Lecture 7:

Feynman's Path integral formulation of Q.M.

I. Background:

Invented by Richard Feynman (1940), motivated by original work by P.A.M. Dirac.
"Role of Lagrangian in Q.M.", Dirac.
$\Rightarrow$ now modern language for QFT & Stat Mech

Dirac:

Q. canonical quantization

$$H(p,q) \to H(\hat{p},\hat{q})$$

classical quantum

$$\{\ ,\ \} \to [\ \ ]$$

$$\text{H-J Eqn} \to \text{Sch. Eqn.}$$

... but what role is played by Lagrangian that is so prominent in classical mech. (maybe even more fundamental, since treats t, r on equal footing $\to$ relativity)

$$S = \int dt\, L(q,\dot{q}) \to ???$$

$e^{iS/\hbar}$ A. Connection through canonical transformation from $q(t_0) \to q(t) \to$ evolution in time with S the Hamilton's characteristic fnc.

Feynman "pushed"/developed this idea into a concrete theory from which explicit calc's can be done.

How? $\Rightarrow$ focus on evolution operator

$e^{-i\hat{H}t/\hbar} = \hat{U}(t)$, whose knowledge is

equivalent to solving Schrödinger's Eqn.

Know $\hat{U}(t) \longrightarrow$ everything else follows.

... but need to compute $\hat{U}(t)$ in a specific representation, e.g. coord. repres.

Q: $\langle x|\hat{U}(t)|x'\rangle \equiv \underline{U(x,t;x',t')} = ?$

A: $$U(x,t;x',t') = A \sum_{\text{all paths}} e^{\frac{i}{\hbar} S[x(t)]}$$

where,

- $S[x(t)]$ classical action functional for path $x(t)$ connecting $x' \equiv x(t')$ to $x = x(t)$
- $\sum_{\text{all paths.}}$ — "sum" over all paths $x(t)$ connecting (x',t') to (x,t)

Why? How related to Sch. Eqn? Corresp. princip?

II. Derivation of path-integral form of $U(x,t;x'$

$$U(x,x';t) = \langle x| e^{-\frac{i}{\hbar}\hat{H}t} |x'\rangle \quad \text{(picked } t'=0\text{)}$$

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = \Big(e^{-\frac{i}{\hbar}\hat{H}\underbrace{\tfrac{t}{N}}_{\equiv\varepsilon}}\Big)^N$$

$$= \underbrace{\hat{U}(\varepsilon)\hat{U}(\varepsilon)\ldots\hat{U}(\varepsilon)}_{N\text{-times}} = \big(\hat{U}(\varepsilon)\big)^N$$

infinitesmal evolution operator:

$$\hat{U}(\varepsilon) = e^{-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m}+V(\hat{x})\right)\varepsilon} \approx e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon}\, e^{-\frac{i}{\hbar}V(\hat{x})\varepsilon}$$

even though $e^{A+B} = e^{A}e^{B - \frac{1}{2}[A,B]+\ldots} \neq e^{A}e^{B}$

if $[A,B]\neq 0$

show for hw4

$$\Rightarrow U(x_N,x_0;t) = \langle x_N|\hat{U}(\varepsilon)\underbrace{\hat{U}(\varepsilon)}_{\int_{-\infty}^{\infty}dx_{N-1}|x_{N-1}\rangle\langle x_{N-1}|}\underbrace{\hat{U}(\varepsilon)}_{\int_{-\infty}^{\infty}dx_{N-2}|x_{N-2}\rangle\langle x_{N-2}|}\ldots\hat{U}(\varepsilon)\underbrace{\hat{U}(\varepsilon)}_{\int_{-\infty}^{\infty}dx_1|x_1\rangle\langle x_1|}|x_0\rangle$$

$$\Rightarrow U(x_N,x_0;t) = \int_{-\infty}^{\infty}dx_{N-1}\int_{-\infty}^{\infty}dx_{N-2}\int_{-\infty}^{\infty}dx_{N-3}\ldots\int_{-\infty}^{\infty}dx_1 \times$$

