Superconductivity

May 11, 2015

Contents

1 Background

Superconductivity is a phenomenon that in certain kind of material exactly zero electrical resistance and magnetic field expulsion can happen below a critical temperature. Superconductivity was discovered on April 8, 1911 by Heike Kamerlingh Onnes[1]. At the temperature of $4.2K$, he observed that the resistance abruptly disappeared. In subsequent decades, superconductivity was observed in several other materials. In 1913, lead was found to superconduct at $7K$, and in 1941 niobium nitride was found to superconduct at $16K$. Before 1980, the critical temperature found cannot be higher than $30K$. Those superconductor are called conventional superconductor. Theoretically, they can be fully explained by BCS theory and phenomenological Landau-Ginzburg theory. In the lecture notes, we will discuss the exotic behavior of superconductor and the remarkable theories for deep understanding.

2 Experimental evidences of superconductor

There are universal properties of superconductor listed as follow which can be measured in experiments. The existence of these universal physical properties implies superconductivity is a thermodynamic phase of matter. Thus, the description of those properties does not need microscopic detail of the material.

2.1 Zero resistance

In a normal conductor, an electric current is constantly colliding with the ions in the lattice, and during each collision some of the energy carried by the current is absorbed by the lattice and converted into heat. As a result, the energy carried by the current is constantly being dissipated. The current will finally decay to zero when there is no external power provided. However, superconductors are able to maintain a current with no applied voltage. Experiments have demonstrated that currents in superconducting coils can persist for years without any measurable degradation. The experimental evidence for zero resistance is shown in Fig. (1)[1]. Without external magnetic field, the critical temperature for vanishing resistance is about $4.2K$. Thus, the inner structure of superconductor would be completely different from normal metal.

2.2 Perfect diamagnetism

Zero electrical resistance would lead to perfect diamagnetism. This means if an additional magnetic field is imposed to a superconductor, current loops would be generated to exactly cancel the imposed field according to Lenz's law. But the original magnetic flux through the material would not change in the applied magnetic field. If we have external applied magnetic field $\vec{H}$ and the magnetization inside is $\vec{M}$, they

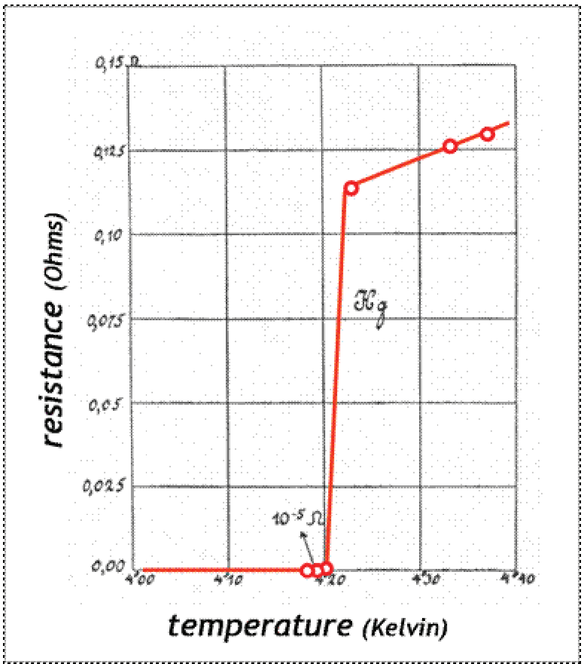


Figure 1: Resistance of mercury drop to zero measured by Onnes. [1]

should satisfy

$$\vec{B} = \vec{H} + 4\pi\vec{M} \tag{1}$$

where $\vec{B}$ is the net magnetic field induced by $\vec{H}$inside the material. For perfect diamagnetism, magnetic susceptibility is $\chi = -1$. Thus,

$$\vec{M} = -\frac{\vec{H}}{4\pi} \Rightarrow \vec{B} = 0 \tag{2}$$

2.3 Meissner-Oschenfeld effect

Another prominent property of superconductor is Meissner effect[2], which is shown in Fig. (2). It reveals that magnetic field is repelled by superconductivity. Although superconductor is a perfect diamagnetism, Meissner effect has different origin other than zero electrical resistance. Because no matter there is magnetic field or not, after cooling down the material, there is always no magnetic flux inside. Thus, the final state at $T < T_c$ is history independent.

