Lecture 15

Solutions in polar and spherical coordinates

Announcements:

- lecture 14 is posted
- homework 11 is due April 25
- please begin reviewing for the final
- reading for this week is:
    - Ch 8 in TZD

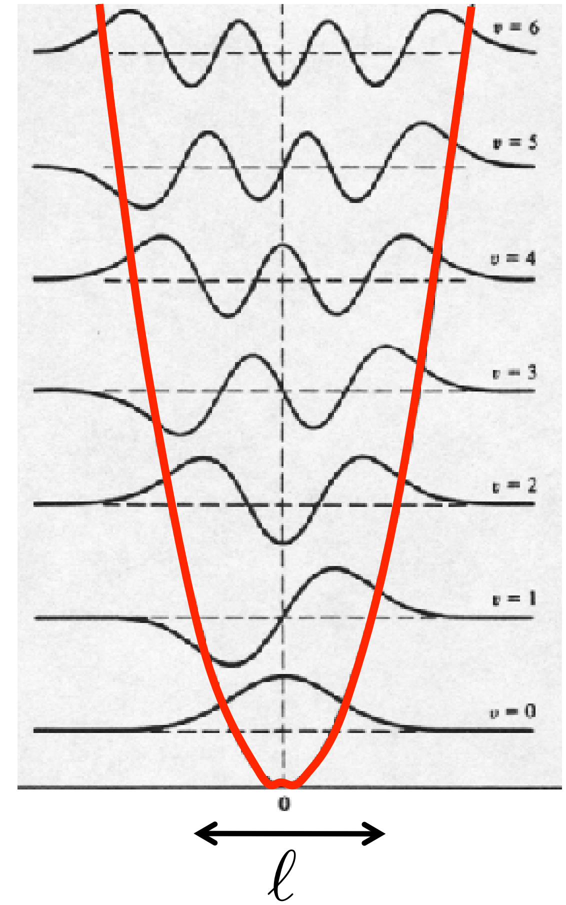
Today

Harmonic oscillator and hydrogen atom

- separation of variables in polar coordinates
- harmonic oscillator in polar coordinates
- separation of variables in spherical coordinates
- Hydrogen atom

Harmonic oscillator in d-dimensions

- S.Eqn: $$-\frac{\hbar^2}{2m}\nabla^2\psi_E + \frac{1}{2}m\omega_0^2 r^2\psi_E = E\psi_E$$



- spectrum: $$E_{n_1,n_2,\ldots} = \hbar\omega_0(n_1 + n_2 + \ldots)$$

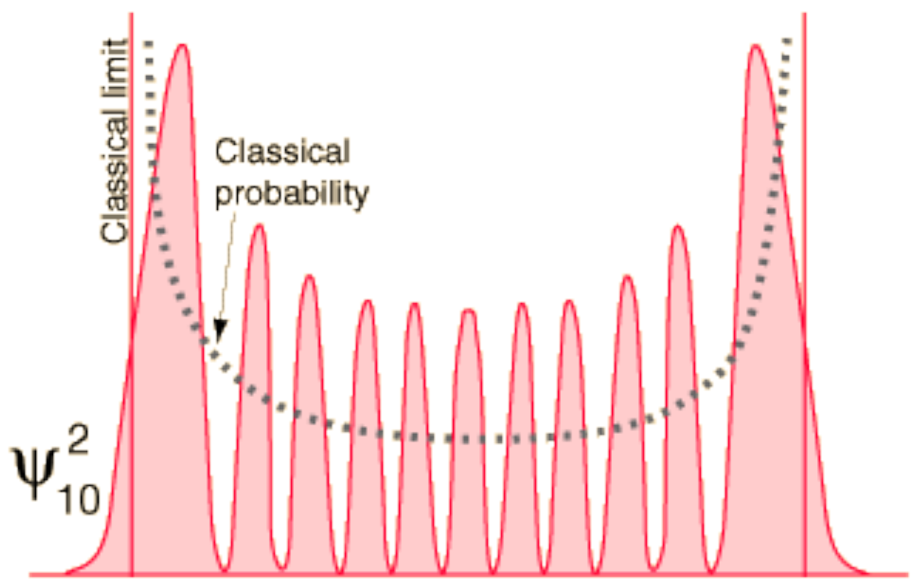
- eigenstates: $\psi_n(x) = H_n(x/\ell)e^{-x^2/2\ell^2}, \ \ell = \sqrt{\frac{\hbar}{m\omega_0}}$

