Lecture 1:

Key ideas in Quantum mechanics

- wave nature of matter with

state of system $\rightarrow \Psi(\vec{r}, t)$ – amplitude (complex in general)

Born interprett. $\rightarrow |\Psi(\vec{r}, t)|^2 =$ Intensity $\propto$ Prob$(\vec{r}, t)$ density.

$\rightarrow$ modified by measurement

dispersion $\rightarrow E_p = \frac{p^2}{2m}, \quad p = \hbar k = \frac{h}{\lambda}$

$$\Rightarrow \quad \Psi(x) = \int dk\, \tilde{\Psi}_k\, e^{ikx}$$

$P(x,t) \neq \delta(x - x(t))$

"particle" is distributed in space

all wave like phenomena follow!

$\Delta k \gtrsim \frac{1}{\Delta x}$

$$\Delta x \Delta k \gtrsim 1 \iff \Delta x \Delta p \gtrsim \hbar$$

$\Rightarrow$ Heisenberg (1927) uncertainty principle & Bohr's complementery principle for mutually exclusive (canonically conjugate) observables.

$$\Delta x \Delta p \gtrsim \hbar \quad , \quad \Delta \varphi \Delta L \gtrsim \hbar, \quad \Delta t \Delta E \gtrsim \hbar$$

$\rightarrow$ interference & superposition

$$\Psi_{12} = \Psi_1 + \Psi_2 \Rightarrow |\Psi_{12}|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2\mathrm{Re}\,\Psi_1^*\Psi_2$$

$\Rightarrow P_{12} \gtrless P_1 + P_2$
(double slit exp)

E&M field

$$\nabla \times H = \frac{1}{c}\partial_t E, \quad \nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \nabla \cdot E = \nabla \cdot B = 0$$

$$F = E + iH, \quad \bar{F} = E - iH.$$

$$\Rightarrow \nabla \times F = \frac{i}{c}\partial_t F, \quad \nabla \times \bar{F} = -\frac{i}{c}\partial_t \bar{F}$$

$$\nabla \cdot F = \nabla \cdot \bar{F} = 0$$

$$\nabla \times F = -i\,\partial_\beta(-i\,\epsilon_{\beta\alpha\gamma})F_\gamma$$

$$\Rightarrow \quad p_\beta = -i\hbar\partial_\beta$$

$$\Rightarrow (p \cdot S)F = i\hbar\partial_t F$$

where $S^\beta_{\alpha\gamma} = -i\,\epsilon_{\beta\alpha\gamma}$ (3 x 3 matrix).

spin 1 operator.

(3.1a)

All these experiments/"paradoxes" can be understood with one leap

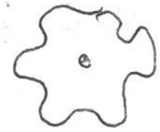
particle is a wave with:

- amplitude: $\psi(r,t)$ ← field
- dispersion: $\omega_k = \dfrac{\hbar k^2}{2m}$
- intensity $|\psi|^2$ = prob. density of finding particle at point (r,t)

Everything follows:

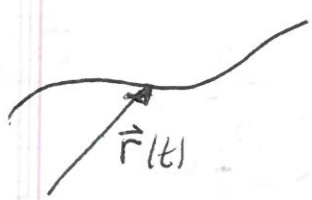
→ interference/diffraction $\psi_{12} = \psi_1 + \psi_2 \Rightarrow P_{12} = P_1 + P_2$

→ spectrum quantization like E&M modes in a cavity, notes on a guitar string

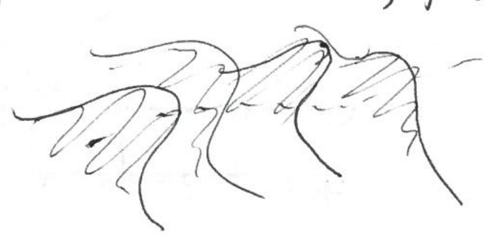


→ classical physics as ray optics, $\lambda/d \to 0$ limit, Sch. Eqn → H-J Eqn.

particles are wave packets at $\vec{r}(t)$, $\vec{p}(t) = -i\hbar\vec{\nabla}\psi$



vs.



$P(\vec{r},t) = \delta^3(\vec{r} - \vec{r}(t))$

$P(\vec{r},t)$ arbitrary fnc of $\vec{r}, t$.

