\ell+\delta_\ell)} - e^{-i(kr-\frac{\pi}{2}\ell+\delta_\ell)}}{r} P_\ell(\cos\theta)$$

require form: $= \underbrace{e^{ikz}}_{\text{incoming \& outgoing.}} + f(\theta) \underbrace{\frac{e^{ikr}}{r}}_{\text{only outgoing}}$

Choose A_ℓ s.t. incoming in $\psi_k(\vec{r})$ gives incoming of $e^{ikz} = \sum_\ell \frac{(2\ell+1)}{2ik}\left[-\frac{e^{-ikr}}{r} e^{+i\pi\ell} + \dots\right]$

$$\Rightarrow \boxed{A_\ell = \frac{2\ell+1}{2ik} e^{i(\frac{\pi}{2}\ell+\delta_\ell)}}$$

$$\Rightarrow \psi_k(\vec{r}) \underset{r\to\infty}{=} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik}\left[\frac{e^{ikr}}{r} e^{i2\delta_\ell} - \frac{e^{-i(kr-\pi\ell)}}{r}\right] P_\ell(\cos\theta)$$

$$= e^{ikz} + \underbrace{\left(\sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{i2\delta_\ell}-1}{2ik}\right)P_\ell(\cos\theta)\right)}_{\equiv f(\theta)} \frac{e^{ikr}}{r}$$

$$\boxed{f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\, \underbrace{f_\ell(k)}_{\text{partial wave } \ell \text{ scattering amplitude.}}\, P_\ell(\cos\theta)}$$

- **Partial wave scattering amplitude**

$$f_\ell(k) = \frac{e^{i2\delta_\ell} - 1}{2ik} \equiv \frac{S_\ell - 1}{2ik}$$

$$f_\ell(k) = \frac{e^{i\delta_\ell}}{k} \sin\delta_\ell$$

- **Scattering matrix**

$$S = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} U(t_f, t_i)$$

For rot. invnt. potential, each partial wave ℓ scatters independently

$$-\frac{e^{-i(kr - \pi\ell)}}{r} + \underbrace{e^{i2\delta_\ell}}_{\equiv S_\ell} \frac{e^{ikr}}{r}$$

Note: as in $V(r) = 0$ case ratio of incoming to outgoing amplitudes is $|S_\ell| = 1$, simply by particle conservation (unitarity)

$$\Rightarrow S_\ell = \frac{e^{i\delta_\ell}}{e^{-i\delta_\ell}} = \frac{1 + i\tan\delta_\ell}{1 - i\tan\delta_\ell}$$

$$\Rightarrow f_\ell = \frac{1 + i\tan\delta_\ell - 1 + i\tan\delta_\ell}{1 - i\tan\delta_\ell} \frac{1}{2ik}$$

$$\boxed{f_\ell = \frac{1}{\frac{k}{\tan\delta_\ell} - ik}}$$

- **Cross section in partial waves & σ_ℓ**

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \, |f(\theta)|^2$$

$$= 2\pi \int_{-1}^{1} \underline{d(\cos\theta)} \sum_{\ell,\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) \, f_\ell^*(k) f_{\ell'}(k) \underbrace{P_\ell(\cos\theta) P_{\ell'}(\cos\theta)}_{\delta_{\ell,\ell'} \frac{2}{2\ell+1}}$$

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2\delta_\ell} \equiv \sum_\ell \sigma_\ell$$

unitarity bound on $\sigma_\ell \leq \sigma_\ell^{max} = (2\ell+1) \frac{4\pi}{k^2}$

saturated by $\delta_\ell = \frac{\pi}{2} n$, n odd $\mathbb{Z}$.

- **Optical Thm:**

$$\sigma = \frac{4\pi}{k} \operatorname{Im} f(0)$$

$\left(P_\ell(\cos\theta) = 1 \text{ for } \theta = 0 \right)$

total scattered of probability $\propto$ decrease in current density behind target $(\theta = 0)$

- **Finding $\delta_\ell \Rightarrow f_\ell, S_\ell, \sigma, \ldots$**

Must solve Schrodinger's Eqn to determine asymptotic $\psi_\ell(r)$

$$\psi_\ell(r) \underset{r \gg r_0}{\propto} B_\ell\, j_\ell(kr) + C_\ell\, n_\ell(kr)$$

$$\underset{kr \gg \ell}{=} B_\ell \frac{\sin(kr - \frac{\pi}{2}\ell)}{kr} - C_\ell \frac{\cos(kr - \frac{\pi}{2}\ell)}{kr}$$

$$\underset{\text{require:}}{=} A_\ell \frac{\sin(kr - \frac{\pi}{2}\ell + \delta_\ell)}{r}$$

$$\Rightarrow B_\ell = (kA_\ell)\cos\delta_\ell, \quad C_\ell = -(kA_\ell)\sin\delta_\ell$$

$$\Rightarrow \boxed{-\frac{C_\ell}{B_\ell} = \tan\delta_\ell}$$

(Recall: A_ℓ is then chosen so that $\psi \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$)

B. Calculations of $f(\theta)$, etc...

1. Born Approximation

(a) via Fermi's golden rule

Amplitude for starting in state $|\vec{p}_i\rangle$ and ending in state $|\vec{p}_f\rangle$ is

$$\langle \vec{p}_f | \hat{U}(t_f, t_i) | \vec{p}_i \rangle$$

scattering matrix $\hat{S} \underset{\substack{t_i \to -\infty \\ t_f \to +\infty}}{\equiv} \hat{U}(t_f, t_i)$

$$\Rightarrow P(\vec{p}_i \to d\Omega) = \sum_{p_f \text{ in } d\Omega} |\langle \vec{p}_f | S | p_f \rangle|^2$$

prob. of starting in state $|\vec{p}_i\rangle$ & ending in $|\vec{p}_f\rangle$ with $\vec{p}_f$ in cone at (θ, φ) size $d\Omega$

To 1^{st} order in $V(r) \rightarrow$ Fermi's golden rule:

$$R_{\vec{p}_f \to d\Omega} = \frac{dP(\vec{p}_i \to d\Omega)}{dt} = \frac{2\pi}{\hbar} \sum_{p_f \in \Omega} |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 \delta(E_f - E_i)$$

$$R_{\vec{p}_f \to d\Omega} = \frac{2\pi}{\hbar} L^3 \int \frac{d^3 p_f}{(2\pi\hbar)^3} \delta\Big(\underbrace{\frac{p_f^2}{2m} - \frac{p_i^2}{2m}}_{(p_f - p_i)\frac{(p_f + p_i)}{2m}}\Big) |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2$$

$$\left(\psi_p = \frac{1}{L^{3/2}} e^{i\vec{p}\cdot\vec{r}/\hbar}\right)$$

$$R_{\vec{p}_f \to d\Omega} = \frac{L^3}{\hbar^4 (2\pi)^2} \int dp_f \, p_f^2 \frac{m}{p_f} \delta(p_f - p_i) |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2$$

