Lecture 6

E&M and Quantum Mechanics

Outline:

- Classical E&M
  - Maxwell's Eqns
  - gauge invnce with quantum matter
  - energy & momentum density
  - Bohm - Aharonov effect
- Atom - E&M interaction (classical)
  - $A\cdot p + \frac{1}{2}A^2 \rightarrow \vec{E}\cdot\vec{r}$ dipolar approx.
  - photoelectric effect via Fermi golden rule
  - Einstein's A & B coefficients. (laser)
- Quantization of E&M radiation
  - $[A_{(r)}, E_{(r')}] = i\hbar\,\delta(r-r')$
  - $H = \frac{1}{8\pi}\int d^3r\left[E^2 + (\nabla\times A)^2\right]$ cf $H = \frac{p^2}{2m} + \frac{1}{2}B\psi^2$
  - spontaneous emission via F.G. rule ...
    $\Rightarrow$ A & B coeff's.

• Classical E&M

- Maxwell's eqns:

$$\nabla\cdot E = 4\pi\rho \quad , \quad \nabla\cdot B = 0$$

$$\nabla\times E + \frac{1}{c}\partial_t B = 0 \quad , \quad \nabla\times B - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j$$

$$\Rightarrow \quad \nabla\cdot j + \partial_t \rho = 0 \qquad \text{local charge conservation}$$

Solve $\nabla\cdot B = 0$ & $\nabla\times E + \frac{1}{c}\partial_t B = 0$

Via $\underline{B = \nabla\times A}$ & $\nabla\times\left(E + \frac{1}{c}\partial_t A\right) = 0$

$$\Rightarrow \quad \underline{E = -\frac{1}{c}\partial_t A - \nabla\phi}$$

- Gauge invnce:
E, B are invnt under gauge transf. on A, ϕ:

$$A' = A - \nabla\chi \quad , \quad \phi' = \phi + \frac{1}{c}\partial_t \chi$$

- Maxwell's Eqn $\rightarrow$ $\underbrace{\text{Wave eqn}}_{\text{transverse}}$ + $\underbrace{\text{Coulomb's law}}_{\text{longitudinal}}$

▲ $\nabla^2\phi + \frac{1}{c}\partial_t(\nabla\cdot A) = -4\pi\rho$

▲ $\nabla^2 A - \frac{1}{c}\partial_t^2 A - \nabla\left(\nabla\cdot A + \frac{1}{c}\partial_t\phi\right) = -\frac{4\pi}{c} j$

For nonrelativistic probs. Coulomb gauge is conven.

$\rightarrow$ $\underline{\nabla\cdot A = 0 \quad , \quad \phi = 0}$ $\quad (\rho = j = 0)$

otherwise ϕ coulomb $\rightarrow$ t-independ.

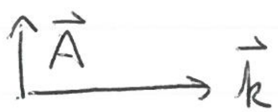
others e.g. Lorentz gauge
$\nabla\cdot A - \frac{1}{c}\partial_t\phi = \partial_\mu A^\mu = 0.$

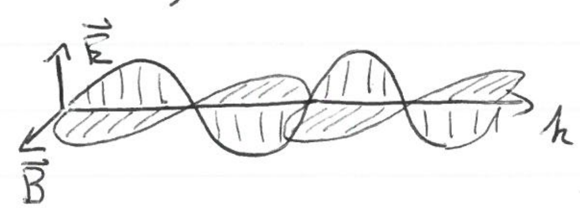
$$\Rightarrow \quad \nabla^2 \vec{A} - \frac{1}{c} \partial_t^2 \vec{A} = 0$$

<u>transverse</u> wave with $\omega_k = ck$

$$\vec{A} = \hat{\epsilon} A_0 \cos(\vec{k}\cdot\vec{r} - \omega t), \quad \hat{\epsilon} \perp \vec{k}$$

<u>polarization</u>





only <u>two</u> $\Leftrightarrow$ transverse

- $H_{EM} = \frac{1}{8\pi} \int d^3r \left[E^2 + B^2 \right] = \int_r \mathcal{H}_{EM}$.

- $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = c \hat{k}\, \mathcal{H}_{EM}$. cf $\boxed{nev = j}$

  energy current density. charge current density.

- E & M – Matter coupling:

  - $\vec{F} = q\underbrace{\left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}\right)}$ $\longrightarrow$ gauge inv't! <u>Classical matter</u>

    equivalent to:

$$H = \frac{1}{2m}\underbrace{\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2}_{\text{"minimal" coupling}} + q\phi(\vec{r})$$

scalar forces/potentials.

<u>Note</u>: all know forces of nature (EM, weak, strong gravity) arise from gauge fields that couple to matter through such gauge/minimal coupling.

$$H = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2 + q\phi \iff L = \frac{1}{2}mv^2 + \frac{q}{c}v\cdot A - q\phi$$

Hamilton's eqns $\iff$ Lagrange's Eqns

$\Rightarrow$ classical trajectories $\vec{r}(t)$ - gauge invnt.

- Quantum matter

Schrodinger's Eqn for $\psi(\vec{r},t)$

$$\left[\frac{1}{2m}\left(\hat{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi\right]\psi = i\hbar\partial_t\psi$$

Path integral for evolution op. $U(\vec{r},\vec{r}';t)$

$$U(\vec{r},\vec{r}';t) = e^{\frac{i}{\hbar}\int_0^t L\,dt'}$$

Note: not gauge invnt, i.e. after gauge transf. $A \to A'$, $\phi \to \phi'$,

$\psi(r,t)$ & $U(r,r';t)$ change!

cf. classical physics.

$$\psi' = e^{-i\frac{q}{\hbar c}\chi}\,\psi$$

inverse unit of magnetic flux

$$U'(\vec{r},\vec{r}';t) = e^{\frac{iq}{\hbar c}[\chi(\vec{r}',0) - \chi(\vec{r},t)]} \cdot U(\vec{r},\vec{r}';t)$$

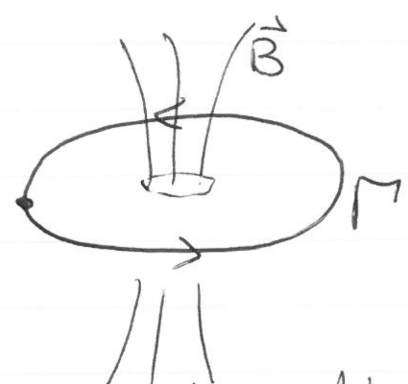
$$\frac{q}{\hbar c} \equiv \frac{2\pi}{\Phi_0}$$

- Aharonov - Bohm effect.