• Wave equation for matter

[Schrödinger, Ann. Physik $\underline{79}$, 361, 489 (1926)
$\underline{81}$, 109 (1926)]

"Photon wave" $\vec{E}, (\vec{B})$ obeys Maxwell eqns

i.e. $\partial_t^2 \vec{E} - c^2 \nabla^2 \vec{E} = 0$

massless limit of $E_p = \sqrt{p^2c^2 + m^2c^4}$

$$E \sim e^{i\vec{k}\cdot\vec{r} - i\omega t} \iff \omega_k^2 = c^2 k^2 \iff E_p = cp$$

$$(\hbar\omega_k) = c(\hbar k)$$

wave eqn for matter wave $\Psi(\vec{r}, t)$?

nonrel. limit of $E_p = mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2} \simeq$

look at dispersion $E_p = \dfrac{p^2}{2m}$ (non relativistic)

$$\iff \hbar\omega_k = \frac{\hbar^2 k^2}{2m}$$

$\Rightarrow$ Schrödinger's Eqn for $\Psi(\vec{r}, t) \sim e^{i\vec{k}\cdot\vec{r} - i\omega t}$

free particle

$$\boxed{i\hbar\,\partial_t \Psi = -\frac{\hbar^2\nabla^2}{2m}\Psi}$$

In a potential:

$$\boxed{i\hbar\,\partial_t \Psi = \underbrace{\left(-\frac{\hbar^2\nabla^2}{2m} + V(r)\right)}_{\frac{\hat{p}^2}{2m} + V(\hat{r}) \;=\; \hat{H}(\hat{p}, \hat{r})}\Psi}$$

$$\underbrace{i\hbar\,\partial_t \Psi = \hat{H}\Psi}_{\text{must be linear}}, \quad \text{Hermitian } H^\dagger = H$$

$\Psi \sim e^{iS/\hbar} \Rightarrow -\partial_t S = H$ H-J eqn

Schrödinger's Eqn plausability

(ultimate justification: <u>experiments</u>)

- correct classical limit (correspondence principle)

▲ $\hbar \omega_k = \frac{\hbar^2 k^2}{2m} \Rightarrow E_p = \frac{p^2}{2m}$

▲ $\Psi \sim e^{i\vec{k}\cdot\vec{r} - i\omega t} \Rightarrow \Psi \sim e^{iS/\hbar}$ ⟵ rays $\frac{\nabla k}{k} \ll 1$

$\left(-\frac{\hbar^2 \nabla^2}{2m} + V\right)\Psi = i\hbar \partial_t \Psi$ with $\omega \simeq -\frac{1}{\hbar}\partial_t S$

$\vec{k} \simeq \frac{1}{\hbar}\vec{\nabla} S$

$-\frac{i\hbar}{2m}\nabla^2 S + \boxed{\frac{(\vec{\nabla} S)^2}{2m} + V = -\partial_t S}$ ⟵ $\frac{\hbar \nabla^2 S}{(\nabla S)^2} = \frac{\nabla k}{k}$

$\hbar \to 0 \Rightarrow$ Hamilton-Jacobi Eqn

with S - Hamiltons characteristic fnc

= classical action

$\Longleftrightarrow$ Newton's Eqns.

- interference of matter waves (e- diffraction)
  Davisson-Germer.

- Heisenberg uncertainty

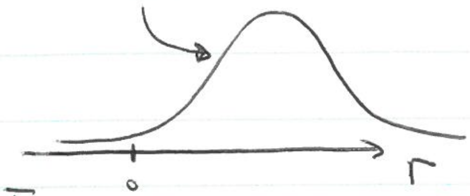
... more later via connection to Heisenberg & Feynman's formulation with their simple classical limits.

Important points:

- in measurement of observerable $\hat{O}$ only one of eigenvalues of $\hat{O}$, o_n can be found !!! → very strange if $\hat{O}$ has discrete eigenvalues.
(cf sound with only discrete set of frequencies / colors)
- overall constant factor in Ψ has no physical content, including overall phase factor $e^{i\varphi}$.