$$\psi_0(x) = e^{-\hat{x}^2/2}, \quad E_0 = \frac{1}{2}\hbar\omega_0,$$

$$\psi_1(x) = \hat{x}e^{-\hat{x}^2/2}, \quad E_1 = \frac{3}{2}\hbar\omega_0,$$

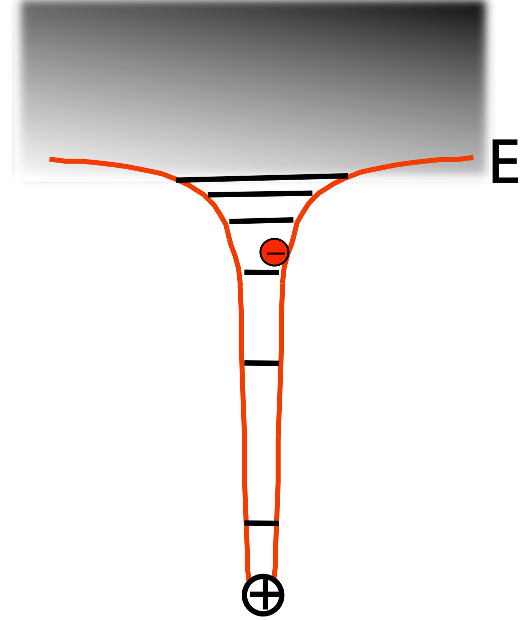
$$\psi_2(x) = (1-2\hat{x}^2)e^{-\hat{x}^2/2}, \quad E_2 = \frac{5}{2}\hbar\omega_0$$

- degeneracy for states n_x n_y n_z:
  - $D_{2d} = n+1$, $D_{3d} = (n+1)(n+2)/2$



- "crude" path to solution: p ≈ n h/x and minimize E(x) over x

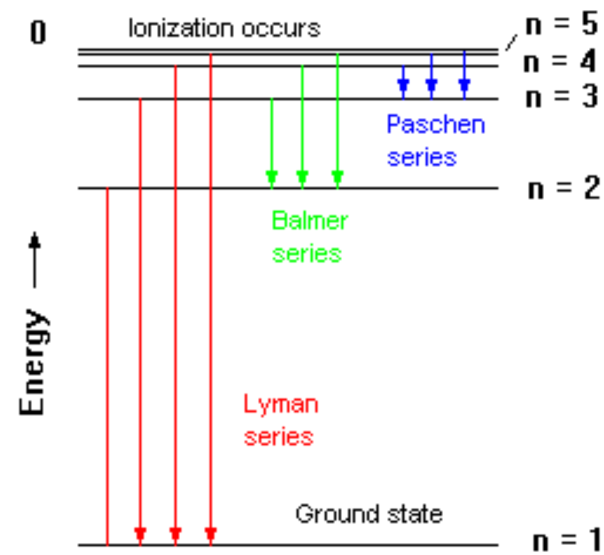
Hydrogen atom: V= -1/r



- S.Eqn: $$-\frac{\hbar^2}{2m}\nabla^2\psi_E - \frac{e^2}{r}\psi_E = E\psi_E$$

- spectrum: $$E_n = -\frac{e^4 m}{2\hbar^2}\frac{1}{n^2} = -\frac{13.6eV}{n^2}, \quad n = 1, 2, 3, \ldots$$
$$\ell = 0, 1, 2, \ldots, n-1, \quad -\ell \leq m \leq \ell$$