In the Meissner state, since we have $\vec{B} = 0$. In type I superconductor, where the magnetic field is totally repelled, the relation among $\vec{B}$, $\vec{H}$ and $\vec{M}$ is shown in Fig. (3). When $H > H_c$, $\vec{B} = \vec{H}$, there is no superconductivity and magnetic field can penetrate material.

2.4 Heat capacity jump at SC-Normal state transition

As we know, as we increase temperature, there would be a phase transition from superconductivity to normal state. Experimentally, at critical temperature, the heat

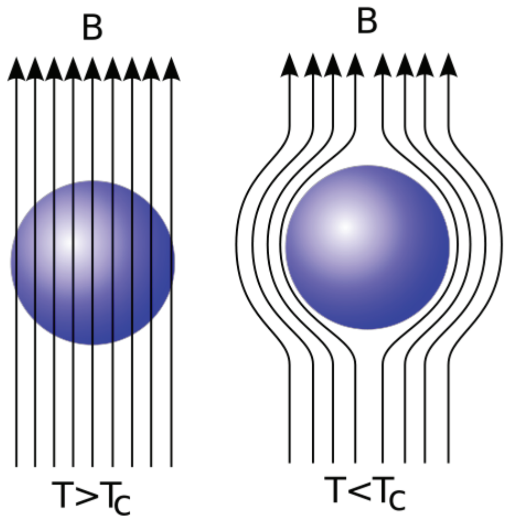


Figure 2: Diagram of Meissner effect. Magnetic field lines, represented as arrows, are excluded from a superconductor when it is below its critical temperature.

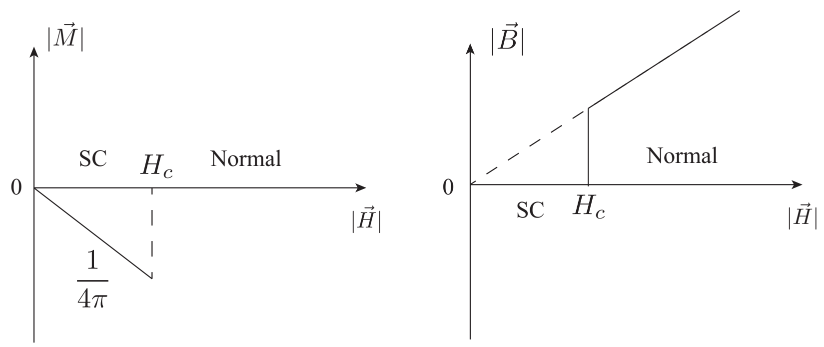


Figure 3: Relation among $\vec{B}$, $\vec{H}$ and $\vec{M}$ in type I superconductivity below and above critical magnetic field.

capacity has a discontinuity as shown in Fig. (4).

When $T < T_c$, the gapped excitations exponentially suppressed heat capacity. The quasiparticle density is approximately

$$n \sim e^{-\Delta/T} \tag{3}$$

Where Δ is the gap above the ground state. Thus, heat capacity is

$$C_v^{T<T_c} \sim k_B e^{-2T_c/T} \tag{4}$$

While in normal metallic state, at small temperature, only a fraction of electron $k_B T/E_F$ can be excited thermally. The total thermal energy is

$$\Delta F = N(\frac{k_B T}{E_F})k_B T \tag{5}$$

Thus, the heat capacity is

$$C_v = \frac{\Delta F}{T} = Nk_B(\frac{k_B T}{E_F}) \propto T \tag{6}$$

which is linear.