- Physical observables $\leftrightarrow$ q.m. operators:

$\rightarrow$ prob. density at $\vec{r}$ — $P(\vec{r}) = |\Psi(\vec{r})|^2$



$\rightarrow$ average position:

$$\langle \hat{\vec{r}} \rangle = \int d^3r \; \vec{r} \; \Psi^*(\vec{r}) \Psi(\vec{r})$$

$$\langle V(\hat{\vec{r}}) \rangle = \int d^3r \; V(\vec{r}) \Psi^*(\vec{r}) \Psi(\vec{r})$$

$\rightarrow$ momentum:

$\vec{p}_k = \hbar \vec{k}$ for k-plane wave $\Psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r} - i\omega t} \tilde{\Psi}_k$

$$\langle \vec{p} \rangle = \sum_k \vec{p}_k |\tilde{\Psi}_k|^2 = \sum_k \hbar \vec{k} \, \tilde{\Psi}^*(k) \tilde{\Psi}(k)$$

$$\langle \vec{p} \rangle = \int d^3r \; \Psi^*(\vec{r}) \left(-i\hbar \vec{\nabla}\right) \Psi(\vec{r})$$

$$\Rightarrow \boxed{\hat{\vec{p}} = -i\hbar \vec{\nabla}}$$

cf $\vec{J} = \frac{-i\hbar}{2m}(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*) = \mathrm{Re}\left(\psi^* \frac{\hat{\vec{p}}}{m} \psi\right)$

$\rightarrow$ energy: $\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(\vec{r})$

$\rightarrow$ angular momentum:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

etc.

$$\langle \hat{\Theta} \rangle = \int d^3r \; \Psi^*(r) \, \Theta \, \Psi(r)$$

<u>Note:</u> $[\hat{r}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

i.e. some operators do <u>not</u> commute!
act like matrices.

compare to CM: $\{r_i, p_j\}_{PB} = 1\,\delta_{ij}$

suggests connection of $[A, B]\frac{1}{i\hbar} \leftarrow \{A, B\}_{P.B.}$

... more on this later when discuss
Heisenberg formulation of q.m.

<u>Also</u>: $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$ (i.e. $[\hat{L}_x, \hat{L}_y] = i\hbar L_z$)

(check for hwk.)

cf. $\{L_i, L_j\}_{P.B.} = \epsilon_{ijk} L_k$.

<u>cf vector</u> <u>vs. its</u> <u>components</u>

<u>operators</u> are <u>physical</u> quantities (cf tensors)
→ that have many equivalent representations (matrices)
→ in some repres. is diagonal
but not in another.

$V(\hat{r})$ diagonal in r-representation
$T(\hat{p})$ diagonal in p-representation.

operators that do not commute, cannot be
diagonalized simult. ⇒ incompatible observables
→ Heisenberg uncert. principle

More generally: operator $\leftrightarrow$ matrix.

depends on basis.

- Coordinate representation:

$$\rightarrow \left(\hat{\vec{r}}\right)_{r,r'} = \vec{r}\,\delta(\vec{r}-\vec{r}\,')$$

$$\rightarrow \left(\hat{\vec{p}}\right)_{r,r'} = \delta(\vec{r}-\vec{r}\,')\left(-i\hbar\vec{\nabla}_{r'}\right)$$

etc.

with $\langle\hat{O}\rangle = \int d^3r\, d^3r'\, \Psi^*(r')\, O_{r'r}\, \Psi(r)$

$V(\hat{\vec{r}})$ - diagonal in $\vec{r}$ representation.

$\hat{\vec{p}}$ - diagonal in $\vec{p}$ (k) repres.

- momentum representation:

$$\langle\hat{O}\rangle = \int d^3k\, d^3k'\, \tilde{\Psi}^*(k')\, O_{k'k}\, \tilde{\Psi}(k)$$

$$\int d^3r\, d^3r'\, e^{-ik'\cdot r'}\, O_{r'r}\, e^{ik\cdot r}$$

$$\Rightarrow \left(\hat{\vec{r}}\right)_{p'p} = \int d^3r\, e^{i(k-k')\cdot r}\,\vec{r} = i\vec{\nabla}_{k'}\delta(\vec{k}-\vec{k}')$$

$$= 2\pi\hbar^3\delta^{(3)}(p-p')\left(i\hbar\vec{\nabla}_{\vec{p}\,'}\right)$$

$$\Rightarrow \left(\hat{\vec{p}}\right)_{p',p} = \delta^{(3)}(\vec{p}-\vec{p}\,')\,\vec{p}$$

(3.7a)

- collapse of Ψ

$$|\psi\rangle = \sum_{\omega} \langle\omega|\psi\rangle |\omega\rangle$$

$$\xrightarrow[\text{measure } \hat{\Omega} \text{ and find } \omega_1]{} |\psi\rangle' = |\omega_1\rangle$$

"collapse" of the wavefnc.