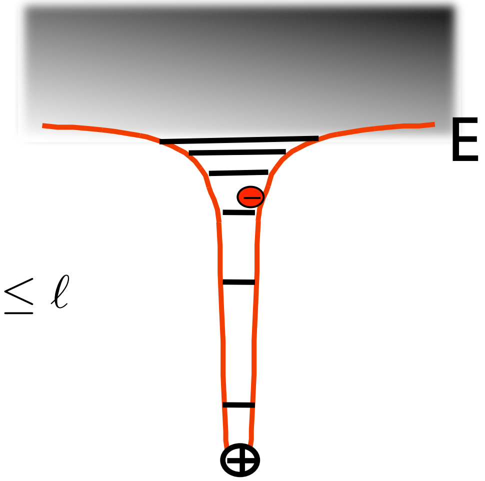
- eigenstates: $$\psi_{n\ell m}(r,\theta,\phi) = Y_\ell^m(\theta,\phi)\hat{r}^\ell e^{-\hat{r}/2} L_{n-\ell-1}^{2\ell+1}(\hat{r}) \qquad \hat{r} = \frac{2r}{na_0}$$
  - $Y_l^m(\theta,\phi)$ – *spherical harmonics*
  - $L_a^b(r)$ – *generalized Laguerre polynomials*



- degeneracy for state n, l, m:
  - *m degeneracy: 2l+1*
  - *total degeneracy: n^2 ($2n^2$, with spin)*