$$\times \underbrace{\langle x_N|\hat{U}(\varepsilon)|x_{N-1}\rangle\langle x_{N-1}|\hat{U}(\varepsilon)|x_{N-2}\rangle\ldots\langle x_1|\hat{U}(\varepsilon)|x_0\rangle}_{N\text{-factors}}$$

focus of generic one $\langle x_{n+1}|\hat{U}(\varepsilon)|x_n\rangle \equiv U(x_{n+1},x_n,\varepsilon) = ?$

$$U(x_{n+1}, x_n; \varepsilon) = \langle x_{n+1} | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon} e^{-\frac{i}{\hbar}V(\hat{x})\varepsilon} | x_n \rangle = ?$$

$$= \underbrace{\langle x_{n+1} | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon} | x_n \rangle}_{\equiv U_0 = ?} e^{-\frac{i}{\hbar}V(x_n)\varepsilon}$$

$\rightarrow$ free particle evolution operator for $t = \varepsilon$.

$$U(x_{n+1}, x_n; \varepsilon) = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_{n+1}-x_n)^2} e^{-\frac{i}{\hbar}V(x_n)\varepsilon}$$

$$= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} e^{\underbrace{\frac{i}{\hbar}\underset{\Delta t}{\underset{\sim}{\varepsilon}} \underbrace{\left(\frac{1}{2}m\left(\overbrace{\frac{x_{n+1}-x_n}{\varepsilon}}^{\dot{x}_n}\right)^2 - V(x_n)\right)}_{L(x_n)}}_{\frac{i}{\hbar}S_\varepsilon = \frac{i}{\hbar}\int_{t_n}^{t_{n+1}=t_n+\varepsilon} dt\, L(x(t))}}$$

$$\boxed{\begin{aligned} U(x_N, x_0; t) &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \int_{-\infty}^{\infty} dx_{N-1} A \int_{-\infty}^{\infty} dx_{N-2} A \ldots \int_{-\infty}^{\infty} dx_1 A \times \\ &\quad \times e^{\frac{i}{\hbar}\varepsilon \sum_{n=1}^{N} \left[\frac{1}{2}m\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)^2 - V(x_{n-1})\right]} \\ U(x_N, x_0; t) &\equiv \int_{x_0, 0}^{x_N, t} \underbrace{\mathcal{D}x(t)}_{\substack{\text{sum over all paths} \\ \text{connecting } x_N, t \text{ \& } x_0, 0}} e^{\frac{i}{\hbar} S(x(t))} \end{aligned}}$$

Action functional

<u>coordinate form of a path-integral</u>
(configuration space)

Show:

A. $U_0(x_{n+1}, x_n; \varepsilon) = \langle x_{n+1} | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon} | x_n \rangle$

B. phase-space path-integral form

C. How it works for: • free particle

hw4 { • harmonic oscillator
• linear potential

D. Physical picture

E. Classical limit & F. Semiclassical expansion

A. $U_0(x_{n+1}, x_n; \varepsilon) = \langle x_{n+1} | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon} \overbrace{}^{\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} |p_n\rangle\langle p_n|} | x_n \rangle$

$$= \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \underbrace{\langle x_{n+1} | p_n \rangle}_{e^{\frac{i}{\hbar} p_n x_{n+1}}} \underbrace{\langle p_n | x_n \rangle}_{e^{-\frac{i}{\hbar} p_n x_n}} e^{-\frac{i}{\hbar}\frac{p_n^2}{2m}\varepsilon}$$

$$\Rightarrow U_0(x_{n+1}, x_n; \varepsilon) = \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{p_n^2}{2m}\varepsilon + \frac{i}{\hbar} p_n (x_{n+1} - x_n)}$$

$$= \left(\frac{m}{2\pi i \hbar \varepsilon}\right)^{1/2} e^{\frac{i}{\hbar} \underset{\Delta t = t_{n+1} - t_n}{\varepsilon} \frac{1}{2} m \Big(\underbrace{\frac{x_{n+1} - x_n}{\varepsilon}}_{\dot{x}_n} \Big)^2}$$