7.12

$$R_{\vec{p}_f \to d\Omega} = L^3 \frac{p_f m}{(2\pi)^2 \hbar^4} \left| \frac{1}{L^3} \underbrace{\int d^3r\, e^{-i\vec{q}\cdot\vec{r}} V(r)}_{\tilde{V}(\vec{q})} \right|^2 d\Omega$$

$$\hbar \vec{q} = p_f - p_i$$

$$\Rightarrow |\vec{q}| = 2k \sin \theta/2$$

$$j_{inc} = \frac{p_f}{m} L^{-3}$$

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{R_{\vec{p}_f \to d\Omega}}{j_{inc}} = \frac{m\, p_f}{(2\pi)^2 \hbar^4} \frac{m}{p_f} \frac{L^3}{L^{-3}} L^{-6} |\tilde{V}(\vec{q})|^2 d\Omega$$

$$\boxed{\Rightarrow \frac{d\sigma}{d\Omega} = \left| \frac{-m}{2\pi\hbar^2} \tilde{V}(\vec{q}) \right|^2 = |f(\theta,\varphi)|^2 \qquad f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2} \tilde{V}(\vec{q})}$$

<u>Born approximation.</u>

For $V(\vec{r}) = V(r)$ spher. symm.

$$f(\theta,\varphi) = \frac{-m}{2\pi\hbar^2} \int e^{-iqr\cos\theta} V(r)\, d(\cos\theta)\, d\phi\, r^2 dr$$

$$= -\frac{2m}{\hbar^2} \int \frac{\sin qr}{q} r\, dr$$

$$= f(\theta) \quad , \varphi \text{ independent}$$

Ex. Yukawa potential: $V(r) = g\,\dfrac{e^{-\kappa r}}{r}$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\, g \int_0^\infty \frac{e^{iqr} - e^{-iqr}}{2i}\, e^{-\kappa r}\, dr$$

$$f(\theta) = \frac{-2mg}{\hbar^2(\kappa^2+q^2)}\ , \quad q = 2k\sin\theta/2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{4m^2 g^2}{\hbar^4(\kappa^2 + 4k^2\sin^2\theta/2)^2}$$

For Coulomb potential scattering $\kappa \to 0$

$g = Ze^2$

$$\Rightarrow \quad \left.\frac{d\sigma}{d\Omega}\right|_{\text{Coulomb}} = \frac{(Ze^2)^2}{16E^2\sin^4\theta/2}$$ Rutherford cross section.

(exact q.m. & classically, Rutherford)

Subtleties with Coulomb potential $\frac{1}{r}$ since vanishes too slowly with $r\to\infty$ (need faster than $1/r$) to have $\psi_k \underset{r\to\infty}{\sim} e^{ikz} + f(\theta,\varphi)\dfrac{e^{ikr}}{r}$.

Note for example: $\sigma = \int d\Omega \dfrac{d\sigma}{d\Omega} \propto \int \dfrac{d\theta}{\theta^3} \to \infty$

$$\frac{e^{ikr}}{r} \xrightarrow[\text{for Coulomb.}]{} \frac{e^{i(kr - \gamma\ln kr)}}{r}\ , \quad \gamma = \frac{Ze^2 m}{\hbar^2 k}$$

In general:

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(r)$$

Low E

at low E, $q \to 0$

$$\Rightarrow \underset{\text{low E}}{f(\theta)} \simeq -\frac{\mu V_0 r_0^3}{\hbar^2}$$

High E $\quad e^{-iqr\cos\theta}$ oscillates as $q \to \infty$.

$\Rightarrow$ averages to 0

$\Rightarrow f(\theta) \simeq 0$ for $\theta \gtrsim \frac{\pi}{kr_0}$ except for $q\underbrace{r\cos\theta}_{\simeq r_0} \lesssim \pi$

$\Rightarrow \theta \simeq \frac{\pi}{kr_0}$

$\Rightarrow$ Expect validity of Born approx. at high E s.t. $f(\theta)$ is small.

(b) Via Green's fnc:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi_k = \frac{\hbar^2 k^2}{2m}\psi_k$$

$$\Rightarrow (\nabla^2 + k^2)\psi_k = \frac{2m}{\hbar^2} V(r)\psi_k(r)$$

Solve via Green's fnc: integral eqn.:

$$\psi_k(r) = \underbrace{\psi_k^0}_{\text{boundary conditions}} + \frac{2m}{\hbar^2}\int d^3r'\, G^0(r,r')V(r')\psi_k(r')$$

where $(\nabla^2 + k^2)\psi_k^0 = 0$ & $(\nabla^2 + k^2)G^0(r,r') = \delta^3(r-r')$

(0) <u>for</u> $V(r) = 0 \Rightarrow \psi_k(r) = \psi_k^0(r) = e^{ik\cdot r}$

$$\Rightarrow \psi_k(r) = \underset{\substack{\Uparrow \\ \text{incident} \\ \text{wave}}}{e^{ik\cdot r}} + \underbrace{\frac{2m}{\hbar^2}\int G^0(r,r')V(r')\psi_k(r')\,d^3r}_{\text{scattered wave.}}$$

Iterate in powers of $V(r)$: Dyson's Eqn.

(0) $\psi_k(r) = \psi_k^0 \quad , \quad V(r) = 0$

(1) $\psi_k(r) = \psi_k^0 + \frac{2m}{\hbar^2}\int_{r'} G^0(r,r')V(r')\psi_k^0(r')$

(2) $\psi_k(r) = \psi_k^0 + \frac{2m}{\hbar^2}\int_{r'} G^0(r,r')V(r')\psi_k^0(r') +$

$$+ \frac{2m}{\hbar^2}\int_{r'r''} G^0(r,r')V(r')G^0(r',r'')V(r'')\psi^0(r'')$$

t - matrix

$$+ \mathcal{O}(V^3)$$

(N) $V\psi_k \equiv T\psi_k^0 = V\left(1 + \left(\frac{2m}{\hbar^2}\right)(G^0V) + \left(\frac{2m}{\hbar^2}\right)^2 G^0V(G^0V)\right.$

$$\left. + \left(\frac{2m}{\hbar^2}\right)^3(G^0V)(G^0V)(G^0V) + \dots\right)\psi_k^0 = \frac{V}{1-\frac{2m}{\hbar^2}G^0V}\psi_k^0$$

$$\Longrightarrow \;=\; \longrightarrow \;+\; \rightarrow\!\times\!\rightarrow \;+\; -\!\times\!-\!\times\!- \;+\; -\!\times\!-\!\times\!-\!\times\!-$$

$$T_k = -\frac{2\pi\hbar^2}{m}f$$

<u>Born Approx: stop at 1^{st} order in V</u>

$$\psi_k(r) = e^{ik\cdot r} + \frac{2m}{\hbar^2}\int_{r'} G^0(r,r')V(r')e^{ik\cdot r'}$$

