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PHYS 7450: Advanced Solid State Physics

Homework Set 3

Issued February 10, 2015
Due February 26, 2015

Reading Assignment: My lecture notes 3 on bosonic matter, X-G Wen textbook, and Appendix H on "second quantization" in J. Solyom, Solids II.

1. Wick's theorem
   (a) Perform normal ordering (actually commuting $a, a^\dagger$) for the operator $O = aaa^\dagger a^\dagger$ and show that the Wick's theorem produces the correct result
   (b) Use the Wick's theorem to calculate $\langle 0|aa^\dagger aa^\dagger|0\rangle$.

2. Noninteracting Boson Green's function

   Consider a one-dimensional free (noninteracting) boson system with N bosons. Let $|\Psi_0\rangle$ be the ground state. Calculate, for $N = 0$ and finite N, the time-ordered propagator

   $$iG(x,t) = \langle GS|T\left(\psi(x,t)\psi^\dagger(0,0)\right)|GS\rangle,$$

   where $\psi(x,t) = e^{\frac{i}{\hbar}Ht}\psi(x)e^{-\frac{i}{\hbar}Ht}$. Show that $iG(x,t) \neq 0$ in the $x \to \infty$ limit, indicating "off-diagonal" long-range order, illustrating that the "off-diagonal" long-range order exists only in the Bose-condensed state.

   Hint: What is the ground state $|GS\rangle$ for noninteracting bosons?

3. Derive a real-time evolution operator $G_0(\mathbf{r}, t)$ for a free particle via a coordinate path integral for $t > 0$. Verify your result by directly finding a Green's function of the Schrödinger's equation

   $$i\hbar\partial_t G(\mathbf{r},t) + \frac{\hbar^2\nabla^2}{2m}G(\mathbf{r},t) = \delta(t)\delta^d(\mathbf{r}),$$

   with the initial condition $G_0(\mathbf{r}, 0^+) = \delta^d(\mathbf{r})$, with a boundary condition $G(\mathbf{r}, t < 0) = 0$.

   Hint:

   The solution can be obtained through Fourier transformation of the equation from $\mathbf{r}$ to $\mathbf{k}$, integrating the resulting ODE over time t, applying the initial condition and Fourier transforming back to the coordinate space.

   Note that this is an inhomogeneous equation, so the solution must involve a part that reproduces $\delta(t)$ on the right hand side; this is a multiplicative $\theta(t)$ function.

4. Noninteracting Bose gas

   (a) Derive the expression for the total number of bosonic atoms $N(\mu, T)$ as a function of chemical potential and temperature from the imaginary time-ordered Green's function of a noninteracting Bose gas

$$G(\mathbf{r}, \tau) = \langle T_\tau \left(\psi(\mathbf{r}, \tau)\overline{\psi}(0, 0)\right)\rangle.$$

   Hint:

   Note that for specific values of $\mathbf{r}$ and τ, $G(\mathbf{r}, \tau)$ gives the number density.

   It is helpful to use the coherent-state path-integral (trivial for noninteracting theory) G, that in Fourier space is given by $G(\mathbf{k}, \omega_n; \mathbf{k}', \omega_{n'}) = \langle \psi(\mathbf{k}, \omega_n)\overline{\psi}(\mathbf{k}', \omega_{n'}\rangle = \frac{\hbar}{-i\hbar\omega_n + \epsilon_k}(2\pi)^d \delta^d(\mathbf{k} - \mathbf{k}')\beta\hbar\delta_{\omega_n, \omega_n'}$.

   (b) For $T = 0$, verify explicitly by doing a simple contour integration over ω that $G(\mathbf{r}, \tau) \propto \theta(\tau)$, i.e., vanishes in the vacuum for $\tau < 0$ and gives a number of states for $\tau > 0$, as is clear from the corresponding operator form, since in one time-ordering it is normal ordered and thus annihilates the vacuum.

   Hint: For simplicity ignore the Bose-condensation here.