▲ gauge invnt effects other than $\vec{E}$ & $\vec{B}$:

"Wilson loop" $e^{i\frac{q}{\hbar c}\oint_\Gamma \vec{A}\cdot d\vec{r}} = W_\Gamma$

$W_\Gamma = e^{i\,2\pi\,\frac{1}{\Phi_0}\iint \vec{B}\cdot d\vec{a}}$ } fraction of flux $\frac{\Phi}{\Phi_0}$ piercing area spanned by Γ

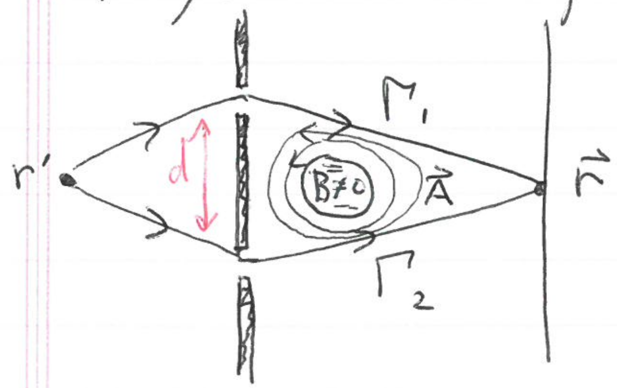


Note: does not directly depend on $\vec{B}$ on path Γ!

Eg. $\vec{B}$ is confined to solenoid, s.t. $\vec{B}=0$ on path Γ. Classically no Lorentz force, i.e. no effect on particle confined to Γ (where $\vec{B}=0$).
QM'lly phase factor $W_\Gamma = e^{i2\pi\frac{\Phi}{\Phi_0}} \neq 1$

[$2\pi\frac{\Phi}{\Phi_0} = \gamma_{Berry's}$, for ψ' after gauge transformation to eliminate $\vec{A}$ from S.Eqn.

▲ Physical consequences: A-B effect.



$$U(r,t;r',0) = \sum_{\text{paths}} e^{\frac{i}{\hbar}\int_0^t dt'(L_0 + \underset{\uparrow\ \frac{q}{c}\vec{A}\cdot\vec{v}}{L_{EM}})}$$

$$= \sum_{\substack{(A)\ \text{paths} \\ \underline{\text{above}} \\ \text{solenoid}}} e^{\frac{i}{\hbar}\int_0^t dt' L_0}\, e^{i\frac{2\pi}{\phi_0}\int^r \vec{A}\cdot d\vec{r}} \; +$$

$$+ \sum_{\substack{(B)\ \text{paths} \\ \underline{\text{below}} \\ \text{solenoid}}} e^{\frac{i}{\hbar}\int^t dt' L_0}\, e^{i\frac{2\pi}{\phi_0}\int^r \vec{A}\cdot d\vec{r}}$$

$$= e^{i\frac{2\pi}{\phi_0}\int_{\Gamma_1} A\cdot d\vec{r}} \Bigg[\sum_{\{A\}} e^{\frac{i}{\hbar}S_0 + i\frac{2\pi}{\phi_0}\underbrace{\oint_{A-\Gamma_1} \vec{A}\cdot d\vec{r}}_{\iint \vec{B}\cdot d\vec{a} \,=\, 0}} \; +$$

$$+ \sum_{\{B\}} e^{\frac{i}{\hbar}S_0 + i\frac{2\pi}{\phi_0}\oint_{\Gamma_1 - \Gamma_2} \vec{A}\cdot dr} \Bigg]$$

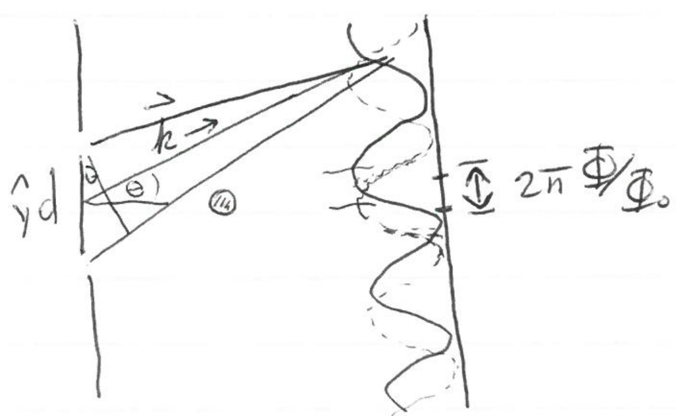
$$= e^{i\gamma}\left[\psi_A(r) + \underbrace{e^{i\frac{2\pi}{\phi_0}\oint_{\Gamma} \vec{A}\cdot d\vec{r}}}_{\text{shift interference pattern!}} \psi_B(r)\right]$$

Interference pattern <u>shifts</u> by $2\pi\frac{\Phi}{\Phi_0}$:

$$|\Psi|^2 = \text{const} + \cos\left[\vec{k}\cdot d\hat{y} - \underline{2\pi\frac{\Phi}{\Phi_0}}\right]$$

<u>Note:</u> periodic in Φ with period $\underline{\Phi = \frac{hc}{q}}$

Classical particle only senses <u>local</u> effect of $\vec{A}$ i.e. $\vec{\nabla}\times\vec{A} = \vec{B}$ on classical trajectory.
Quantum particle "measures" global props of $\vec{A}$ as <u>well</u> e.g. $\oint \vec{A}\cdot d\vec{r}$



- EM field (classical) – matter interaction

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + V(\vec{r})$$

$$= \underbrace{\frac{1}{2m}p^2 + V(\vec{r})}_{H_0} + \underbrace{\frac{-q}{2mc}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right) + \frac{q^2}{2mc^2}A^2}_{H_1}$$

$$H_1\Psi = -\frac{q}{2mc}\left[-i\hbar\underbrace{\vec{\nabla}\cdot(\vec{A}\Psi)}_{(\vec{\nabla}\cdot\vec{A})\Psi + \vec{A}\cdot\vec{\nabla}\Psi} + \vec{A}\cdot(-i\hbar\nabla\Psi)\right] + \frac{q^2}{2mc^2}A^2\Psi$$