$G^\circ(r-r') = ?$

$$(\nabla^2 + k^2) G^\circ(r) = \delta^3(r)$$

<u>soln:</u>

(i) $G^\circ(r) = \frac{u(r)}{r} \Rightarrow u'' + k^2 u = 0$, for $r \neq 0$

$$\Rightarrow G^\circ(r) = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}$$

b.c. $B = 0$

$$(\nabla^2 + k^2)\left(\frac{A}{r} e^{ikr}\right) = \delta^3(r)$$

$$\Rightarrow A = -\frac{1}{4\pi}$$

$$\Rightarrow G^\circ(r) = -\frac{1}{4\pi r} e^{ikr}$$

<u>Note:</u> consistent with $k \to 0$ limit

Coulomb's law $\nabla^2 G^\circ = \delta^3(r) \Rightarrow G^\circ = -\frac{1}{4\pi r}$

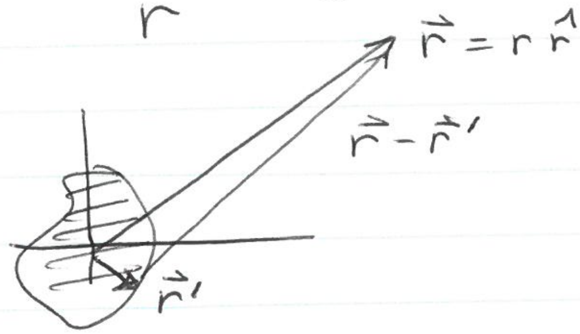
$$\Rightarrow \psi_k(r) = e^{ik\cdot r} - \frac{2m}{4\pi\hbar^2}\int d^3r' \frac{e^{ik|\vec{r}-\vec{r}'|}}{|r-r'|} V(\vec{r}') e^{ik\cdot r'}$$

Look at detector at $r >> r_0$ - extent of $V(r)$

$$|\vec{r}-\vec{r}'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2} = r\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}\right]^{1/2}$$

$$\simeq r\left(1 - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

$$\Rightarrow \frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|} \simeq \frac{e^{ikr}}{r}\, e^{-i\underbrace{(k\hat r)}_{\equiv \vec k_f}\cdot \vec r'}$$



Note: $\underline{\vec k_f = k\hat r}$

$$\Rightarrow \psi_k(r) \xrightarrow[r\to\infty]{} e^{ik\cdot r} + \frac{e^{ikr}}{r}\underbrace{\left(\frac{-2m}{4\pi\hbar^2}\int_{r'} e^{-ik_f\cdot r'} V(r')\psi_k(r)\right)}_{\equiv f(\theta,\phi)}$$

<u>Born Approx</u>:

$$\Rightarrow f(\theta,\varphi) = -\frac{2m}{4\pi\hbar^2}\int_{\vec r'} e^{-i(\vec k_f-\vec k_i)\cdot\vec r} V(\vec r)$$

(ii) $(\nabla^2+k^2)\, G_k^0(r) = \delta^3(r)$

F.T. $(-q^2+k^2)\,\tilde G_k^0(q) = 1$

$$\Rightarrow \tilde G_k^0(q) = \frac{1}{k^2-q^2}$$

$$G_k^0(r) = \int \frac{d^3q}{(2\pi)^3}\, \frac{e^{i\vec q\cdot\vec r}}{k^2-q^2+i\varepsilon}$$

↑ added to ensure an outgoing wave G_0 only (b.c.)

<u>Why?</u> $\frac{1}{q^2-(k+i\varepsilon)^2}$

$= \oint \underline{e^{iqr}} \to e^{ikr-\varepsilon r}$

$$\frac{1}{q^2-(k+i\varepsilon)^2} = \frac{1}{q^2-k^2-i\underbrace{k2\varepsilon}_{\varepsilon_k}}$$

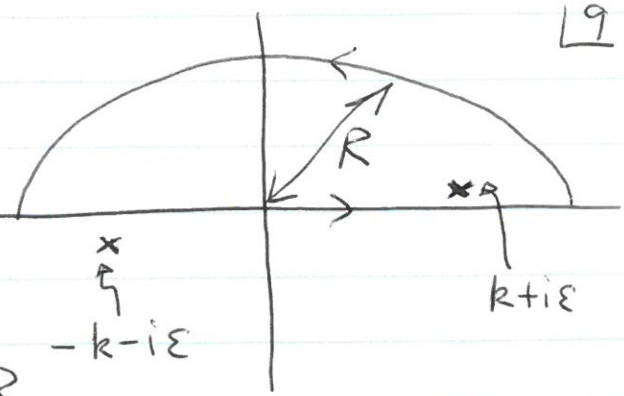
7.18

$$G_k^0(\vec{r}) = \frac{2\pi}{(2\pi)^3}\int_{-1}^{1} d\mu \int_0^\infty e^{iq\mu r} q^2 dq \frac{1}{(k-q+i\varepsilon)(k+q+i\varepsilon)}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dq\, q^2 \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{(k-q+i\varepsilon)(k+q+i\varepsilon)}$$

$$= \frac{-i}{4\pi^2 r}\int_{-\infty}^{\infty} dq \frac{q\, e^{iqr}}{(k-q+i\varepsilon)(k+q+i\varepsilon)}$$

- choice of $k+i\varepsilon$ ensures that contour is closed in upper-half plane so that $e^{iq\cdot r} = e^{i(k+i\varepsilon)r}$
  $= \left(\frac{e^{ikr}}{r}\right)e^{-\varepsilon r} \to 0 \quad r \to \underset{\infty}{R}$



- choice of $k-i\varepsilon$ selects instead
  $e^{iq\cdot r} = \left(\frac{e^{-ikr}}{r}\right)e^{-\varepsilon r}$ incoming wave.

