Leo Radzihovsky

PHYS 7450: Introduction to Solid State Physics

Homework Set 5

Issued March 30, 2015
Due April 13, 2015

Reading Assignment: Reading Assignment: My Electron Liquid lecture notes 5 on, textbook J. Solyom, Solids II, Ch. 16, 22.

1. Ground-state energy of a noninterating electron gas

   (a) Show that the ground state energy of a three-dimensional gas of N electrons at $T = 0$K is given by $E = \frac{3}{5}N\epsilon_F$.

   Notice that even $T = 0$K because of Pauli exclusion principle, electron gas has anormous kinetic energy even at zero temperature.

   (b) Generalize this result to d dimensions, to show that $E = \frac{d}{d+2}N\epsilon_F$.

   Hint: To do part (b), write down expressions for the number of particles and the total kinetic energy in terms of a d-dimensional integral over all the states $\vec{k}$ (remembering factor of 2 for two spin states per each orbital state $\vec{k}$) occupied by the electrons. Then use the fact that $\int d^dk \ldots = S_d \int_0^{k_F} dk k^{d-1} \ldots$ (where S_d is the surface area of a d-dimensional unit radius sphere, e.g., $S_2 = 2\pi$, $S_3 = 4\pi$) to relate the expressions for N and E to each other, thereby obtaining result above. You do *not* need to compute S_d, but it is given in earlier lecture notes.

2. Pressure and bulk modulus of a noninteracting electron gas

   (a) Show that the equation of state (relation between pressure P and volume V) for an electron gas at $T = 0$K can be written as $P = (2/3)E/V$.

   Notice that unlike classical Boltzmann gas, with $PV = k_BT$, here the pressure is finite even at $T = 0$. This is what's called Pauli's degeneracy pressure that, for not too large of a mass supports a neutron star (made up of neutrons that similarly to electrons in a metal form a degenerate liquid of fermions) from collapsing into a black hole.

   Hint: At $T = 0$K pressure is given by $P = -\partial E/\partial V$, keeping N fixed; answer to problem 1 should be useful to you.

(b) Show that the bulk modulus $B = -V\partial P/\partial V$ of an electron gas at $T = 0\text{K}$ is given by $B = 5P/3 = 10E/9V$ in 3d and generalize it to d-dimensions using results from problem 1.

(c) Estimate bulk modulus B and fermi velocity v_F for Potassium, using for example Table 1 in Kittel.

3. Consider a nearly free Fermi gas confined by 3d isotropic harmonic potential $V_{\text{trap}} = \frac{1}{2}m\omega_t^2 r^2$, as now is routinely done in JILA, in e.g,. Debbie Jin's lab. For this system, calculate:

   (a) Fermi energy, E_F as function of ω_t and number N of fermions in the trap.

   (b) Generalize this to finite temperature computing the chemical potential $\mu(T)$ (with, by definition, $\mu(0) = E_F$).

   (c) Compute the energy $E(T, N)$.

   (d) Compute the heat capacity $C_v(T, N)$.

   Simplifications:

   (i) Ignore spin degree of freedom, as spin is usually frozen out by the strong magnetic field creating the harmonic trap. (ii) Consider $N >> 1$, which will allow you to ignore small (order 1) shifts in the principle quantum number n.

   Hint:

   Your analysis for this problem is a straightforward generalization of the fermion in a "box" analyzed in class, with the only complication that single particle states are no longer plane waves labelled by $\vec{k}$, but are 3d harmonic oscillator states labelled by a discrete set of eigenvalues $\vec{n} = (n_x, n_y, n_z)$. To solve the problem it will be useful for you to figure out the density of states, i.e., degeneracy of the 3d harmonic oscillator at the principle quantum number $n = n_x + n_y + n_z$ (although you do not need to do this). Also for the purposes of convenience (this is just a redefinition of a zero of the chemical potential) ignore the zero-point energy. Finally, consider the case of a very shallow trap, i.e., small ω_t, which will allow you to replace sums with integrals (analogous to looking at $L \to \infty$ for the "box" problem, which makes $k = n2\pi/L$ a continuous variable).

   The computation in (a) can be done exactly, but for (b),(c),(d) it is sufficient to compute the results to lowest nontrivial order in T.

4. Density of states and the chemical potential.

   (a) Compute the density of states for a 1,2, and 3-dimensional noninteracting electron gas (with $\epsilon_k = \hbar^2 k^2/2m$). Note that in 2d the density of state is actually (ϵ-independent) constant.

(b) Generalize part (a) to d-dimensions, showing that $D(\epsilon) \propto \epsilon^{(d-2)/2}$, expressing the proportionality constant in terms of the surface area factor S_d as in problem 1. You are not required to compute S_d for general d. (Compare this result to the density of states for acoustic phonons studied previously).

(c) In general dimensions there is no simple closed form expression for the chemical potential $\mu(n,T)$ (with n the average electron density). However, in $d = 2$ it happens that the integral determining the chemical potential can be easily calculated exactly. Show that in 2d the chemical potential for a electron gas of 2d density n (number per area) at temperature T is given by

$$\begin{aligned} \mu(n,T) &= k_B T \ln[e^{\pi n \hbar^2/m k_B T} - 1], \\ &= k_B T \ln[e^{\epsilon_F/k_B T} - 1]. \end{aligned} \tag{1}$$

Hint: You can do this either directly by evaluating the 2d integral over k or instead using density of states $D(\epsilon)$ from part (a) and integrating over energies ϵ.

(d) Verify that $\mu(n,T)$ reduces (as it must) to the Fermi energy ϵ_F in the limit of $T \to 0$.

(e) Plot $\mu(n,T)$ as function of T for few densities n.

(f) Plot using Mathematica or another plotting type of a program that the 2d Dirac-Fermi distribution function $n_F(\epsilon)$ as a function of ϵ for a few values of temperature $T = \epsilon_F/100, T = \epsilon_F/10, T = \epsilon_F/10, T = 10\epsilon_F$. Make sure to include temperature-dependent $\mu(n,T)$ found above.

5. Heat capacity

(a) Derive heat capacity $C(T)$ for a 3d Fermi gas

(b) Plot the full exact expression as a function of T.

(c) Show that at high temperature $C(T)$ reduces to the Boltzmann gas equipartition result.

(d) Show that at low temperature $C(T) \approx \gamma T$, with $\gamma = \frac{\pi^2}{3} g(\epsilon_F) k_B^2$, $g(\epsilon_F)$ is the density of states at the Fermi level.

6. Plasmon in two dimensions

Using hydrodynamic equations for the mass density, ρ and momentum density, $\mathbf{g}$ (mass continuity and momentum continuity (Newton's equation)),

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{g} = 0, \tag{2}$$

$$\partial_t \mathbf{g} = en\mathbf{E} - \kappa^{-1} \boldsymbol{\nabla} \rho, \tag{3}$$

where $\mathbf{E}$ is the 2d electric field induced by 2d charge fluctuation and κ is mass compressibility, derive the dispersion of a 2d plasmon collective mode in an electron liquid, showing that its dispersion at small momenta is given by

$$\omega_k^{2d} = \alpha\sqrt{k},$$

to be contrasted with the gapped dispersion in 3d.

Hint:

The Coulomb electric field at $z = 0$, $\mathbf{E}(\mathbf{r}_\perp, z = 0)$, due to a 2d charge fluctuation $\rho(\mathbf{r}_\perp, z) = \rho(\mathbf{r}_\perp)\delta(z)$ can be obtained from the standard 3d Coulomb law relation (easiest formulated in terms of gradient of the scalar potential), working in Fourier space and focussing on $z = 0$.