- measurement changes the system.
- $|\psi\rangle = \sum_{\omega} \langle\omega|\psi\rangle |\omega\rangle$

every particle in same superposition same state, unlike classical case where $|\langle\omega|\psi\rangle|^2 \propto$ # in state $|\omega\rangle$

- imperfect analogy of quantum uncertainty and classical prob. of e.g. six-sides die tossed & covered by a cup.

- In any one measurement of $\hat{\Omega}$ only ω_n can be measured (eigenvalues of $\hat{\Omega}$), never anything in-between. Average value of $\hat{\Omega}$ after many measurements can be fractional ω_n, as 2.1 children per household.

Observation → Measurement → projection of Ψ

⟺ • interaction with a "classical" system. (see L&L)
otherwise just q.m. evolution

• most subtle, least understood (philosophical) aspect of Q.M.

... but does not spoil success of Q.M. to predict/explain observations.

Simplest view:

(classical system!)

<u>measur. apparatus</u> — projection operator on eigenstates of $\hat{\mathcal{O}}$
$\mathcal{O}$

<u>Ex's</u> • position measurement $\hat{P}_r$

$$\hat{P}_r[\Psi] = \psi(r) \rightarrow \langle \hat{r} \rangle = \int_r r\,|\psi(r)|^2$$

$$\psi(r) \rightarrow \delta(r - r_0)$$

• momentum measurement $\hat{P}_p$

$$\hat{P}_p[\Psi] = \tilde{\Psi}(h) \rightarrow \langle \hat{p} \rangle = \int_k \hbar k\,|\tilde{\psi}_k|^2$$

many measurem.

explicitly: $\psi(\vec{r}) = \sum_k \tilde{\psi}(k)\, e^{i\vec{k}\cdot\vec{r}}$

$$\Rightarrow \hat{P}_p[\psi(\vec{r})] = \hat{\Psi}(h) \rightarrow \tilde{\psi}(h) \rightarrow \delta(h - k)$$

<u>Key point</u>: <u>no uncertainty</u> if system (ψ) is in an eigenstate of measurement apparatus.

simultaneously in a combination of eigenstates, not unlike E&M wave in polarization  going through a $\hat{y}$-oriented polarizer

(see Sakurai)

Uncertainty (prob. nature of Q.M.)

- due to system not being in an eigenstate of measurement apparatus $\hat{O}$
- not described by QM — Schrödinger's eqn **is deterministic**

$P_{\hat{O}}$ projects onto any φ_n state $\hat{O}\varphi_n = O_n\varphi_n$ with prob $|a_n|^2$

- system in state Ψ expandable in φ_n basis.

$$\Psi = \sum_n a_n \varphi_n$$

$$\Rightarrow \langle \hat{O} \rangle = \int d^3r\, d^3r'\, \Psi^*(r') O_{r'r} \Psi(r)$$

diagonal in φ_n basis.

$$= \sum_{n,n'} a^*_{n'} a_n \int_{r,r'} \varphi^*_{n'}(r') \underbrace{O_{r'r}\varphi_n(r)}_{O_n \varphi_n(r')}$$

$$= \sum_{n,n'} a^*_{n'} O_{nn'} a_n \quad O_{nn'} \to O_n \delta_{nn'} \quad \Big(O_n \underbrace{\int_r \varphi^*_{n'}(r)\varphi_n(r)}_{\delta_{n,n'}} \Big)$$

$$\langle \hat{O} \rangle = \sum_n |a_n|^2 O_n$$

$\Rightarrow$ $|a_n|^2$: prob. of system in state Ψ to be found in φ_n eigenstate of $\hat{O}$

O_n: eigenvalue of $\hat{O}$ in state φ_n

e.g. $\langle \hat{p} \rangle = \int_r \Psi^*(-i\hbar)\nabla\Psi = \sum_k |\tilde{\Psi}(k)|^2 \hbar k$

if in definite state $\hbar K$ of $\hat{p}$ $\Rightarrow$ $\tilde{\Psi}(k) = \delta_{k,K}$

$\Rightarrow \langle \hat{p} \rangle = \hbar K$ with certainty 1

- Physical observables →

→ (Hermitian) operators. $\hat{O}$

- $\hat{O}\varphi_n = O_n \varphi_n$

- - Unitary, <u>deterministic</u> evolution via S.E.

$$\Psi(r,0) \xrightarrow{\hat{U}(t)} \Psi(r,t)$$

- Measurement of $\hat{O}$ (due to contact with "classical" system)

$$\Psi(r,t) \xrightarrow[P_O[\Psi(r,t)]]{} \varphi_n(r)$$

large incoherent

grey areas are subtle.