$$G_k^0(r) = \frac{-i}{4\pi^2 r} 2\pi i \frac{e^{ikr-\varepsilon r}\cancel{(k+i\varepsilon)}}{-2\cancel{(k+i\varepsilon)}}$$

$$\boxed{G_k^0(r) = -\frac{e^{ikr}}{4\pi r}}$$ ✓

- **Validity of Born approx**:

$$\psi_k = e^{ik\cdot r} + \psi_{sc}$$

require $|\psi_{sc}| \ll |e^{ik\cdot r}| = 1$

look at origin where $\psi_{sc}(r)$ is largest.

$$|\psi_{sc}(0)| = \left| \frac{m}{2\pi\hbar^2} \int \frac{e^{ikr'}}{r} V(r')\, e^{-i\vec{k}_i\cdot\vec{r}'}\, d^3r' \right| \ll 1$$

$$\frac{m}{2\pi\hbar^2}\,\frac{2\pi\cdot 2}{k} \left| \int_0^\infty dr'\, e^{ikr'} \sin kr'\, V(r') \right| \ll 1$$

Low E:

$k \to 0 \Rightarrow$

$$\Rightarrow \quad \frac{2m}{\hbar^2} \left| \underbrace{\int_0^\infty dr'\, r'\, V(r')}_{} \right| \ll 1$$

to be convergent $V(r') \to 0$, as $r \to \infty$ faster than $\frac{1}{r'}$

$$\Rightarrow \quad \frac{2m}{\hbar^2} V_0 r_0^2 \ll 1$$

$$\boxed{V_0^{Low} \ll \frac{\hbar^2}{2m r_0^2}}$$

← must be shallow enough so as not allow a bound state.

High E: fast oscillations $\Rightarrow$

$$\frac{2m}{k\hbar^2} \left| \int dr'\, V(r') \right| \ll 1 \Rightarrow \boxed{\frac{2m V_0 r_0^2}{\hbar^2} \ll k r_0}$$

i.e. $V_0^{high} \ll \left(\frac{\hbar^2}{2m r_0^2}\right)(k r_0)$; Note: If good at low E → good at all E!

- **Back to Dyson's Eqn. : T-matrix.**

$$(\nabla^2 + k^2)\psi_k = \frac{2m}{\hbar^2} V \psi_k$$

$$\Rightarrow \psi_k(r) = \psi_k^0(r) + \frac{2m}{\hbar^2}\int_{r'} G_k^0(r-r') \underbrace{V(r')\psi_k(r')}$$

why? because this formally solves the prob. exactly:

$$\equiv T(r')\psi_k^0(r')$$

$\Uparrow$ T-matrix

$$\psi_k(r) = \psi_k^0(r) + \frac{2m}{\hbar^2}\int_{r'} G_k^0(r-r')\, T(r')\,\psi_k^0(r')$$

$$= e^{i\vec{k}\cdot\vec{r}} + \underbrace{\left(\frac{-m}{2\pi\hbar^2}\tilde{T}(\vec{q})\right)}\frac{e^{ikr}}{r}$$

$$\equiv \boxed{f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\tilde{T}(\vec{q})}$$

Note: as $k \to 0$, $f(\theta,\varphi) = -a \Rightarrow \boxed{\tilde{T}(0) \simeq \frac{2\pi\hbar^2}{m} a}$

How to find $T(r)$?

$$\psi = \left(1 + \left(\frac{2m}{\hbar^2}\right) G_0 V + \left(\frac{2m}{\hbar^2}\right)^2 G_0 V G_0 V + \dots\right)\psi_0$$

formally:

$$V\psi = \frac{V}{1 - G_0 V\left(\frac{2m}{\hbar^2}\right)}\psi_0 \equiv T\psi_0 \Rightarrow \boxed{T = \frac{V}{1 - G_0 V\left(\frac{2m}{\hbar^2}\right)}} \simeq V$$

to lowest order. (Born approx.)

$$T(r)\,\psi^{\circ}_{k}(r) = V(r)\,\psi_{k}(r) = V(r)\,\psi^{\circ}_{k}(r) +$$

$$+ \frac{2m}{\hbar^2} V(r) \int_{r_1} G^{\circ}_{k}(\vec{r}-\vec{r}_1)\, V(r_1)\, \psi^{\circ}_{k}(r_1)$$

$$+ \left(\frac{2m}{\hbar^2}\right)^2 V(r) \int_{r_1 r_2} G^{\circ}_{k}(\vec{r}-\vec{r}_1)\, V(r_1)\, G^{\circ}_{k}(\vec{r}_1-\vec{r}_2)\, V(r_2)\, \psi^{\circ}_{k}(r_2)$$

$$+ \dots$$

$$= V_0\, \delta^3(r) \Big[\psi^{\circ}_{k}(0) + \frac{2m}{\hbar^2} V_0\, G^{\circ}_{k}(0)\, \psi^{\circ}_{k}(0) +$$

$$+ \left(\frac{2m}{\hbar^2}\right)^2 G^{\circ}_{k}(0)\, V_0\, G^{\circ}_{k}(0)\, V_0\, \psi^{\circ}_{k}(0) + \dots \Big]$$

$$\boxed{T(r) = \frac{V_0\, \delta^3(r)}{1 - \frac{2m}{\hbar^2} V_0\, G^{\circ}_{k}(0)}}$$

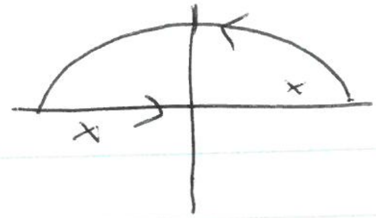
$$\hat{T}(\vec{q}) = \frac{V_0}{1 - \frac{2m}{\hbar^2} V_0\, G^{\circ}_{k}(0)}$$

$$G^{\circ}_{k}(0) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{k^2 - q^2 + i\varepsilon} = \frac{-4\pi}{(2\pi)^3} \int_0^{\Lambda = \frac{2\pi}{r_0}} dq\, q^2 \frac{1}{q^2 - k^2 - i\varepsilon}$$

$$= -\frac{1}{2\pi^2} \underbrace{\int_0^{\Lambda} dq} - \frac{k^2}{2\pi^2} \int_0^{\Lambda} dq \frac{1}{q^2 - k^2 - i\varepsilon}$$

$$= -\frac{\Lambda}{2\pi^2} - \frac{k^2}{4\pi^2} \int_{-\infty}^{\infty} dq \frac{1}{q^2 - k^2 - i\varepsilon}$$

$$\int_{-\infty}^{\infty} \frac{dq}{q^2 - k^2 - i\varepsilon}$$



$$= \oint \frac{dq}{[q-(k+i\varepsilon)][q+(k+i\varepsilon)]} = \frac{2\pi i}{2(k+i\varepsilon)} = \frac{\pi i}{k+i\varepsilon} = i\frac{\pi}{k}.$$