$(\vec{\nabla}\cdot\vec{A}) = 0$ in Coulomb gauge

diamagnet term small for weak intensity since A^2 v.s. A

$$H_1 \approx -\frac{q}{2mc}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right)$$

$$\underline{H_1(r,t) = -\frac{q}{mc}\vec{A}\cdot\vec{p}}$$

$$= -\frac{q}{2mc}\left[e^{i(k\cdot r - \omega t)} + e^{-i(k\cdot r - \omega t)}\right]\vec{A}_0\cdot\vec{p}$$

feed into 1^{st} order p.t.

$$d_f(t) = -\frac{i}{\hbar}\int_0^t dt' \langle f|H_1|i\rangle e^{\frac{i}{\hbar}(E_f - E_i)t'}$$

$$= -\frac{i}{\hbar}\left(\frac{-q}{2mc}\right)\int_0^t dt'\left[\langle f|e^{ik\cdot r}\vec{A}_0\cdot\vec{p}|i\rangle e^{\frac{i}{\hbar}(E_f - E_i - \hbar\omega)t'} + \langle f|e^{-ik\cdot r}\vec{A}_0\cdot\vec{p}|i\rangle e^{\frac{i}{\hbar}(E_f - E_i + \hbar\omega)t'}\right]$$

absorption: $E_f = E_i + \hbar\omega$

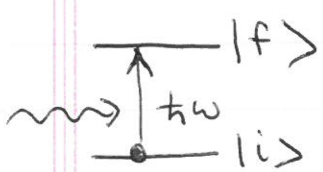
emission: $E_f = E_i - \hbar\omega$

$$\Rightarrow R_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f|\left(\frac{-q}{2mc}\right)e^{ik\cdot r}\vec{A}_0\cdot\vec{p}\,|i\rangle\right|^2\delta(E_f-E_i-\hbar\omega)$$

$$+\frac{2\pi}{\hbar}\left|\langle f'|\left(\frac{-q}{2mc}\right)e^{-ik\cdot r}\vec{A}_0\cdot\vec{p}\,|i'\rangle\right|^2\delta(E_f'-E_i'+\hbar\omega)$$

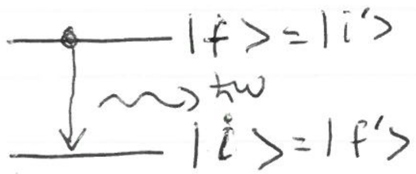
<u>Comments</u>:

1. rate of <u>stimulated</u> (proportional to $|A_0|^2$) <u>absorption</u>



$\langle f|H_1|i\rangle$

equals rate of <u>stimulated</u> emission



$$\langle i|H_1|f\rangle = \langle f|H_1^{+}|i\rangle^*$$
$$= \langle f|H_1|i\rangle^*$$

<u>ensured by</u> $\underline{H_1^{+} = H_1}$

more explicitely:

absorption $|\langle f|e^{ik\cdot r}\vec{A}_0\cdot\vec{p}\,|i\rangle|^2\delta(E_f-E_i-\hbar\omega)$

emission $\underbrace{|\langle i|e^{-ik\cdot r}\vec{A}_0\cdot\vec{p}\,|f\rangle|^2}_{|\langle f|e^{ik\cdot r}\vec{A}_0\cdot\vec{p}^{\,+}|i\rangle^*|^2}\delta(\underbrace{E_i-E_f+\hbar\omega}_{-(E_f-E_i-\hbar\omega)\ \checkmark})$

$\vec{p}^{\,+} = \vec{p}$

$$= |\langle f|e^{ik\cdot r}\vec{A}_0\cdot\vec{p}\,|i\rangle|^2\delta(E_f-E_i-\hbar\omega)\ \checkmark$$

2. $\langle e^{ik\cdot r} \rangle \simeq \langle 1 + ik\cdot r + \frac{1}{2!}(ik\cdot r)^2 + \ldots + \frac{1}{n!}(ik\cdot r)^n + \ldots \rangle$

Electric dipole approx — E2, M2 quadrupole — n+1 -pole

note: $\langle k\cdot r \rangle \simeq k a_0 << 1$ in atom

why? $\hbar\omega \simeq \frac{e^2}{a_0} = E_{Ry}$.

$$\Rightarrow k = \frac{\omega}{c} = \frac{e^2}{\hbar c}\frac{1}{a_0}$$

$$\Rightarrow k a_0 \simeq \frac{e^2}{\hbar c} = \alpha \approx \frac{1}{137} << 1$$

more physically $e^{ik\cdot r} \simeq 1$ over extent of atom

$$\langle k\cdot r \rangle \simeq 2\pi\frac{a_0}{\lambda} << 1.$$

also: Zeeman energy is small, ignored:

$$\frac{\frac{q}{2mc}\langle \vec{S}\cdot\vec{B} \rangle}{\frac{q}{mc}\langle \vec{A}\cdot\vec{p} \rangle} \simeq \frac{\langle \hbar\,\vec{\sigma}\cdot\vec{\nabla}\times\vec{A} \rangle}{\langle A\cdot\hbar\nabla \rangle} \simeq \frac{\hbar k}{P} << 1$$

γ's momentum

electron's momentum

3. <u>Dipole approximation</u>: $e^{ik\cdot r} \simeq 1$

$$R_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{q^2}{4m^2c^2}\right)|\langle f|\hat{\varepsilon}\cdot\vec{p}|i\rangle|^2|A_0|^2\delta(E_f-E_i\mp\hbar\omega)$$

(if vanishes by selection rules e.g. $|1s\rangle \to |2s\rangle$ then need to go to higher multipole order. Also for high energy photon → high E electron)

- Connection to $\vec{E}\cdot\vec{\mu}_e = q\vec{E}\cdot\vec{r}$ coupling:

→ $\vec{E} = -\frac{1}{c}\partial_t\vec{A} = \frac{i\omega}{2c}\vec{A}_0 e^{-i\omega t}$

$$H_{dipole} = -\vec{E}\cdot q\vec{r} = -i\omega\frac{q}{2c}e^{-i\omega t}\vec{A}_0\cdot\vec{r}$$

compare:

$$H_{A\cdot p} = -\frac{q}{2mc}\vec{A}_0 e^{-i\omega t}\cdot\vec{p} = -\frac{q}{mc}\vec{A}\cdot\vec{p}$$

$$= -\frac{q}{\cancel{m}c}\vec{A}\cdot\cancel{m}\,\partial_t\vec{r} \underset{\substack{\text{integrate}\\ \text{by parts}\\ (\text{under } \int^t \text{ in p.t.})}}{=} \frac{1}{c}\partial_t\vec{A}\cdot(q\vec{r})$$

$$H_{A\cdot p} = -\vec{E}\cdot\vec{\mu}_e \quad \checkmark$$

→ more generally: $H_1 = -\frac{q}{mc}\vec{A}\cdot\vec{p}$

Coulomb gauge: $\vec{p} = \left[\vec{r}, \frac{p^2}{2m}\right]\frac{m}{i\hbar}$

$$\vec{p} = \frac{m}{i\hbar}[\vec{r}, H_0] = \frac{m}{i\hbar}[\vec{r}H_0 - H_0\vec{r}]$$

$$\Rightarrow \langle f|\vec{p}|i\rangle = \frac{m}{i\hbar}\underbrace{(E_i^0 - E_f^0)}_{-\hbar\omega}\langle f|\vec{r}|i\rangle$$

$$\Rightarrow H_{A\cdot p} = -\frac{q}{2mc}\vec{A}_0 e^{-i\omega t}\cdot\vec{p}$$

$$= -\frac{q}{2mc}\vec{A}_0 e^{-i\omega t}\cdot\frac{m}{i\hbar}[\vec{r}, H_0]$$

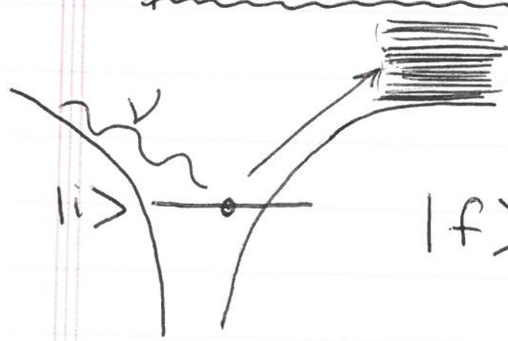
$$\Rightarrow \underline{\langle f|H_{A\cdot p}|i\rangle} = -\frac{q}{2mc}\vec{A}_0 e^{-i\omega t}\, im\omega\cdot\langle f|\vec{r}|i\rangle$$

$$= \frac{q}{c}\partial_t(\vec{A}_0 e^{-i\omega t})\cdot\langle f|\vec{r}|i\rangle$$

$$= \underline{\langle f|-\vec{E}\cdot q\vec{r}|i\rangle} \checkmark$$

4. Full transition rate $R_{i\to ?}$ constructed from $R_{i\to f}$ depending on physical situation:

A. <u>photoemission of e</u> (photoelectric effect (Einstein 1905))



} continuum states for $H_0 = \frac{p^2}{2m} - \frac{q^2}{r}$

$|f\rangle \simeq |p_f\rangle$ + corrections.

$$\Updownarrow$$

$$\psi_{p_f}(r) \equiv \langle r|p_f\rangle = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\vec{p}_f\cdot\vec{r}}$$

$|i\rangle$ – one of bound states of interest e.g. $|nlm\rangle$

e.g. for Hydrogen $\langle r|i\rangle = \psi_{100}(r) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$

Must sum over final states of ejected e.

$$R_{i\to d\Omega_{p_f}} = \sum_{p_f} R_{i\to p_f}$$

$$R_{|100\rangle\to|p_f\rangle} = \frac{2\pi}{\hbar}\frac{q^2}{4m^2c^2}\left|\langle p_f|\vec{A}_0\cdot\vec{p}\,|100\rangle\right|^2\delta(E_{p_f}-E_0-\hbar\omega)$$

$$\langle p_f|\vec{A}_0\cdot\vec{p}\,|100\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{\pi a_0^3}}\int e^{-ip_f\cdot r/\hbar}\vec{A}_0\cdot(-i\hbar\nabla)e^{-r/a_0}d^3r$$

$$\underset{\substack{\text{integrate}\\\text{by parts}}}{=} \frac{1}{\sqrt{Va_0^3\pi}}\vec{A}_0\cdot\vec{p}_f\underbrace{\int e^{-ip_f\cdot r/\hbar - r/a_0}d^3r}_{\frac{8\pi/a_0}{\left[(1/a_0)^2+(p_f/\hbar)^2\right]^2}}$$

$$\Rightarrow R_{|100\rangle\to d\Omega_{p_f}} = \sum_{p_f}\frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2\frac{1}{Va_0^3\pi}\frac{|\vec{A}_0\cdot\vec{p}_f|^2(8\pi/a_0)^2}{[\quad]^4}$$

$$= \int_{d\Omega_{p_f}}\frac{d^3p_f}{(2\pi\hbar)^3}\frac{32\pi^2e^2a_0^3|\vec{A}_0\cdot\vec{p}_f|^2\delta\left(\frac{p_f^2}{2m}-E_0-\hbar\omega\right)}{\hbar m^2c^2\left[1+(p_fa_0/\hbar)^2\right]^4}$$

$$= \frac{m}{p_f}\delta\left(p_f-\sqrt{2m(E_i+\hbar\omega)}\right)$$

$$R_{|100\rangle\to d\Omega_{p_f}} = \int d\Omega_{p_f}\frac{4a_0^3e^2p_f^3|A_0|^2\cos^2\theta}{m\pi\hbar^4c^2\left[1+(p_fa_0/\hbar)^2\right]^4} = \frac{2\pi}{\hbar}|H'_{fi}|^2mp_f\,d\Omega.$$