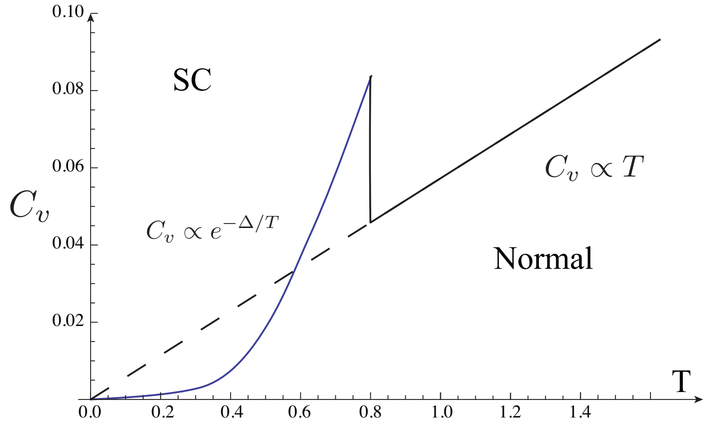


Figure 4: Heat capacity jump at critical temperature and exponential decay for $T < T_c$.

2.5 Isotope effect

The critical temperature of a superconductor is found depending on the isotopic mass of the constituent element[3]. Their relation is given as

$$T_c \propto M^{-\beta} \tag{7}$$

Experimentally, for Zinc, Lead, Mercury and a number of other elements, their β values agree with theoretical value, which is 0.5. This phenomenon can be reproduced by BCS theory. In BCS theory, the Debye frequency of phonons in a lattice is proportional to the inverse of the square root of the mass of lattice ions. Meanwhile, the critical temperature is proportional to the Debye frequency:

$$T_c \propto \hbar\omega_D e^{-1/N(0)V_0} \tag{8}$$

where $N(0)$ is density of state at Fermi energy. Then the relation

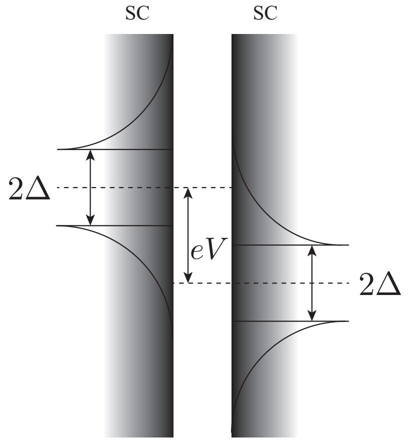
$$T_c \propto \frac{1}{\sqrt{M}} \tag{9}$$

can be verified.

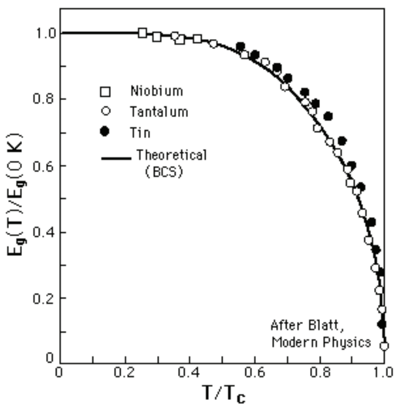
2.6 Energy gap

Besides the experiment of heat capacity shows the evidence of an energy gap, electric tunneling experiment directly measured the existence of energy gap in superconducting phase. In 1960, Ivar Giaever measured relation between tunneling current and bias voltage in superconductor-insulator-superconductor (SIS) tunnel junction as shown in Fig. (5(a)).