B. phase-space P-I.:

$$U(x_N, x_0; t) = \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_{N-2} \cdots \int_{-\infty}^{\infty} dx_1 \times$$

$$\times \prod_{n=1}^{N} \left[\underbrace{\langle x_n | e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\varepsilon} | x_{n-1}\rangle}_{U_0(x_n, x_{n-1}; \varepsilon)} e^{-\frac{i}{\hbar} V(x_{n-1})\varepsilon} \right]$$

<u>insert</u> $\prod_{n=1}^{N} \left(\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} |p_n\rangle\langle p_n| \right)$

$$\boxed{\begin{gathered} \Rightarrow U(x_N, x_0; t) = \int_{-\infty}^{\infty} dx_{N-1} \frac{dp_{N-1}}{2\pi\hbar} \int_{-\infty}^{\infty} dx_{N-2} \frac{dp_{N-2}}{2\pi\hbar} \cdots \int \overbrace{dx_1 \frac{dp_1}{2\pi\hbar}}^{\text{dimensionless}} \cdot \\ \prod_{n=1}^{N} \left(e^{\varepsilon \frac{i}{\hbar}\left[p_n \underbrace{\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)}_{\dot{x}_n} - \frac{p_n^2}{2m} - V(x_{n-1}) \right]} \right) \\ N \to \infty, \quad \varepsilon \to 0 \\ U(x_N, x_0; t) = \int \mathcal{D}x(t)\mathcal{D}p(t)\, e^{\frac{i}{\hbar}\int_0^t dt \left[p\dot{x} - H(x(t), p(t)) \right]} \end{gathered}}$$

phase-space path-integral.

$$H = \frac{p^2}{2m} + V(x)$$

integrating over $\int \mathcal{D}p(t) \rightarrow$ configurational path-integral.

<u>Note</u>: (for the most part) can <u>only</u> do Gaussian path integrals $\Rightarrow$ can only solve probs with $\underline{H = \frac{p^2}{2m} + ax + bx^2}$ i.e. harmonic oscillators!

C. How it works for free particle.
(more examples later)

$$U_0(x_N, x_0; t) = \int_{x_0, 0}^{x_N, t} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}\int_0^t dt' \frac{1}{2} m \dot{x}(t)^2}$$

Actual integral done in discrete form:

Note "closure" property of Gaussian P-I:

$$\int_{-\infty}^{\infty} dx_n U_0(x_{n+1}, x_n; t_{n+1}, t_n)\, U_0(x_n, x_{n-1}, t_n, t_{n-1})$$

$$= \int_{-\infty}^{\infty} dx_n \left(\frac{m}{2\pi i \hbar (t_{n+1}-t_n)}\right)^{1/2} e^{\frac{i}{\hbar}\frac{1}{2} m \frac{(x_{n+1}-x_n)^2}{t_{n+1}-t_n}} \times$$

$$\times \left(\frac{m}{2\pi i \hbar (t_n - t_{n-1})}\right)^{1/2} e^{\frac{i}{\hbar}\frac{1}{2} m \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}}$$

$$= \left(\frac{m}{2\pi i \hbar (t_{n+1}-t_{n-1})}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_{n+1}-x_{n-1})^2}{t_{n+1}-t_{n-1}}}$$

↑ easiest to show through phase space P-I (hw 4)

$$= U_0(x_{n+1}, x_{n-1}; t_{n+1}, t_{n-1})$$

This way integrate all $\int dx_{N-1}, \int dx_{N-2}, \ldots \int dx_1$

$$\Rightarrow \boxed{U_0(x_N, x_0; t) = \left(\frac{m}{2\pi i \hbar t}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\frac{(x_N-x_0)^2}{t}}}$$