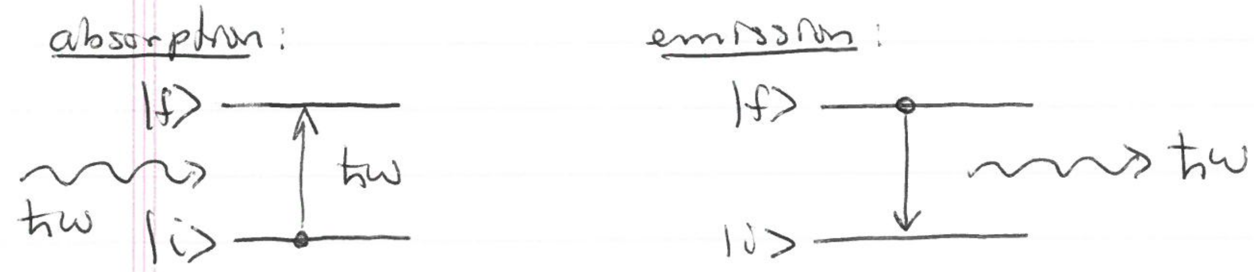
$$R_{|100\rangle\to all} = \int d\Omega\, R_{\to d\Omega}\, d\phi\, d(\cos\theta) \qquad |\vec{S}| = \frac{c}{4\pi}|E\times B|$$

$$\frac{dE_{abs}}{dt} = \hbar\omega\cdot R_{i\to all} \equiv \int d\sigma_{abs}\underbrace{\frac{\omega^2}{8\pi c}|A_0|^2}$$

$$\Rightarrow \boxed{d\sigma_{abs} = d\Omega_{p_f}\frac{32a_0^3e^2p_f^3\cos^2\theta}{m\omega\hbar^3c\left[1+(p_fa_0/\hbar)^2\right]^4}}$$

where $p_f = \sqrt{2m(E_0+\hbar\omega)}$

B. Transitions between discrete energy levels (as in a laser)



$$dR_{|i\rangle\to|f\rangle} = \frac{2\pi}{\hbar}\frac{e^2}{4m^2c^2}\left|\langle f|\vec{A}_0\cdot\vec{p}\,|i\rangle\right|^2\delta(E_f-E_i-\hbar\omega)$$

$$= \frac{2\pi e^2}{4m^2c^2\hbar}\left|\langle f|\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2|A_0|^2\delta(E_f-E_i-\hbar\omega)$$

$$\vec{A} = A_0\hat{\varepsilon}\cos(k\cdot r-\omega t)$$

$$\Rightarrow \underset{\text{at } \omega,\ \omega+d\omega}{\text{Intensity}} = \frac{\text{Power}}{\text{Area}} \equiv I(\omega)d\omega = |\vec{S}| = \frac{c}{4\pi}|\vec{E}\times\vec{B}|$$

$$\vec{E}:\ \frac{1}{c}\partial_t A \to \frac{i\omega}{c} \qquad \vec{B}:\ \nabla\times A \to ik \to \frac{\omega}{c}$$

$$I(\omega)d\omega = \frac{\omega^2}{8\pi c}|A_0|^2$$

$$\Rightarrow dR^{\omega}_{|i\rangle\to|f\rangle} = \frac{4\pi^2e^2}{m^2c\hbar\omega^2}I(\omega)d\omega\left|\langle f|\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2\delta(E_f-E_i-\hbar\omega)$$

$$R_{|i\rangle\to|f\rangle} = \int dR^{\omega}_{|i\rangle\to|f\rangle} = \frac{4\pi^2e^2}{m^2c(\hbar\omega)^2}\underset{\substack{\uparrow\\ \text{power spectrum}\\ \text{of radiation}\\ \omega=\omega_{fi}}}{I(\omega)}\underbrace{\left|\langle f|\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2}_{\propto(\hbar\omega)^2}$$

where $\hbar\omega = E_f - E_i$

$$R_{|i\rangle\to|f\rangle} \equiv B_{fi}\, I(\omega_{fi})$$

$\uparrow$ Einstein's B coefficient.

$$R_{|f\rangle\to|i\rangle} = B_{if}\, I(\omega_{fi})\ (+\ A)$$

missing in classical E&M theory.

Einstein's A & B coefficient:

in equilibrium:

$$\underbrace{n_i R_{|i\rangle \to |f\rangle}}_{\text{absorption}} = \underbrace{n_f R_{|f\rangle \to |i\rangle}}_{\text{emission}}$$

$$n_i B I(\omega_{fi}) = n_f \big(\underbrace{B I(\omega_{fi})}_{\substack{\text{stimulated}\\ \text{emission}}} + \underbrace{A}_{\substack{\text{spontaneous}\\ \text{emission}}} \big)$$

$n_{i,f}$ – # of atoms in state $|i\rangle, |f\rangle$ at temperature T

$$\frac{n_f}{n_i} = e^{-(E_f - E_i)/k_B T}$$

$$B I(\omega_{fi}) \left(1 - \frac{n_f}{n_i}\right) = \frac{n_f}{n_i} A$$

$$\underline{I(\omega_{fi})} = \frac{A/B}{\frac{n_i}{n_f} - 1} = \underline{\frac{(A/B)}{e^{E_{fi}/k_B T} - 1}}$$

Bose – Einstein distribution: # of photons with freq. ω.

$$\int I(\omega)d\omega = \int_k \hbar\omega_k \, n_{BE}(\omega_k) \frac{d^3k}{(2\pi)^3} c \equiv \int dS = \frac{\text{Power}}{\text{Area}}$$

(Intensity)

$$I(\omega)d\omega = \frac{\hbar\omega c\, \omega^2}{8\pi^3 c^3} \frac{4\pi\, d\omega}{e^{\hbar\omega/k_B T} - 1}$$

$$\Rightarrow A/B = \frac{\hbar\omega^3}{2\pi^2 c^2}$$

To have laser action (amplification), need population inversion $n_f \gg e^{-E_{fi}/T} n_i$

- E&M field quantization: QFT (QED)

→ unification: quantum treatment of radiation as well as matter

→ experiments: finite rate of spontaneous (free space) emission/decay rate in atoms. eg $|2\ell m\rangle \to |100\rangle$ $R \simeq 10^9\ \mathrm{sec}^{-1}$, due to quantum nature of $\vec{A}$, i.e. zero-point fluctuations of E&M field cf. oscillator $\langle x\rangle = \langle p\rangle = 0$ but not $\langle \Delta x^2\rangle, \langle \Delta p^2\rangle \neq 0$. (definite $\vec{A}$ is not an eigenstate)