Which $\varphi_n(r)$ does system end up

ANY! single measurement → any O_n, φ_n. !

... but can predict probability P_n ⇒ average props.

$$\Psi = \sum_n a_n \varphi_n \qquad \text{(complete basis)}$$

$$\langle\Psi|\hat{O}|\Psi\rangle = \int \Psi^* O \Psi = \sum_{n_1 n_2} a_{n_1}^* a_{n_2} \underbrace{\int \varphi_{n_1}^* \underbrace{\hat{O}\varphi_{n_2}}_{O_{n_2}\varphi_{n_2}}}_{O_{n_2}\delta_{n_1,n_2}}$$

$$= \sum_n |a_n|^2 O_n$$

$$\equiv \sum_n P_n O_n$$

If in an eigenstate of $\hat{O}$, e.g $\Psi(r) = \varphi_3$

$\Rightarrow$ every measurement (nondemolition) of $\hat{O}$ gives O_3.

$$P_n = |a_n|^2 = \delta_{n,3}$$

Ex. $\hat{O} = \hat{p} = -i\hbar\vec{\nabla}$

$$\text{state} \quad \Psi(r) = \sum_p a_p \varphi_p$$

$$\varphi_p = ?$$

$$\hat{p}\varphi_p = p\varphi_p \Rightarrow -i\hbar\nabla\varphi_p = p\varphi_p$$

$$\Rightarrow \varphi_p(r) = e^{i\frac{p}{\hbar}\cdot r}$$

$$(\text{just a FT}) \Rightarrow \Psi(r) = \sum_p a_p e^{i\frac{p}{\hbar}\cdot r}$$

$$\Rightarrow \langle\Psi|\hat{p}|\Psi\rangle = \sum_p |a_p|^2 p$$

if a system is measured (and therefore put) in a state $\Psi(r) = \varphi_{p_0}$

$$\Rightarrow \langle\Psi|\hat{p}|\Psi\rangle = \langle\varphi_{p_0}|\hat{p}|\varphi_0\rangle = p_0 \text{ with certainty}$$

Key point:

QM predicts deterministic evolution of a state $\psi(r,t)$ via S. Eqn. (as e.g. Newton's Eqn)

Stochastic/probabilistic nature comes from measurement of incompatible observables (op's) that intrinsically disturbs/changes the system (state).

Why Uncertainty principle?

Consider operators $[\hat{A}, \hat{B}] \neq 0$

can we find state Ψ that is an eigenst. of $\hat{A}$ & of $\hat{B}$? NO!

suppose: $\hat{A}\Psi_{a,b} = a\Psi_{a,b}$ & $\hat{B}\Psi_{a,b} = b\Psi_{a,b}$

$$\Rightarrow \hat{A}\hat{B}\Psi_{a,b} = ab\Psi_{a,b}$$
$$\hat{B}\hat{A}\Psi_{ab} = ab\Psi_{ab}$$
$$\Rightarrow [\hat{A}, \hat{B}] = 0 \ !$$

$\rightarrow$ contradiction of noncommutativity.

Perform measurement of $\hat{A}$ and find value $\underline{a} \Rightarrow \Psi = \Psi_a$ (measurement of a $\Leftrightarrow$ projection onto Ψ_a)

$\Psi_a = \sum_b C_b^a \Psi_b$ ← expand in orthonormal basis Ψ_b, eigenstates of $\hat{B}$

$$\Rightarrow \langle \hat{B} \rangle_a = \int_r \Psi_a^* \hat{B} \Psi_a$$

$$= \int_r \sum_{b,b'} C_b^{a*} C_{b'}^a \Psi_b^* \underbrace{\hat{B}\Psi_{b'}}_{b'\Psi_{b'}} = \sum_b |C_b^a|^2 b$$

$$\Rightarrow \langle \hat{B} \rangle_a = \sum_b |C_b^a|^2 b$$

not quite; only eigenvalues of $\hat{B}$, b_n's

$\Rightarrow$ measuring $\hat{B}$ after $\hat{A}$ can give ANY b value, with average given by weight (prob.) $|C_b^a|^2$.