Note: closing in l.h.p. ⇒ same result.

$$\oint \ldots = \frac{-2\pi i}{-2(k+i\varepsilon)} = \frac{\pi i}{k+i\varepsilon} = \underline{\frac{i\pi}{k}}$$

$$\Rightarrow \boxed{G^0_k(0) = -\frac{\Lambda}{2\pi^2} - ik\frac{1}{4\pi}}$$

$$\Rightarrow \tilde{T}(\vec{q}) = \frac{V_0}{1 + \frac{2m}{\hbar^2} V_0 \frac{\Lambda}{2\pi^2} + \frac{2m}{\hbar^2}\frac{V_0}{4\pi} ik}$$

$$f_0(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\tilde{T}(q)$$

$$f_0(\theta,\varphi) = \frac{-\left(\frac{m}{2\pi\hbar^2}V_0\right)}{1 + \frac{2m}{\hbar^2}V_0\frac{\Lambda}{2\pi^2} + \left(\frac{m}{2\pi\hbar^2}V_0\right) ik}$$

$$f_0(\theta,\varphi) = \frac{1}{\underbrace{-\frac{2\pi\hbar^2}{m}V_0^{-1} - \frac{2}{\pi}\Lambda}_{-a^{-1}} - ik}$$

Note, $-ik$ as required by unitarity $|S|=1$

$$\Rightarrow \boxed{a = \frac{1}{\frac{2\pi\hbar^2}{m}V_0^{-1} + \frac{2}{\pi}\Lambda}}$$

Note: if $V_0 < 0 \Rightarrow$ $a \to 0$ at $\boxed{V_0^c = \frac{\pi^2\hbar^2}{m\Lambda}}$ (E-L³) ✓

Resonant scattering

Recall:

$$\psi_{k\ell}(r) \xrightarrow[r\to\infty]{} A_\ell \frac{e^{ikr}}{r} + B_\ell \frac{e^{-ikr}}{r}(-1)^\ell$$

outgoing wave — incoming wave

$$S_\ell(k) \equiv e^{i2\delta_\ell(k)} = \frac{A_\ell}{B_\ell} \Leftrightarrow f_\ell = \frac{i\tan\delta_\ell}{1-i\tan\delta_\ell}$$

$$= \frac{1+i\tan\delta_\ell}{1-i\tan\delta_\ell}$$

Same poles.

Can extract from $S_\ell(k)$ bound states & resonances! by thinking of k & E as complex #'s.

Recall how boundstate is obtained:

Solve S.Eqn for $E \equiv \frac{\hbar^2 k^2}{2m} < 0$, $k = i\kappa$ and require only exp. decaying soln at ∞.

$$\psi_{k\ell}(r) \sim A_\ell \frac{e^{-\kappa r}}{r} + B_\ell \frac{e^{\kappa r}}{r}$$

$$\Rightarrow \text{demand } B_\ell(i\kappa) = 0$$

$$\Rightarrow S_\ell(k = i\kappa) = \infty$$

i.e. look at poles of $S_\ell(k)$ (k complex) to find poles & resonance.

$\underline{|k}$

must have > 0 real part

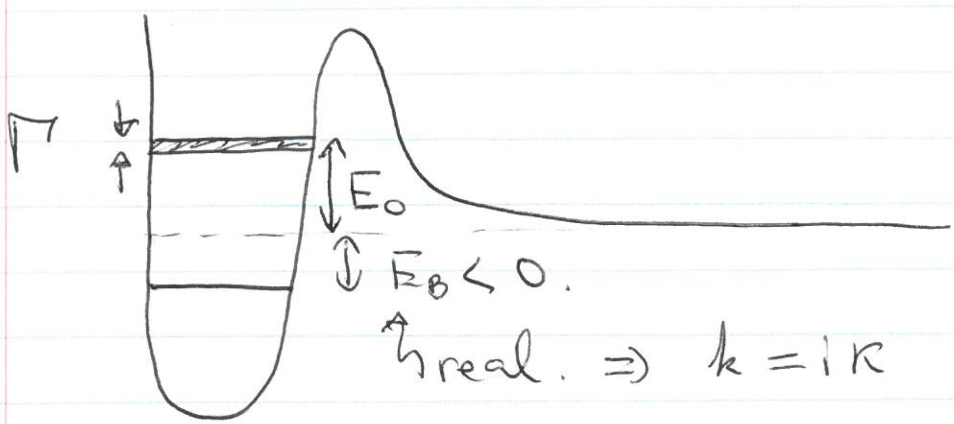
must have < 0 imaginary part

$$k = k_0 - i\eta \Rightarrow \frac{\hbar^2 k^2}{2m} = E = E_0 - i\Gamma/2$$

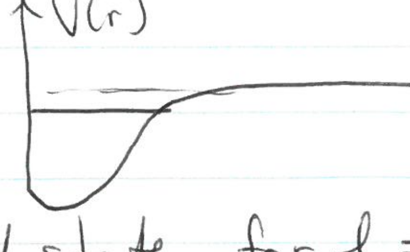
$$\Rightarrow e^{-iE_B t/\hbar} \to e^{-i(E_0 - i\Gamma/2)/\hbar}$$

$$= e^{-iE_0 t/\hbar}\, e^{-\frac{\Gamma t}{2\hbar}}$$

Γ is width of resonance, $\frac{2\pi}{\Gamma} = T$ - lifetime.

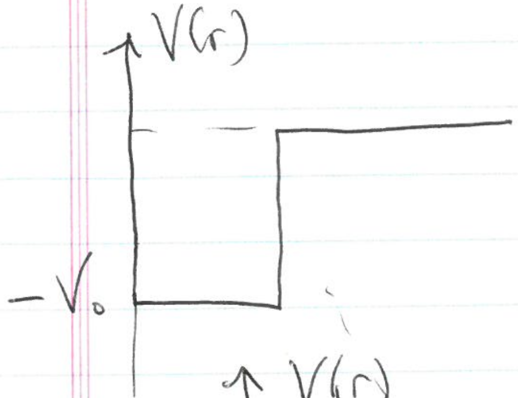


this can never happen for purely attractive potential $\underline{\text{for } l = 0}$