→ canonical quantization of a field $A(\vec{r})$:

▲ field $A(\vec{r})$ – infinite # of d.o.f., one at each point in space, best thought of as discrete. ($\vec{r}$ is just a label for these d.o.f., not operator)

▲ identify canonical momentum field $\Pi(\vec{r})$ through Lagrangian:

$$\Pi(\vec{r}) = \frac{\delta \mathcal{L}}{\delta(\partial_t A(\vec{r}))}$$

▲ impose canonical commutation relation

$$[\hat{A}(\vec{r}), \hat{\Pi}(\vec{r}\,')] = i\hbar\,\delta(\vec{r}-\vec{r}\,')$$

▲ express $H[\hat{A}, \hat{\Pi}]$ and decouple (if necessary) via canonical transformation $(\hat{A}, \hat{\Pi} \to \hat{A}', \hat{\Pi}')$ so that H is a sum of independent oscillators $\Rightarrow H = \sum_k \hbar\omega_k a_k^+ a_k$.

see HW4, prob. 4, for a typical example, demostrated through phonons.

▲ E& M field (as other gauge fields) is slightly more complicated due to gauge-fixing constraint

→ Path-integral quantization of $A(r)$:

$$U(A_f(\vec{r}), t_f; A_i(\vec{r}), t_0) = \mathcal{N} \int_{A_i(r)}^{A_f(\vec{r})} \mathcal{D}A(\vec{r},t)\, e^{\frac{i}{\hbar} S[A(\vec{r},t)]}$$

→ Fields "coordinates" $\phi(r)$, $\vec{A}(r)$
(complication: not fully physical due to gauge freedom)

→ conjugate momentum field:

$$L = \frac{1}{8\pi} \int (|E|^2 - |B|^2) d^3r = \frac{1}{8\pi} \int (F_{\mu\nu})^2$$

$$= \frac{1}{8\pi} \int \left(\left| -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi \right|^2 - |\vec{\nabla} \times \vec{A}|^2 \right) d^3r$$

$$\frac{\delta L}{\delta \partial_t \vec{A}} = \vec{\Pi}_A = \frac{1}{4\pi c} \left(\frac{1}{c}\partial_t \vec{A} + \vec{\nabla}\phi \right)$$

$$\Rightarrow \boxed{\vec{\Pi}_A = -\frac{1}{4\pi c} \vec{E}} \Rightarrow \vec{E}, \vec{A} \text{ canonically conjugate!}$$

$$\frac{\delta L}{\delta \partial_t \phi} = \Pi_\phi = 0 \quad \Rightarrow \text{ no dynamics for } \phi$$

just a constraint:

L does not depend on $\partial_t \phi$.

$\Rightarrow \Pi_\phi = 0$.

$$\Rightarrow \underbrace{\frac{d}{dt}\frac{\delta L}{\partial_t \phi}}_{=0} - \frac{\delta L}{\delta \phi} = 0$$

$$\Rightarrow \frac{\delta L}{\delta \phi} = \frac{1}{4\pi}\left(-\nabla^2 \phi - \vec{\nabla}\cdot \frac{1}{c}\partial_t \vec{A}\right) = 0$$

- $\underline{\vec{\nabla}\cdot \vec{E} = \nabla \cdot \Pi_A = 0}$ ← (Coulomb's law)

- $\dfrac{d}{dt}\dfrac{\delta L}{\delta \partial_t \vec{A}} - \dfrac{\delta L}{\delta \vec{A}} = 0$

$$\Rightarrow \underline{\vec{\nabla}\times\vec{B} + \frac{1}{c}\partial_t \vec{E} = 0} \leftarrow \text{(Ampere's law)} \quad (4\pi j)$$

$$\Leftrightarrow \nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t \phi\right) = 0$$

- $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t B = 0$ automatically satisfied due to $\vec{A}$

<u>Note:</u> Ampere's law ⇒ by taking $\nabla \cdot$

$$\cancel{\nabla^2 \nabla\cdot A} - \frac{1}{c^2}\partial_t^2 \nabla\cdot A - \nabla^2(\cancel{\nabla\cdot A} + \frac{1}{c}\partial_t \phi) = 0$$

$$\Rightarrow \partial_t \left[\nabla\cdot\left(\frac{1}{c}\partial_t A + \nabla\phi\right)\right] = 0$$

i.e. constraint $\nabla\cdot \Pi_A = -\frac{1}{4\pi c}\nabla\cdot E = \underline{\text{const}}$

const. specified by (background) charge density $4\pi\rho(\vec{r})$ of motion

Coulomb's law → const. of motion, constraint on longit. comp.

- implement constraints: $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{\Pi}_A = 0$ in Fourier space, i.e. $\vec{A}$ & $\vec{\Pi}_A$ are transverse to $\vec{k}$ (pick gauge $\phi = 0$)

$$\Rightarrow \quad [A_i^T(\vec{r}), \Pi_j^T(\vec{r}\,')] = i\hbar\, \delta_{ij}^T\, \delta(\vec{r}-\vec{r}\,')$$

$$A_i^T(k) \equiv \underbrace{(\delta_{ij} - \hat{k}_i \hat{k}_j)}_{P_{ij}^T(k)} A_j$$

(T ← transverse components only)

$P_{ij}^T(k)$ — transverse (to $\vec{k}$) projection operator.

$$H_{EM} = \int d^3r \left[\vec{\Pi}\cdot\partial_t \vec{A} - L\right]$$

$$H_{EM} = \frac{1}{8\pi}\int d^3r \left[16\pi^2 c^2 |\Pi|^2 + |\nabla\times A|^2\right] = \frac{1}{8\pi}\int d^3r \left[E^2 + (\nabla\times A)^2\right]$$

$$\text{cf.} \quad H_{oscill.} = \left(\frac{1}{2m}\right)p^2 + \left(\frac{1}{2}m\omega^2\right)q^2$$

$$H_{EM} = \frac{1}{8\pi}\int d^3r \left[16\pi^2 c^2\, \Pi_i^\dagger \Pi_i + A_i^\dagger \underbrace{(-\nabla^2)}_{} A_j\right]$$