D. Physical picture

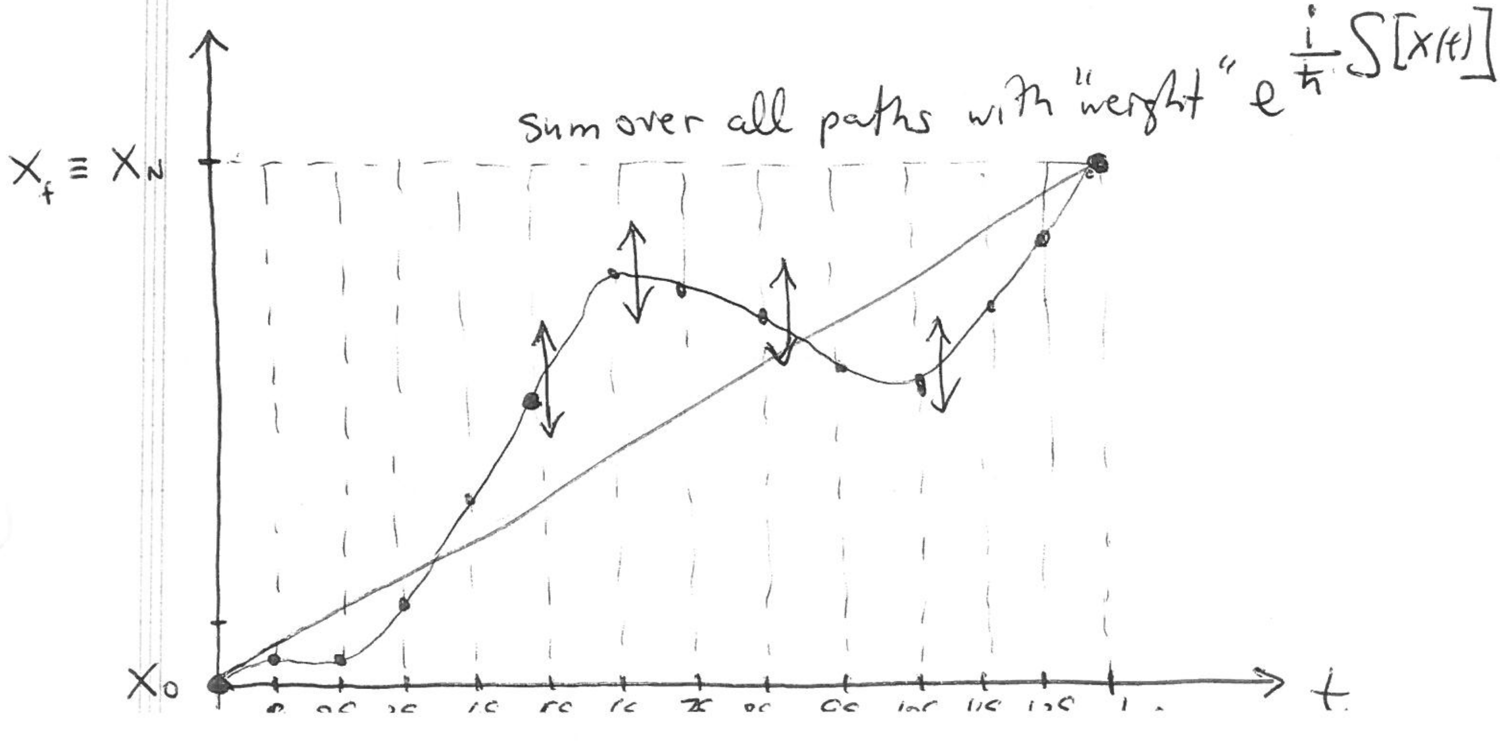
Feynman: There are no waves, only particles

... but these quantum particles can follow *any* trajectory $x(t)$ from $x_0, 0$ to x_N, t *not* just the classical one (from Newton's law) (also true for photons, etc.)