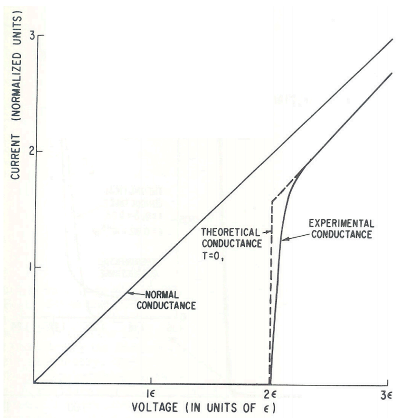
The experimental result for SIS junction is shown in Fig. (5(c)). The current is zero until bias voltage exceeds its critical value. For SIS junction, the critical value is 2Δ where Δ is the energy gap. For NIS junction, the critical voltage is only one Δ. The gap and temperature relation is also obtained experimentally as shown in Fig. (5(b)).



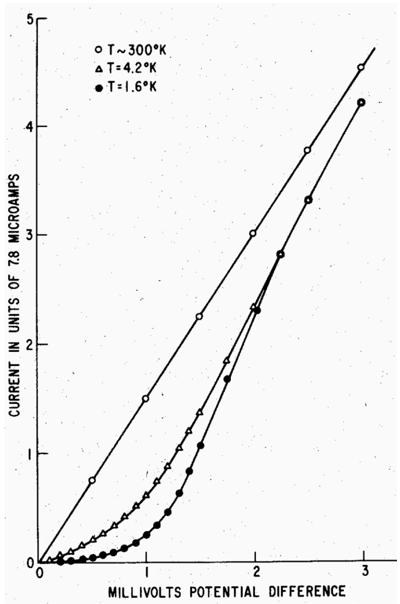
(a) *SIS* junction. The vertical axis is energy, and the horizontal axis shows the density of states.



(b) Gap vs. temperature. [4]



(c) *I*-*V* relation measured on Sn/SnO/Sn junction by STM. [5]



(d) *I*-*V* relation measured on Al/Al_2O_3/Pb junction by STM. [6]

Figure 5: Gap measured by SIS and NIS junction.

2.7 Two types of superconductor

There are two types of conventional superconductor according to their magnetic property. As the applied magnetic field become too large, superconductivity breaks down. Superconductors can be divided into two types according to how this breakdown occurs.

2.7.1 Type I superconductor

For type I superconductors, superconductivity is abruptly destroyed via a first order phase transition when the strength of the applied field rises above a critical value H_c. This type of superconductivity is normally exhibited by pure metals, e.g.

aluminium, lead, and mercury. The only alloy known up to now which exihibits type I superconductivity is $TaSi_2$. The net magnetic field inside and the phase diagram as a function of external magnetic field are shown in Fig. (6).

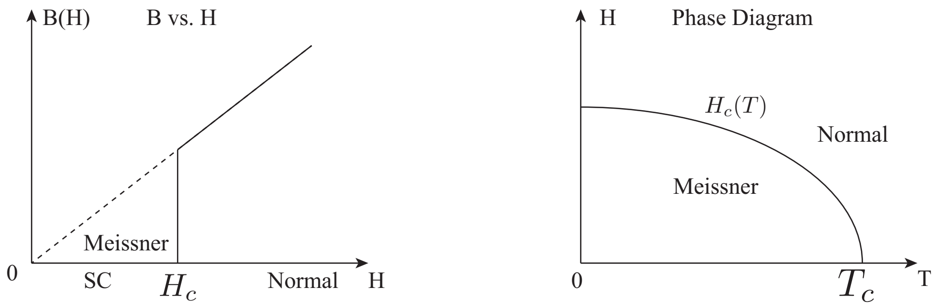


Figure 6: B-H relation and phase diagram of Type I superconductors.

2.7.2 Type II superconductor

The transition from SC to normal state is different for type II superconductors, which exhibit two critical magnetic fields. The first, lower critical field occurs when magnetic flux vortices penetrate the material but the material remains superconducting outside of these microscopic vortices. When the vortex density becomes too large, the entire material becomes non-superconducting; this corresponds to the second, higher critical field. The net magnetic field inside and the phase diagram are also shown in Fig. (7).