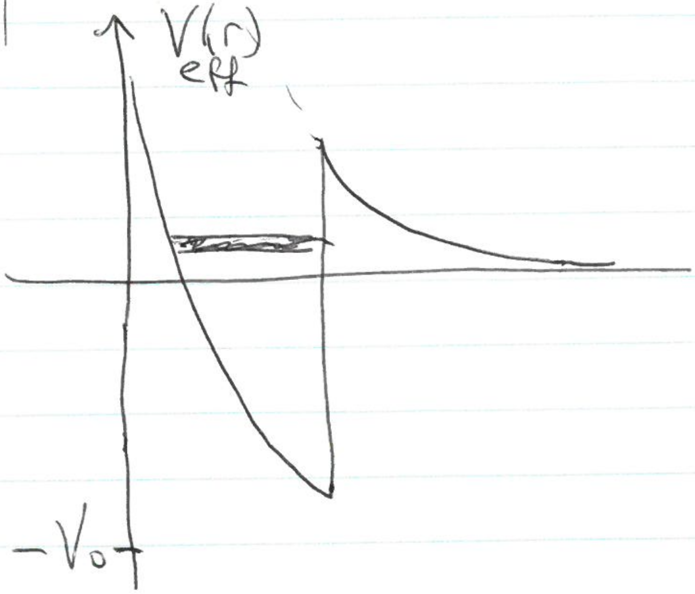
only bound state for $l = 0$ as V_0 is varied.

For $l > 0$ then centrifugal barrier allows for resonances



$$+ \quad \frac{l(l+1)}{2mr^2}$$

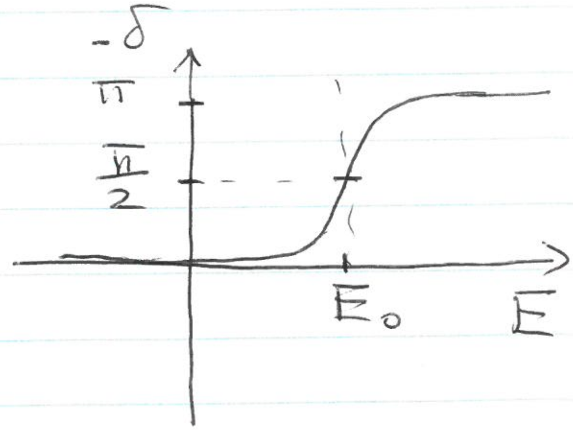
$\Rightarrow$

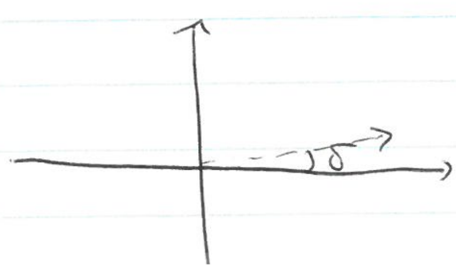


$$S_l(k) = \frac{1 + i\tan\delta_l}{1 - i\tan\delta_l} = e^{i2\delta_l} = \frac{E - (E_0 + i\Gamma/2)}{E - (E_0 - i\Gamma/2)}$$

$$\Rightarrow \tan\delta_l = \frac{\Gamma/2}{E_0 - E}$$

$$\Rightarrow \delta_l = \tan^{-1}\frac{\Gamma/2}{E_0 - E}$$





δ_ℓ change by π across a resonance

Levinson theorem ('49):

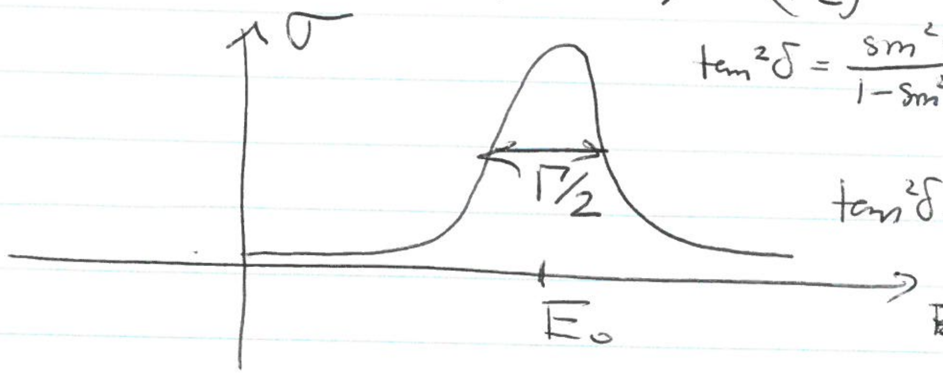
$$\delta_\ell(E) = N_\ell \pi \qquad , \text{ defining } \delta_\ell(E\to\infty)=0$$

N_ℓ: # of bound states with angular momentum $\ell\hbar$.

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$$

$$\underset{E\approx E_0}{\simeq} \frac{4\pi}{k^2}(2\ell+1)\frac{(\Gamma/2)^2}{(E-E_0)+(\Gamma/2)^2}$$

Breit-Wigner form



$$\tan^2\delta = \frac{\sin^2\delta}{1-\sin^2\delta} \Rightarrow \sin^2\delta = \frac{\tan^2\delta}{1+\tan^2\delta}$$

$$\tan^2\delta = \left(\frac{\Gamma/2}{E_0-E}\right)^2$$

$\Gamma/2 \simeq$ const around E_0 but near $E\to 0$

$\Gamma/2 \sim k^{2\ell+1}$; see e.g. hard sphere calculation on hw5.

general result $\leftarrow$ consequence of cent. barrier, recall WKB.

$$f_\ell(k) = \frac{1}{\frac{k}{\tan\delta_\ell}-ik}$$

$$\tan\delta_\ell = -\frac{B_\ell}{A_\ell} = \frac{j_\ell(kr_0)}{n_\ell(kr_0)} \sim (kr_0)^{2\ell+1}$$

$$f_\ell(k) = \frac{1}{\frac{k}{\tan\delta_\ell} - ik} = \frac{1}{\frac{k}{k^{2\ell+1}}F_\ell(k^2) - ik}$$

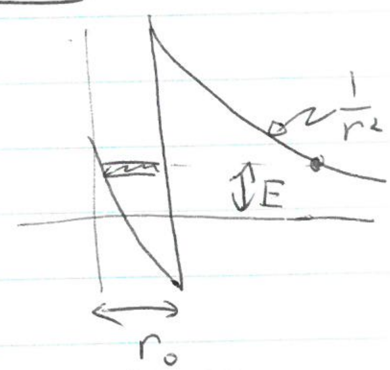
$$f_\ell(k) = \frac{k^{2\ell}}{F_\ell(k^2) - ik^{2\ell+1}}, \quad F_\ell(k^2) = c_1 + c_2 k^2 + \dots$$
$$= \gamma(E_0 - E)$$