$$U(x_f, x_0; t) = \int_{x(0)=x_0}^{x(t)=x_f} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}\int_0^t dt'\, L(x(t), \dot{x}(t))}$$

$$= \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dx_{N-2} \cdots \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1\; U(x_N, x_{N-1}; \varepsilon)\, U(x_{N-1}, x_{N-2}; \varepsilon) \times \cdots U(x_2, x_1; \varepsilon)\, U(x_1, x_0; \varepsilon)$$

"sum" over all intermediate $x_1, x_2, \ldots, x_{N-1}$, i.e., sum over all possible paths:



E. Classical limit:

$$U(x_f, x_0; t) = \int_{x(0)=x_0}^{x(t)=x_f} \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]}$$

All paths contribute in general →
sum bunch of complex #'s $e^{\frac{i}{\hbar} S[x(t)]}$

... but for $\hbar \to 0$ this sum is that of fast "oscillating" complex #'s, i.e.
small change in path → small change in $S[x(t)]$
⇒ great change in phase of $\frac{1}{\hbar} S[x(t)]$

⇒ $e^{\frac{i}{\hbar} S}$ oscillates a lot

⇒ sum → ≈ 0 from most paths' contributions cancelling out.

Path-integral is dominated by $x(t)$ that extremizes $S[x(t)]$, since then $S[x(t)]$ changes more slowly ⇒
main contribution from paths $x_c(t)$

s.t. $$\boxed{\left.\frac{\delta S}{\delta x(t)}\right|_{x_c(t)} = 0}$$ ⟺ Euler-Lagrange Eqn
⟺ Newton's 2nd law

Saddle pnt approx.
= Method of steepest descent

$$\Rightarrow U(x_f, x_o; t) = \int_{x(0)=x_o}^{x(t)=x_f} \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]}$$

$$\boxed{U \approx e^{\frac{i}{\hbar} S[x_c(t)]}}$$

satisfies Newton's / Euler-Lagrange eqns. with b.c. $x(t) = x_f$, $x(0) = x_o$

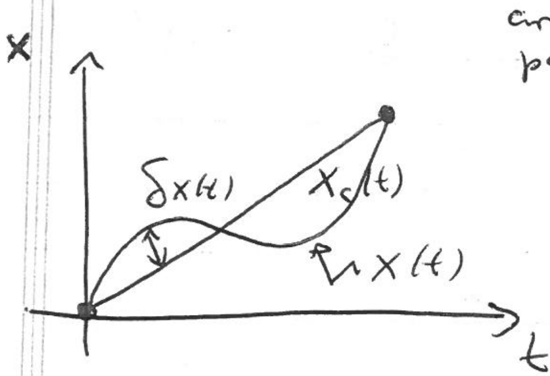
wow !!!

F. Semiclassical expansion:

$$U(x_f, x_o; t) = \int_{x(0)=x_o}^{x(t)=x_f} \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]}$$

change vars: $x(t) = x_c(t) + \delta x(t)$

arbitrary path — classical path — fluct. about classical path



$\Rightarrow$ Note: $\delta x(t_f) = 0$, $\delta x(0) = 0$. $\}$ simple b.c.'s.

$$\Rightarrow U(x_f, x_o, t_f) = \int_{\delta x(0)=0}^{\delta x(t_f)=0} \mathcal{D}\delta x(t)\, e^{\frac{i}{\hbar} \underbrace{S[x_c(t) + \delta x(t)]}_{\text{Taylor expand in } \delta x(t)}}$$

$$S[x_c + \delta x] = \underbrace{\int_0^{t_f} dt \left[\frac{m}{2}\dot{x}_c^2 - V(x_c)\right.}_{S_c[x_c(t)]} + \frac{m}{2}\delta\dot{x}^2 - \frac{1}{2}V''(x_c(t))\,\delta x(t)^2 + \ldots \Big]$$

Note: $\left.\frac{\delta S}{\delta x(t)}\right|_{x_c} = 0 \Rightarrow$ no linear terms in $\delta x(t)$!