   (c) Calculate it analytically in 2d and numerically in 1d showing that that the behavior at finite temperature as a function of T and μ is smooth, i.e., no phase transition takes place in 2d and 1d.

   (d) Show that generically, in the high temperature limit, for fixed N, $\mu(T)$ is negative and is given by the classical Boltzmann result $\mu(T) \sim k_B T \ln T$. What are the actual arguments of the logarithmic, setting the scale for T.

5. Bogoluibov theory

   (a) By starting with a 2nd-quantized interacting Bose gas and following the steps in the posted notes (expanding in small quadratic fluctuations about the condensate), fill in the derivation details of the Bogoluibov Hamiltonian

$$H_{Bogol} = \frac{1}{2}\sum_{\mathbf{k}\neq 0}\left[(\epsilon_k + gn)a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + gna_{-\mathbf{k}}a_{\mathbf{k}} + h.c.\right].$$

   (b) Show explicitly that diagonalization in terms of u_k, v_k

$$\begin{aligned} a_k &= u_k\alpha_k + v_k\alpha^\dagger_{-k}, \\ a^\dagger_{-k} &= v^*_k\alpha_k + u^*_k\alpha^\dagger_{-k}, \end{aligned}$$

requires that $|u_k|^2 - |v_k|^2 = 1$ if the new operators $\alpha_k, \alpha_k^\dagger$ are also to be bosons. Show that the solution is given by:

$$
\begin{aligned}
u_k^2 &= \frac{1}{2}(\frac{\tilde{\varepsilon}_k}{E_k} + 1), \\
v_k^2 &= \frac{1}{2}(\frac{\tilde{\varepsilon}_k}{E_k} - 1), \\
E_k &= \sqrt{\tilde{\varepsilon}_k^2 - g^2 n^2} = \sqrt{\epsilon_k^2 + 2gn\epsilon_k} = c\hbar k\sqrt{1+\xi^2 k^2},
\end{aligned} \tag{1}
$$

where E_k is the Bogoluibov spectrum, $\tilde{\varepsilon}_k = \epsilon_k + gn$, $\epsilon_k = \hbar^2 k^2/(2m)$, zeroth-sound velocity $c = \sqrt{gn/m}$ and correlation length $\xi = \hbar/\sqrt{4mgn}$.

Hint: note that $u_k = \cosh\chi_k$, $v_k = \sinh\chi_k$ satisfy above constraint.

(c) Show that the $k = 0$ state condensate depletion $N_d = \sum_{\mathbf{k}\neq 0}\langle a_k^\dagger a_k\rangle$ (the average is the trace over the Bogoluibov Hamiltonian eigenstates at temperature T, i.e., $\alpha_k^\dagger \alpha_k$ number eigenstates),

$$
\begin{aligned}
N_d &= \sum_{\mathbf{k}\neq 0}\langle a_k^\dagger a_k\rangle, \\
&= \sum_{\mathbf{k}\neq 0}\left[|v_k|^2 + (|u_k|^2 + |v_k|^2)n_{BE}(T)\right], \\
&= N\frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2}, \quad \text{at } T = 0,
\end{aligned} \tag{2}
$$

where n_{BE} is the Bose-Einstein distribution for the Bogoluibov quasi-particles $\alpha_{\mathbf{k}}$.

Hint:

To compute the final expression note that at $T = 0$, Bogoluibov ground state is a vacuum of α_k particles, i.e., there are no $\alpha_{\mathbf{k}}$ excitations at $T = 0$, allowing you to set $n_{BE} = 0$.

The final $\mathbf{k}$ sum must be done in the thermodynamic limit by converting into a $\mathbf{k}$ integral.

6. Minimize the coherent-state real-time action to get the Gross-Pitaevskii equation. Then by looking at real and imaginary parts in the density-phase representation derive quantum "hydrodynamics" for number density $n(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m}\boldsymbol{\nabla}\theta$, quoted in the notes.