- "crude" path to solution: p ≈ n h/r and minimize E(r) over r

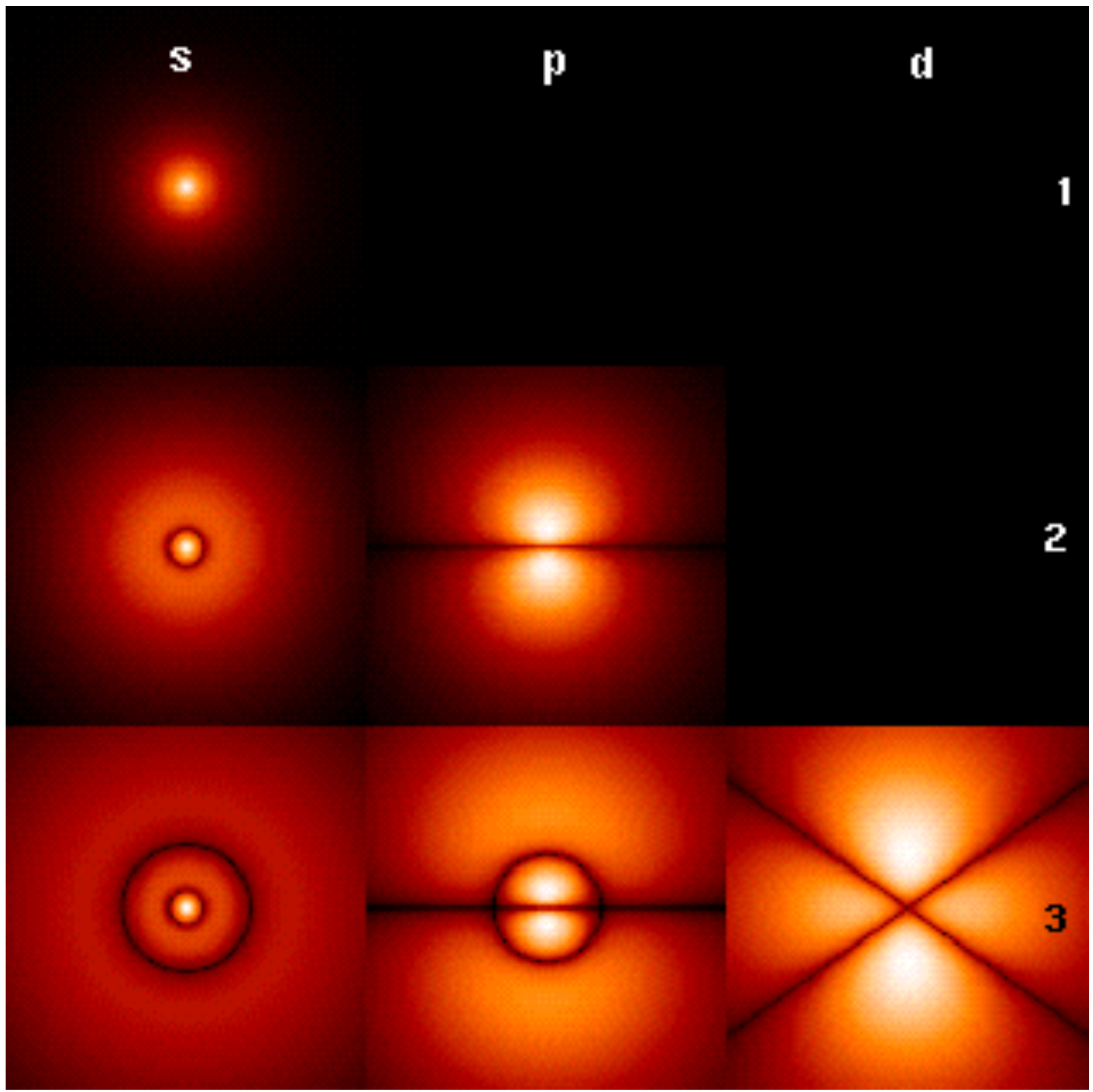
Hydrogen atom wavefunctions



- eigenstates labeled by n, l, m quantum numbers

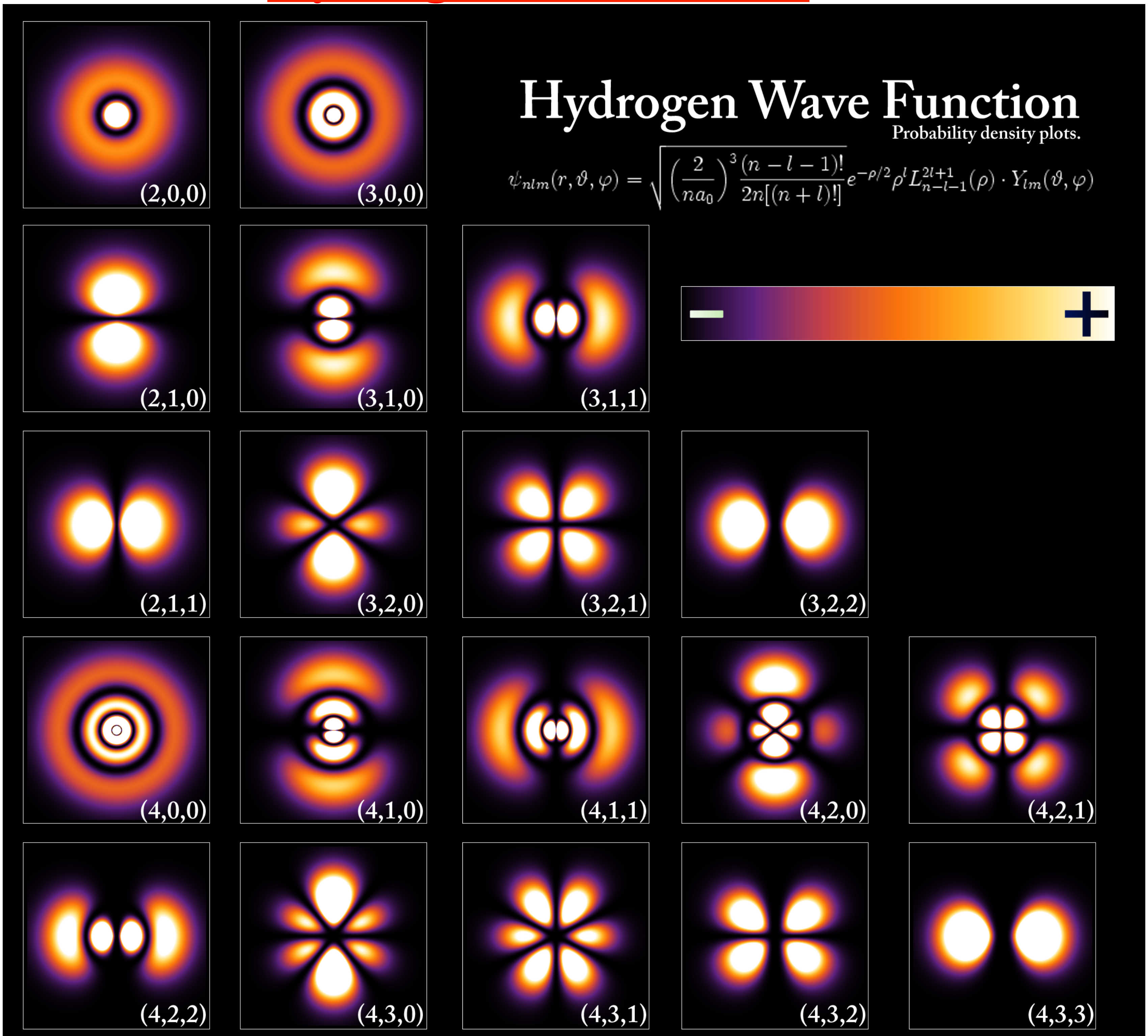
$$E_n = -\frac{e^4 m}{2\hbar^2}\frac{1}{n^2} = -\frac{13.6eV}{n^2}, \quad n = 1, 2, 3, \ldots \quad \ell = 0, 1, 2, \ldots, n-1, \quad -\ell \le m \le \ell$$