$$U(x_f, x_0; t_f) \simeq e^{\frac{i}{\hbar} S[x_c(t)]} \times \underbrace{\int_0^0 \mathcal{D}\delta x(t)\, e^{\frac{i}{\hbar}\int_0^{t_f} dt' \left[\frac{1}{2} m \delta\dot{x}^2 - \frac{1}{2} V''(x_c(t))\delta x(t)^2 + \cdots\right]}}$$

effective harmonic oscillator about classical path $x_c(t)$ → do Gaussian integral over $\delta x(t)$. If want to go to higher order expand exponential in $\delta x^3, \delta x^4, \ldots$

⇒ moments of Gaussian integral

→ path-integral pert. theory.

Ex: free particle

$$S[x(t)] = \int_0^{t_f} dt' \, \frac{1}{2} m \dot{x}(t)^2$$

$$\ddot{x}_c(t) = 0 \Rightarrow x_c(t) = \frac{x_f - x_0}{t_f} t + x_0$$

Note: $x_c(0) = x_0, \; x_c(t_f) = x_f$

exactly:

$$S[x(t)] = \underbrace{\int_0^{t_f} dt' \, \frac{1}{2} m \dot{x}_c(t)^2}_{S[x_c(t)]} + \underbrace{\int_0^{t_f} dt' \, \frac{1}{2} m \delta\dot{x}(t)^2}_{\text{Gauss. fluctuations.}}$$

$$\Rightarrow U(x_f, x_0; t_f) = \underbrace{e^{\frac{i}{\hbar} S[x_c(t)]}}_{= ?} \underbrace{\int_0^0 \mathcal{D}\delta x(t)\, e^{\frac{i}{\hbar}\int_0^{t_f} dt' \frac{1}{2} m \delta\dot{x}(t)^2}}$$

$N(t_f)$ just a #, independent of x_0, x_f; just depends on t_f.

$$S[x_c(t)] = \int_0^{t_f} dt' \frac{1}{2} m \left(\frac{x_f - x_0}{t_f}\right)^2 = \frac{1}{2} m \left(\frac{x_f - x_0}{t_f}\right)^2 t_f$$

$$\Rightarrow \boxed{U(x_f, x_0; t_f) = N(t_f)\, e^{\frac{i}{\hbar} \frac{m}{2} \frac{(x_f - x_0)^2}{t_f}}}$$

$= \left(\frac{m}{2\pi i \hbar t_f}\right)^{1/2}$, fix will loo to

G. Connection/equivalence with Schrödinger Eqn

local { Schrödinger Eqn. $\longleftrightarrow$ Feynman Path integral } global
similar to:
Newton's Eqn $\longleftrightarrow$ Least action principle. $\frac{\delta S}{\delta x(t)} = 0$

$$H|\psi\rangle = i\hbar\, \partial_t |\psi\rangle$$

infinitesimal evolution of time ε:

$$\Leftrightarrow \quad -\frac{i\varepsilon}{\hbar} H|\psi(0)\rangle = |\psi(\varepsilon)\rangle - |\psi(0)\rangle$$

In $|x\rangle$ basis to lowest order in ε $(\Theta(\varepsilon))$:

$$\psi(x,\varepsilon) - \psi(x,0) = \frac{-i\varepsilon}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,0)\right]\psi(x,0)$$

$$\psi(x,\varepsilon) = \psi(x,0) - \frac{i\varepsilon}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,0)\right]\psi(x,0)$$

compare to evolution from 0 to ε

$$\psi(x,\varepsilon) = \int_{-\infty}^{\infty} U(x,\varepsilon;x')\,\psi(x',0)\,dx'$$

where:

$$U(x;\varepsilon;x') = \left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2} e^{i\left[\frac{m(x-x')^2}{2\varepsilon} - \varepsilon V\left(\frac{x+x'}{2},0\right)\right]/\hbar}$$