$$(AB)^\dagger = B^\dagger A^\dagger \qquad \text{(cf with matrices)}$$

why? $$(AB|\psi\rangle)^\dagger = (A|B\psi\rangle)^\dagger$$

$$= \langle B\psi|A^\dagger$$

$$= \langle\psi|B^\dagger A^\dagger \quad \checkmark$$

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Main ideas in QM

$\boxed{\text{Matter wave} \quad \psi(\vec{r}, t)}$ – describes state of system.

- dispersion: $E = \frac{p^2}{2m}$, $p = \hbar k \Rightarrow E_k = \frac{\hbar^2 k^2}{2m}$

- Schrödinger wave eqn: $i\hbar \partial_t \psi = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(r)\right)\psi$

- "wave" intensity: $I(r) = |\psi(r)|^2$ = Prob. of finding a wave/particle in state/position $\vec{r}$.

consistency check – particle conservation:

$$\frac{d}{dt}\int_r |\psi(r,t)|^2 = \int_r \partial_t P(r,t) = \int_r \left(\psi^* \partial_t \psi + \partial_t \psi^* \psi\right)$$

$$= \frac{1}{i\hbar}\int_r \left[\psi^*\left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi - \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi^*\psi\right]$$

$$= -\int_r \vec{\nabla} \cdot \underbrace{\frac{-i\hbar}{2m}\left(\psi^* \nabla \psi - \nabla\psi^* \psi\right)}_{\equiv \vec{J}_p}$$

$\equiv \vec{J}_p$ – matter wave probability current

$$\Rightarrow \quad \partial_t P(r,t) + \vec{\nabla} \cdot \vec{J}_p = 0$$ ← local particle conservation.

- Observables: average of Q.M. operators in state ψ

$$\langle \hat{O} \rangle = \int_r \psi^*(r)\, O\, \psi(r)$$

- Not all operators/observables are compatible (commute)
- superposition: $\psi_{12} = \psi_1 + \psi_2$

ex. $\psi(r) = \sum_k \tilde{\psi}_k e^{ik\cdot r}$ $\Rightarrow$ $P_{12} \neq P_1 + P_2$

$\psi_{\rightarrow} = \frac{1}{\sqrt{2}}\psi_{\uparrow} + \frac{1}{\sqrt{2}}\psi_{\downarrow}$

- $\langle \psi_1 | \psi_2 \rangle = \int \psi_1^* \psi_2$

  $\langle \psi_2 | \psi_1 \rangle = \int \psi_2^* \psi_1$

  $\Rightarrow \langle \psi_1 | \psi_2 \rangle^* = \langle \psi_2 | \psi_1 \rangle$

- Adjoint (Hermitian conjugate) of an operator $\mathcal{O}$

  $\mathcal{O}|\psi\rangle \equiv |\mathcal{O}\psi\rangle$ ket.

  corresponding bra dual to $|\mathcal{O}\psi\rangle$ is

$|\mathcal{O}\psi\rangle^+ = \langle \mathcal{O}\psi| = \langle \psi | \mathcal{O}^+$

- Note: $\rightarrow \langle \psi_1 | \hat{\mathcal{O}} | \psi_2 \rangle = \langle \psi_1 | \mathcal{O}\psi_2 \rangle$

$$= \langle \mathcal{O}^+ \psi_1 | \psi_2 \rangle$$

$$\rightarrow \langle \psi_1 | \hat{\mathcal{O}} | \psi_2 \rangle^* = \langle \psi_1 | \hat{\mathcal{O}} \psi_2 \rangle^* = \langle \hat{\mathcal{O}} \psi_2 | \psi_1 \rangle$$

$$= \langle \psi_2 | \mathcal{O}^+ | \psi_1 \rangle$$

i.e. $\langle n | \hat{\mathcal{O}}^+ | m \rangle = \langle m | \hat{\mathcal{O}} | n \rangle^*$

$$(\mathcal{O}^+)_{nm} = \mathcal{O}^*_{mn}$$

- Hermitian ops : $\mathcal{O}^+ = \mathcal{O}$ (represented by Hermitian matrix)