l



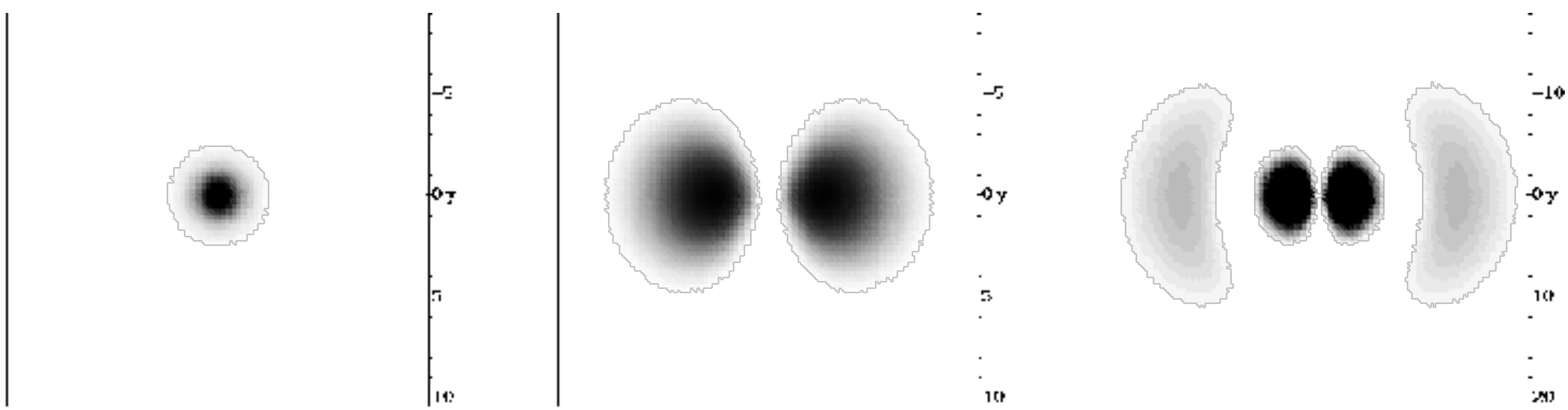
n

Hydrogen atom details



Hydrogen atom details

n	ℓ	m	$R_{n\ell}$	$Y_{\ell m}$	$\psi_{n\ell m} = R_{n\ell} Y_{\ell m}$
1	0	0	$2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
2	0	0	$\left(\frac{1}{2a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	± 1	$\left(\frac{1}{2a_0}\right)^{3/2} \frac{1}{\sqrt{3}}\frac{r}{a_0} e^{-r/2a_0}$	$\pm\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$	$\frac{1}{8}\sqrt{\frac{1}{\pi}}\left(\frac{1}{a_0}\right)^{3/2} \frac{r}{a_0} e^{-r/2a_0}\sin\theta e^{\pm i\phi}$



Hydrogen atom details