$$\Psi(x,\varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}\underbrace{(x-x')}_{\eta}{}^2}\, e^{-\frac{i\varepsilon}{\hbar}V\left(\frac{x+x'}{2},0\right)}\,\Psi(x',0)\,dx'$$

$$x' = x+\eta$$

change vars $x' \to \eta$

$$\Psi(x,\varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\int_{-\infty}^{\infty} d\eta\, e^{\underbrace{\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2}_{\lesssim \mathcal{O}(\pi)}}\, e^{-\frac{i\varepsilon}{\hbar}V\left(x+\underbrace{\tfrac{\eta}{2}}_{\text{does not matter}},0\right)}\,\Psi(x+\eta,0)$$

i.e. $\eta/2 = 0$ since $\varepsilon \to 0$ & $\eta \ll \varepsilon$

$$\Rightarrow \eta \lesssim \sqrt{\frac{\pi\varepsilon\hbar}{m}} \to 0 \quad \text{as } \varepsilon \to 0$$

Taylor expand in ε to $\mathcal{O}(\varepsilon^1)$

$$\Psi(x,\varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\int_{-\infty}^{\infty} d\eta\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2}\, e^{-\frac{i\varepsilon}{\hbar}V(x,0)} \times$$

$$\times\left(\Psi(x,0)+\eta\frac{\partial}{\partial x}\Psi(x,0)+\frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\Psi(x,0)+\dots\right)$$

$$\Psi(x,\varepsilon)=\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\int_{-\infty}^{\infty} d\eta\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2}\left(1-\frac{i}{\hbar}\varepsilon V(x,0)+\dots\right)\times$$

$$\times\left(\Psi(x,0)+\eta\frac{\partial}{\partial x}\Psi(x,0)+\frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\Psi(x,0)+\dots\right)$$

$$\simeq\left(\frac{m}{2\pi\hbar i\varepsilon}\right)^{1/2}\int_{-\infty}^{\infty} d\eta\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}\eta^2}\Bigg[\left(1-\frac{i}{\hbar}\varepsilon V(x,0)\right)\Psi(x,0)+$$

$$+\underbrace{\eta}_{0}\frac{\partial\Psi(x,0)}{\partial x}+\eta^2\frac{1}{2}\frac{\partial^2\Psi(x,0)}{\partial x^2}+\dots\Bigg]$$

$$\psi(x,\varepsilon) = \psi(x,0) - \frac{i}{\hbar}\varepsilon V(x,0)\psi(x,0) + \underbrace{\langle \eta^2 \rangle}_{\frac{\hbar\varepsilon}{im}} \frac{1}{2}\frac{\partial^2\psi(x,0)}{\partial x^2}$$

$$\psi(x,\varepsilon) = \psi(x,0) - \frac{i}{\hbar}\varepsilon\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,0)}{\partial x^2} + V(x,0)\psi(x,0)\right)$$

$$\Rightarrow \quad \underline{-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar\,\partial_t\psi(x,t)}$$

- Harmonic Oscillator:

$$U(x,t;x',0) = A(t)\, e^{\frac{i}{\hbar}S_{cl}}$$

$$S_{cl} = \frac{m\omega_0}{2\sin\omega_0 t}\left[(x^2+x'^2)\cos\omega_0 t - 2xx'\right]$$

$$A(t) = \left(\frac{m\omega_0}{2\pi i\hbar\sin\omega_0 t}\right)^{1/2} \quad \left(\underline{\text{Note}}:\ A(t) \xrightarrow[\omega_0\to 0]{} \left(\frac{m}{2\pi i\hbar t}\right)^{1/2}\right)$$

- Linear potential $V(x) = -fx$:

$$U(x,t;x',0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{\frac{i}{\hbar}S_{cl}}$$

where $S_{cl} = S[x_c(t)]$ & $x_c(t) = x_0 + v_0 t + \frac{1}{2}\left(\frac{f}{m}\right)t^2$

chosen s.t. $x(t_f) = x_f$.