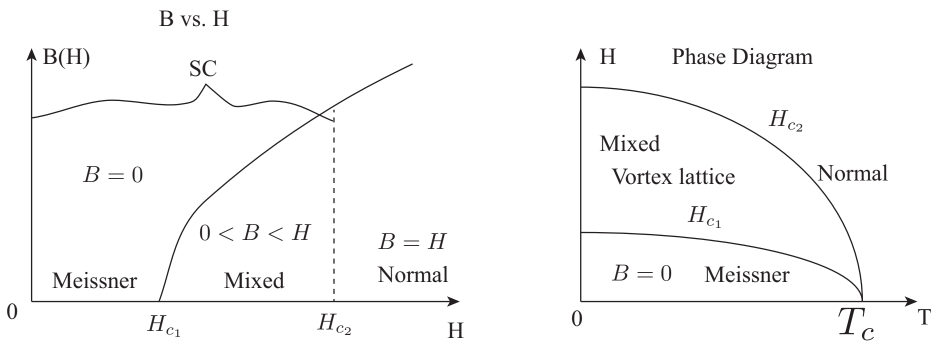


Figure 7: B-H relation and phase diagram of Type II superconductors.

For this type of superconductor, the net magnetic field penetrates into material is quantized into vortices because of Higgs mechanism. In the mixed state, each vortex carry flux

$$\phi_0 = \frac{hc}{2e} = 2 \times 10^{-7} \quad \text{Gauss} \times \text{cm}^2. \tag{10}$$

The total flux is $n\phi_0$.

3 Phenomenological theories

Generally speaking, directly studying microscopic model of corresponding material is extremely hard. Thus, we will introduce some phenomenological theories.

3.1 Two-Fluids model

Superconductors are perfect conductors of electricity only at zero frequency (DC). At any non-zero frequency (AC) they dissipate the charge carrier kinetic energy into heat, although this conversion is often quite small. Consider a superconductor exposed to a time-varying magnetic field parallel to its surface. The response of the superconductor is complex in the sense that currents are induced that are both in-phase and out of phase with the alternating magnetic field. The out-of-phase (with magnetic field B) currents are dissipative and now we believe it arises from the absorption of photons by quasiparticles excited out of the ground state. The in-phase response is due to the Meissner Effect in which the superconductor actively excludes magnetic field from its interior to maintain the condition B = 0. In order to explain the AC dissipation of superconductor, a simple two-fluid model is introduced. The strength of the in-phase current is dictated by the fraction of the charge carriers that condense into the ground state.

Consider a material with total electron density n, consisting superconducting part n_s and normal part n_n. At $T \to 0$, $n_s \approx n$ while $n \approx 0$. When $T > T_c$, we have $n_s = 0$ and $n_n = n$. Between those two cases at $0 < T < T_c$, both density are non-zero and smaller than n. Starting from equation of motion of electron

$$m\dot{\vec{v}}_s = -e\vec{E} \Rightarrow m\frac{\partial(-n_s e\vec{v})}{\partial t} = n_s e^2 \vec{E} \tag{11}$$

and the definition of electrical current

$$\vec{J}_s = -en_s\vec{v}_s \tag{12}$$

Consequently, we obtain

$$\frac{d\vec{J}_s}{dt} = \frac{n_s e^2}{m}\vec{E} \tag{13}$$

It is also called the 1st London's equation.

In addition, use Maxwell equation

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{14}$$

then,

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{J}_s) = (\frac{n_s e^2}{m})\vec{\nabla} \times \vec{E} = -\frac{n_s e^2}{mc}\frac{\partial}{\partial t}\vec{B} \Rightarrow \vec{\nabla} \times \vec{J}_s = -\frac{n_s e^2}{mc}\vec{B} = -\frac{n_s e^2}{mc}\vec{\nabla} \times \vec{A} \tag{15}$$

Since gauge transformation

$$A \to A - \frac{\hbar c}{e}\vec{\nabla}\theta \tag{16}$$

we can have

$$\vec{J}_s = -\frac{n_s e^2}{mc}(A - \hbar c\vec{\nabla}\theta) = \frac{n_s e}{m}(\hbar\vec{\nabla}\theta - \frac{e}{c}\vec{A}) \tag{17}$$

This is the 2nd London's equation.