∇^2 couples A's at different points in space; $\vec{A}(\vec{r})$ not normal modes

- Normal modes, decouple oscillators are Fourier coeff. of $\vec{A}(\vec{r})$, $\vec{\Pi}(\vec{r})$:

$$\vec{A}(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\vec{A}}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{\Pi}(\vec{r}) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\vec{\Pi}}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$[A_i^T(r), \Pi_j^T(r')] = i\hbar\, \delta_{ij}^T\, \delta(\vec{r}-\vec{r}\,') \Rightarrow \underline{[\tilde{A}_i^T(\vec{k}), \tilde{\Pi}_j^T(\vec{k}\,')] = i\hbar\, \delta_{ij}^T (2\pi)^3 \delta(\vec{k}+\vec{k}\,')}$$

<u>Note</u>: b.c. $\vec{A}(\vec{r})$ is a real field, Hermitian,

$$\vec{A}^{+}(r) = \vec{A}(r) \Rightarrow \underline{\tilde{\vec{A}}^{+}(k) = \tilde{\vec{A}}(-k)}$$, i.e. $k, -k$ not independent complex fields.

In terms of $\tilde{\vec{A}}(k)$, $\tilde{\vec{\Pi}}(k)$ H_{EM} decouples:

$$H_{EM} = \int \frac{d^3k}{(2\pi)^3} \left[\underbrace{(2\pi c^2)}_{\equiv \alpha/2} \tilde{\Pi}_i^{+}(k)\, \tilde{\Pi}_i(k) + \underbrace{\left(\frac{k^2}{8\pi}\right)}_{\equiv \beta/2} \tilde{A}_i^{+}(k)\, \tilde{A}_i(k) \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar \sqrt{\alpha\beta} \left[\frac{1}{\hbar}\sqrt{\frac{\alpha}{\beta}}\, \tilde{\Pi}_i^{+} \tilde{\Pi}_i + \frac{1}{\hbar}\sqrt{\frac{\beta}{\alpha}}\, \tilde{A}_i^{+} \tilde{A}_i \right]$$

preserves $[a, a^{+}] = 1$ relation

- Introduce creation & annihilation ops.

$$\vec{a}_k = \frac{1}{\sqrt{2}} \left(\frac{1}{\hbar^{1/2}} \left(\frac{\beta}{\alpha}\right)^{1/4} \vec{A}_k + i \left(\frac{\alpha}{\beta}\right)^{1/4} \frac{1}{\hbar^{1/2}} \vec{\Pi}_k \right)$$

$$\vec{a}_k^{+} = \frac{1}{\sqrt{2}} \left(\frac{1}{\hbar^{1/2}} \left(\frac{\beta}{\alpha}\right)^{1/4} \vec{A}_k^{+} - i \left(\frac{\alpha}{\beta}\right)^{1/4} \frac{1}{\hbar^{1/2}} \vec{\Pi}_k^{+} \right)$$

$\Rightarrow \vec{A}_k = (\vec{a}_k + a_{-k}^{+}) \frac{\hbar}{2}$

$\vec{\Pi}_k = (a_{-k}^{+} - a_k) \frac{\hbar}{2i}$

$$\Rightarrow \underline{[a_{ik}, a_{jk'}^{+}] = \delta_{ij}\, \delta(\vec{k} - \vec{k}')}$$

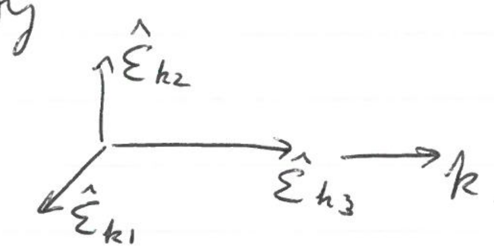
$$\Rightarrow H_{EM} = \sum_i \int \frac{d^3k}{(2\pi)^3} \hbar \underbrace{ck}_{\omega_k} \frac{1}{2} \left[a_{ik} a_{ik}^{+} + a_{ik}^{+} a_{ik} \right]$$

$$H_{EM} = \sum_i \int \frac{d^3k}{(2\pi)^3} \hbar \omega_k \left[a_{ik}^{+} a_{ik} + \frac{1}{2} (2\pi)^3 \delta(k - k = 0) \right]$$

$$\underline{H_{EM} = \sum_i \int \frac{d^3k}{(2\pi)^3} \hbar \omega_k\, a_{ik}^{+} a_{ik}} + \underbrace{\frac{1}{2} \sum_k \hbar \omega_k}_{\text{zero-point energy}}\, \underset{\text{2 transv. modes}}{2}$$

- impose transversality

$$\vec{a}_k = \sum_{\lambda=1}^{3} a_{k\lambda}\, \hat{\varepsilon}_{k\lambda}$$



$\Rightarrow$ take $a_{k3} = 0$

$$H_{EM} = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3}\, \hbar\omega_k\, a^+_{k\lambda} a_{k\lambda} + \sum_k \hbar\omega_k$$

- states of EM field:

$|\{n_{k\lambda}\}\rangle$ where $a^+_{k\lambda} a_{k\lambda} |\{n_{k\lambda}\}\rangle = n_{k\lambda} |\{n_{k\lambda}\}\rangle$

vacuum : $|0\rangle$

$\langle 0|A|0\rangle \sim \langle 0|\, a + a^+ |0\rangle = 0$;

$\langle 0|\Pi|0\rangle \sim \langle 0|\, a - a^+ |0\rangle = 0$

but $\langle 0|A^2|0\rangle$; $\langle 0|\Pi^2|0\rangle \neq 0$.

state of laser field is coherent state
$\hat{E}|\vec{\mathcal{E}}\rangle = \vec{\mathcal{E}}|\vec{\mathcal{E}}\rangle$

$a^+_{k\lambda}|0\rangle = |k\lambda\rangle$

creates a photon with momentum $\hbar\vec{k}$, polariz. $\hat{\varepsilon}_\lambda$.

$$\vec{P} = \frac{1}{4\pi c} \underbrace{\int d^3r\, (\vec{E}\times\vec{B})}_{\frac{1}{c^2}\int d^3r\, \vec{S}} \leftarrow \text{momentum operator.}$$

$$= \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\, a^+_{k\lambda} a_{k\lambda}\, \hbar\vec{k}$$