At low energy: $k \to 0$

$$f_\ell(k) = \frac{\gamma^{-1}k^{2\ell}}{E_0 - E \quad - ik^{2\ell+1}\gamma^{-1}}$$

$$\Rightarrow \quad E_p = E_0 - i\gamma^{-1}k^{2\ell+1}$$

$$\Rightarrow \boxed{\Gamma/2 = \gamma^{-1}k^{2\ell+1} \propto E^{\ell+1/2}} \longrightarrow 0$$
as $E \to 0$



cf WKB

$$\Gamma/2 \sim e^{-2\int_{r_0}^{r(E)} \underbrace{\sqrt{\frac{2m}{\hbar^2}(V-E)}}_{\sqrt{\frac{\ell(\ell+1)}{r^2} - \frac{2mE}{\hbar^2}}}\,dr}$$

$$\Gamma/2 \sim e^{-2\sqrt{\ell(\ell+1)}\,\ln r(E)} = r(E)^{-2\sqrt{\ell(\ell+1)}}$$

$$r(E) \sim \frac{1}{\sqrt{E}}$$

$$\Rightarrow \Gamma(E)/2 \sim E^{\sqrt{\ell(\ell+1)}} \simeq E^{\ell} \quad \text{for } \ell \to \infty$$

centrifugal barrier gets higher with $\ell \gg 1 \Rightarrow \Gamma \searrow$

$\Rightarrow \tau(E) \sim \frac{1}{E^{\ell+1/2}}$ gets longer $\to \infty$ as $E \to 0$.

Two-particle scattering

- A single particle onto target: $V(r)$

$$R = \text{No. of collisions/sec} = \sigma \rho_1 v_1 \equiv \sigma j$$

- 2 particle collision $V(|\vec{r}_1 - \vec{r}_2|)$

$\Rightarrow$

$$R = \text{No. of collisions/sec/vol. of interaction}$$

$$= \sigma \rho_1 \rho_2 v_{relative}$$

$$dR = \frac{d\sigma}{d\Omega_L} d\Omega_L \rho_1 \rho_2 v_{rel}$$

No. of counts – frame independent

$$\Rightarrow \quad \frac{d\sigma}{d\Omega_L} d\Omega_L = \frac{d\sigma}{d\Omega_{cm}} d\Omega_{cm}$$

Two-particle collision:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|)$$

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(r)$$

where $M = m_1 + m_2$, $\vec{P} = \vec{p}_1 + \vec{p}_2$, $\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$

<u>Note:</u> $[r, p] = i\hbar = [R, P]$ $\quad \vec{p} = \frac{m_2}{M}\vec{p}_1 - \frac{m_1}{M}\vec{p}_2$, $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$[r, P] = [r, R] = [p, P] = [R, p] = 0$$

$$H = \overbrace{\frac{P^2}{2M}}^{H_{cm}} + \overbrace{\frac{p^2}{2\mu} + V(r)}^{H_{rel}}$$

$$\Rightarrow \Psi(\vec{r}_1, \vec{r}_2) = \underbrace{e^{\frac{i}{\hbar}\vec{P}_{cm}\cdot\vec{R}}}_{\text{free (unscattered) motion}} \Psi^{rel}(r)$$

with $\quad H_{rel}\Psi^{rel}(r) = E^{rel}\Psi^{rel}(r)$

i.e.,

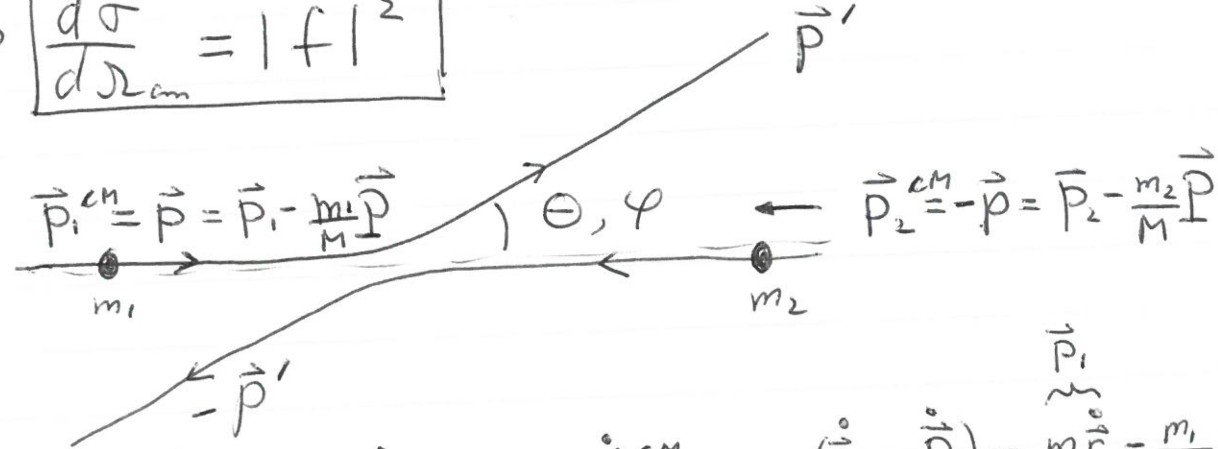
in COM: 2 particle scattering interacting with $V(|\vec{r}_1 - \vec{r}_2|)$ maps onto 1 particle with mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ scattering on potential $V(r)$

$$\Rightarrow \Psi^{rel}(\vec{r}) \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

$$R_{i\to d\Omega} = |f(\theta,\varphi)|^2 \frac{\hbar k}{\mu} d\Omega_{cm}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega_{cm}} = |f|^2}$$

CM



Physically:

$$m_1\dot{\vec{r}}_1 \to m_1\dot{\vec{r}}_1^{\,CM} = m_1(\dot{\vec{r}}_1 - \dot{\vec{R}}) = \underbrace{m_1\dot{\vec{r}}_1}_{\vec{p}_1} - \frac{m_1}{M}\underbrace{M\dot{\vec{R}}}_{\vec{P}}$$

$$m_2\dot{\vec{r}}_2 \to m_2\dot{\vec{r}}_2^{\,CM} = m_2(\dot{\vec{r}}_2 - \dot{\vec{R}}) = m_2\dot{\vec{r}}_2 - \frac{m_2}{M}\underbrace{M\dot{\vec{R}}}_{\vec{P}}$$

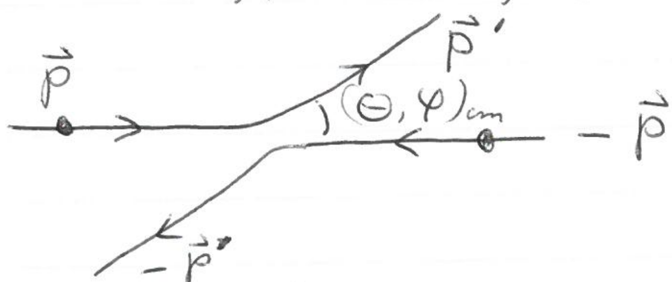
$\Rightarrow$ Calculate in CM $f_{cm}(\theta,\varphi)$, $\frac{d\sigma}{d\Omega_{cm}}$

then transform back to lab frame, if necessary.