The above equation can also be derived from the definition of probability current

$$\vec{J} = e\psi^*\vec{v}\psi = \frac{e}{m}\psi^*(i\hbar\vec{\nabla} - \frac{e}{c}\vec{A})\psi \tag{18}$$

Let $\psi = \rho e^{i\theta(x)}$, then the final result is

$$\vec{J} = \frac{e}{m}(\hbar\vec{\nabla}\theta - \frac{e}{c}\vec{A}) \tag{19}$$

The electrical current is

$$\vec{J}_s = n_s\vec{J} = \frac{n_s e}{m}(\hbar\vec{\nabla}\theta - \frac{e}{c}\vec{A}) \tag{20}$$

We can see the superconducting current is from the condensed charge carrier.

The Meissner effect can also be derived from above two London's equations. Since we already have

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}_s \Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{\nabla} \times \vec{J}_s = -\frac{4\pi n_s e^2}{mc^2}\vec{B} \tag{21}$$

Consequently, it results in

$$\nabla^2\vec{B} = \lambda_L^{-2}\vec{B} \tag{22}$$

where

$$\lambda_L = (\frac{mc^2}{4\pi n_s e^2})^{1/2} \sim \frac{1}{\sqrt{n_s}} \tag{23}$$

This magnetic property of superconductor is pictorial shown in Fig. (8).

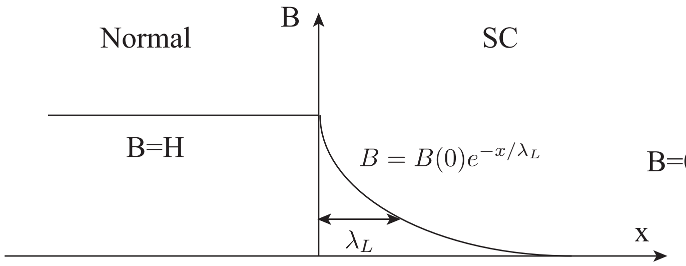


Figure 8: Magnetic property at interface between superconductivity and normal state.

3.2 Ginzburg-Landau Theory

Based on Landau's theory of second-order phase transition, the free energy of superconductor can be expand in terms of complex order parameter fields in the vicinity of critical point since the order parameter is very small compared with that deep inside the superconducting phase. Within each phases, Lagrangian is analytic and can be written polynomially in the order parameter Ψ. The main idea is to minimize the free energy with respect to those complex fields in different parameter regions, although it is not necessary to give a direct interpretation of these parameters. After learning the microscopic BCS theory of superconductor, we will know the complex field $\Psi(\vec{r})$ used in this theory is copper pair field.

The Lagrangian should be imposed by both global and local gauge symmetry. For global symmetry, since superconductor has global U(1) symmetry $\Psi \to e^{i\frac{e^*}{\hbar c}\chi}\Psi$, thus, Lagrangian should have term proportional to $|\Psi|^2$. Besides, the gauge invariance leads to a term $|(\hbar\vec{\nabla} - i\frac{e^*}{c}\vec{A})\Psi|^2$, where $e^* = 2e$ is the charge of Cooper pair.

Taking the external magnetic field into account, the free energy for superconductor can be written as

$$g_s = \frac{1}{2m^*}\left|(\frac{\hbar}{i}\vec{\nabla} + \frac{e^*}{c}\vec{A})\Psi\right|^2 + a|\Psi|^2 + \frac{1}{2}b|\Psi|^4 + \cdots + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B}\cdot\vec{H}}{4\pi} + g_n \tag{24}$$

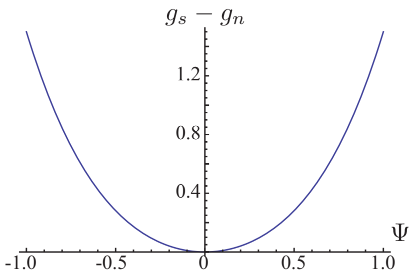
where the first term represents kinetic energy $\vec{J}_s^2 \sim n_s\frac{1}{2}mv^2$.