$\vec{P}|k\lambda\rangle = \hbar\vec{k}|k\lambda\rangle$

$\Leftrightarrow |\{n_k,\lambda\}\rangle$

$a^{+}_{k_1\lambda_1} a^{+}_{k_2\lambda_2} \dots a^{+}_{k_N\lambda_N}|0\rangle \equiv \underbrace{|k_1\lambda_1, k_2\lambda_2, \dots\rangle}_{n \text{ photon state}}$

$\vec{P}|\{k_i\lambda_i\}\rangle = \overbrace{\left(\sum_k \hbar\vec{k}_i\right)}^{\sum_k n_i \hbar k}|\{k_i\lambda_i\}\rangle$

$H|\{k_i\lambda_i\}\rangle = \left(\sum_k n_k \hbar\omega_k\right)|\{k_i\lambda_i\}\rangle$

$\langle k_1\lambda_1|k_2\lambda_2\rangle = \delta_{\lambda_1\lambda_2}(2\pi)^3\delta(k_1-k_2)$

$\langle r, s_i|k\lambda\rangle = \dfrac{\hat{\varepsilon}_{k\lambda}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$

$\vec{k} \parallel \hat{\varepsilon}_{k3} \Leftrightarrow S_z = 0$ state since $e^{\frac{iL_z\varphi}{\hbar}}$ mvmt

$S_z = \pm\hbar$, $S_z = 0$ is not an allowed state due to $\vec{\nabla}\cdot\vec{A} = 0$ constraint $\Leftrightarrow \vec{k}\cdot\vec{A}_k = 0$

$\uparrow$

η

$\Leftrightarrow$ only helicity of $\pm\hbar$ allowed

i.e., positive & negative circularly polarized.

6.23

- Emission / Absorption.

label, operator

$$H_1 = \frac{e}{mc}\hat{\vec{A}}\cdot\hat{\vec{p}} = \left(\frac{e}{mc}\int d^3r\, \hat{\vec{A}}(\vec{r})\,\delta(\vec{r}-\hat{\vec{r}})\cdot\hat{p} \right.$$
$$= \frac{e}{mc}\sum_k \hat{\vec{A}}_k\, e^{i\vec{k}\cdot\hat{\vec{r}}}\cdot\hat{p}$$

state of EM field

Emission: $|i^0\rangle = |2lm\rangle \otimes |\{n_{k_i\lambda_i}\}\rangle$

electronic state

$|f^0\rangle = |100\rangle \otimes |\{n_{k_i\lambda_i} + \underbrace{\delta_{k_i\lambda_i,k\lambda}}\}\rangle$

increase occupation of mode $k\lambda$ by 1.

$$E_f^0 - E_i^0 = (E_{100}+\hbar\omega) - (E_{2lm})$$

Fermi's golden rule:

$$R_{2lm\to100} = \frac{2\pi}{\hbar}\left|\langle f^0|\frac{e}{mc}\vec{A}\cdot\hat{p}|i^0\rangle\right|^2\delta(E_{100}+\hbar\omega-E_{2lm})$$

$$\vec{A} = \sum_\lambda\int_k\left(\frac{\hbar c^2}{4\pi^2\omega_k}\right)^{1/2}\left[a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{ik\cdot r} + h.c.\right]$$

$$= \frac{2\pi}{\hbar}\left|\left(\frac{\hbar c^2}{4\pi^2\omega_k}\right)^{1/2}\int d^3r\, \psi_{100}^* e^{-ik\cdot r_\lambda}\varepsilon_{k\lambda}^*\cdot(-i\hbar\vec{\nabla})\psi_{2lm}\cdot\right.$$
$$\left.\cdot\sqrt{n_{k\lambda}+1}\right|^2\delta(\quad) \propto (n_{k\lambda}+1)$$

Absorption:

$$R_{100\to2lm} = \frac{2\pi}{\hbar}\left|\left(\frac{\hbar c^2}{4\pi^2\omega_k}\right)^{1/2}\int d^3r\,\psi_{2lm}^* e^{ik\cdot r_\lambda}\varepsilon_{k\lambda}\cdot(-i\hbar\vec{\nabla})\psi_{100}\cdot\right.$$
$$\left.\cdot\sqrt{n_{k\lambda}}\right|^2\delta(\ ) \propto (n_{k\lambda})$$

$$R^{\text{emission}} - R^{\text{absorp.}} \propto (n_{k\lambda}+1) - (n_k) \neq 0.$$

spontaneous emission.

For more details see Shankar, pg 519.

→ dipole approx $\hat{\varepsilon}\cdot\vec{p} \rightarrow \hat{\varepsilon}\cdot\vec{r}\, im\omega$

→ selection rule $l=1$ (parity)

→ $\varepsilon\cdot r = \sum_{-1}^{1} (-1)^q \varepsilon_1^q r_1^{-q} = -\varepsilon_1^1 r_1^{-1} + \varepsilon_1^0 r_1^0 - \varepsilon_1^{-1} r_1^{+1}$

$\varepsilon_1^1 = -\frac{(\varepsilon_x + i\varepsilon_y)}{\sqrt{2}}$, $\quad r_1^{\mp 1} = \left(\frac{4\pi}{3}\right)^{1/2} r\, Y_1^{\mp}$, $\quad \varepsilon_1^0 = \varepsilon_z$, $\quad r_1^0 \sim Y_1^0\, r$

→ average over 3 state $m=\pm 1, 0$ in initial electronic state.

→ integrate over final photon momenta k & 2 polarizations, $\lambda = 1, 2$.

$$\int \delta(E_{100} + \underbrace{\hbar\omega}_{kc} - E_{2lm})\, k^2 dk\, d\Omega = \frac{4\pi k^2}{\hbar c}$$

with $k = \frac{1}{\hbar c}\underbrace{(E_{21m} - E_{100})}_{\frac{e^2}{2a_0}\left(1-\frac{1}{4}\right)}$

$k = \frac{3e^2}{8a_0\hbar c}$

$$\Rightarrow \boxed{R_{i\to all} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar} \equiv \frac{1}{\tau} \simeq 0.6 \times 10^9 \text{ sec}^{-1}}$$

✓