CM → Lab transformation

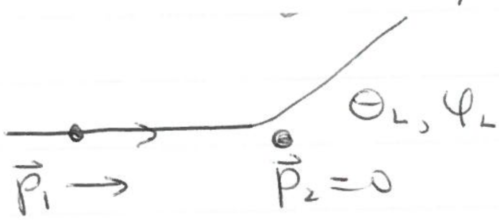
$$\frac{d\sigma}{d\Omega_L} = \frac{d\sigma}{d\Omega_{cm}} \left(\frac{d\Omega_{cm}}{d\Omega_L} \right)$$

look at θ, φ & θ_L, φ_L relation:



COM: $\Rightarrow$ $\tan\theta_{cm} = \dfrac{p_\perp'}{p_z'}$, $\tan\phi_{cm} = \dfrac{p_y'}{p_x'}$

Lab: $\Rightarrow$ with a stationary target particle



$$\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{p}_1^{\,L}, \quad \vec{p}_1^{\,L} = \frac{M}{m_2}\vec{p} \quad (= 2\vec{p} \text{ for } m_1 = m_2)$$

equivalently boost all p_i's by shift in $\vec{v} = \dfrac{\vec{p}}{m_2}$ (to have $\vec{p}_2^{\,L} = 0$)

$$\Rightarrow \underline{\vec{p}_1^{\,L}} = \vec{p} + m_1\vec{v} = \vec{p} + \frac{m_1}{m_2}\vec{p} = \underline{\frac{M}{m_2}\vec{p}} \,! \checkmark$$

$$\Rightarrow \tan\theta_L = \frac{p_\perp'}{p_z' + \frac{m_1}{m_2}p}, \quad \tan\phi_L = \frac{p_y'}{p_x'}$$

$$\boxed{\tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + m_1/m_2}} \quad (\text{using } p = p')$$

$\Rightarrow \underline{\tan\theta_L = \tan\frac{\theta_{cm}}{2}} \Rightarrow \theta_L = \frac{1}{2}\theta_{cm}$.

for $m_1 = m_2$

$\phi_L = \phi_{cm}$ $\qquad \underline{0 < \theta_{cm} \le \pi} \Rightarrow \underline{0 < \theta_L \le \frac{\pi}{2}}$

$$\frac{d\sigma}{d\Omega_L} = \frac{d\sigma}{d\Omega_{cm}}\left(\frac{d\Omega_{cm}}{d\Omega_L}\right)$$

$$\frac{d\Omega_{cm}}{d\Omega_L} = \frac{d(\cos\theta_{cm})\, d\phi_{cm}}{d(\cos\theta_L)\, d\phi_L} = \frac{\sin\theta_{cm}}{\sin\theta_L}\frac{d\theta_{cm}}{d\theta_L}$$

for $m_1 = m_2$ $\quad \theta_L = \theta_{cm}/2$

$$\Rightarrow \frac{d\Omega_{cm}}{d\Omega_L} = 2\,\frac{\sin 2\theta_L}{\sin\theta_L} = 4\cos\theta_L$$

- Scattering of <u>Identical</u> particles: $m_1 = m_2$

$$\psi(r_1, r_2) = \psi_{cm}((r_1+r_2)/2)\, \psi_{rel}(\vec{r}_1 - \vec{r}_2)$$

symmetric.

$\rightarrow$ $\boxed{\text{Bosons, } S=0}$

$$\psi_{symm} \rightarrow e^{ikz} + e^{-ikz} + \left[f(\theta,\varphi) + f(\pi-\theta, \varphi+\pi)\right]\frac{e^{ikr}}{r}$$

$$\Rightarrow \boxed{f_{sym}(\theta,\varphi) = f(\theta,\varphi) + f(\pi-\theta, \varphi+\pi)}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta,\varphi) + f(\pi-\theta,\varphi+\pi)|^2$$
$$= |f(\theta,\varphi)|^2 + |f(\pi-\theta,\varphi+\pi)|^2 + \underbrace{2\mathrm{Re}\left[f(\theta,\varphi) f^*(\pi-\theta,\varphi+\pi)\right]}_{\text{interference term}}$$

$$\sigma = \int' d\Omega \frac{d\sigma}{d\Omega}$$

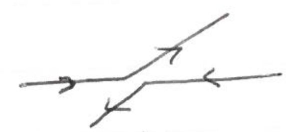
integral only over 2π radians, not 4π, not to double-count.

$\Rightarrow$ $\boxed{\text{Fermions}, \; S = 1/2}$

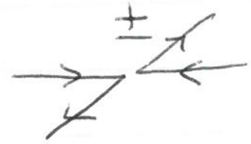
$$\tfrac{1}{2} + \tfrac{1}{2} \to 1 \text{ (triplet)}, \; 0 \text{ (singlet)}$$
$$S_1 + S_2 = S$$

$\Rightarrow$ <u>triplet scattering</u>: orbital $\psi(r_1, r_2)$ <u>antisym.</u>

$$\Rightarrow \psi(r_1, r_2) \Rightarrow \psi_{rel}(\vec{r}) = \frac{1}{\sqrt{2}}\left(e^{ik\cdot r} - e^{-ik\cdot r}\right) +$$
$$+ \frac{1}{\sqrt{2}}\Big(\underbrace{f(\theta,\varphi) - f(\pi-\theta, \varphi+\pi)}_{f_{triplet}}\Big)\frac{e^{ikr}}{r}$$



$\Rightarrow$ <u>singlet scattering</u>:



$$\Rightarrow \underline{f_{singlet} = f(\theta,\varphi) + f(\pi-\theta, \varphi+\pi)}$$

<u>Note</u>: $f_{triplet}(\theta = \frac{\pi}{2}) = 0$.

For unpolarized fermions $\Rightarrow$ $\frac{3}{4}$ triplet + $\frac{1}{4}$ singlet.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{3}{4}\left|f(\theta,\varphi) - f(\pi-\theta, \varphi+\pi)\right|^2$$
$$+ \frac{1}{4}\left|f(\theta,\varphi) + f(\pi-\theta, \varphi+\pi)\right|^2$$