3.2.1 Uniform case: $B = H = 0$

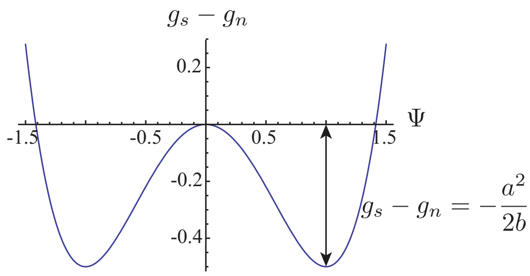
In this case, the Lagrangian becomes

$$g_s = g_n + a|\Psi|^2 + \frac{1}{2}b|\Psi|^4 \tag{25}$$

where a depends on temperature $a = a_0(T - T_c)$.



(a) $g_s - g_n$ vs. Ψ in normal phase



(b) $g_s - g_n$ vs. Ψ in superconducting phase

Figure 9: Free energy as a function of order parameter Ψ. In the normal state, there is only one minimum. While in superconducting state, there are many non-zero Ψ at arbitrary phase.

Minimize g_s, we have

$$\frac{\delta g_s}{\delta \Psi} = a\Psi_0 + b|\Psi_0|^2\Psi_0 = 0 \tag{26}$$

When $a > 0$ and $b > 0$, there is only one solution: $\Psi_0 = 0$. This is the normal state for $T > T_c$. While when $a < 0$ and $b > 0$, there is symmetry breaking solution: $|\Psi_0| = \sqrt{\frac{-a}{b}} \propto \sqrt{T_c - T}$. This is the superconducting phase for $T < T_c$. Since

$\Psi_0 = \sqrt{\frac{-a}{b}} e^{i\theta}$ has an arbitrary and fixed phase, it breaks continuous U(1) symmetry spontaneously. The above two solutions are shown in Fig. (9).

The change of g_s, Ψ_0, and heat capacity C_v as a function of temperature can be pictorially shown as Fig. (10).

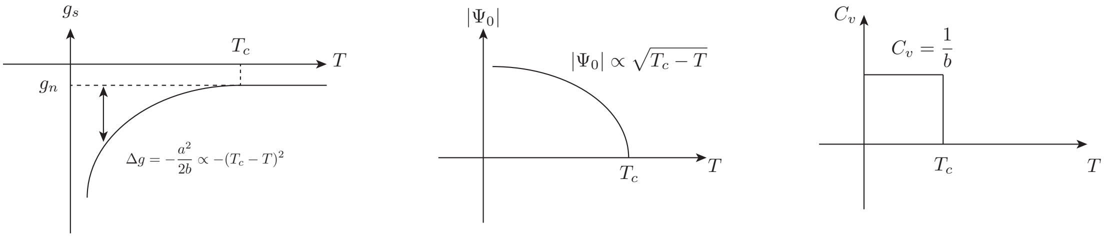


Figure 10: g_s, $|\Psi_0|$ and C_v as a function of temperature.

3.2.2 $H \neq 0$, $B = 0$ and $\vec{J}_s \approx 0$

At fixed temperature,

$$g^{(H)} = g_s - g_n^0 = -g_c + \frac{B^2}{8\pi} - \frac{B \cdot H}{4\pi} \tag{27}$$

where g_c is the magnitude of free energy of condensation.

In Meissner phase, $B = 0$, then $g_s^{(H)} = -g_c$. While in normal phase, $B = H$, then $g_n^{(H)} = -H^2/8\pi$. Pictorially, the phase transition driven by external magnetic field is shown in Fig. (11).

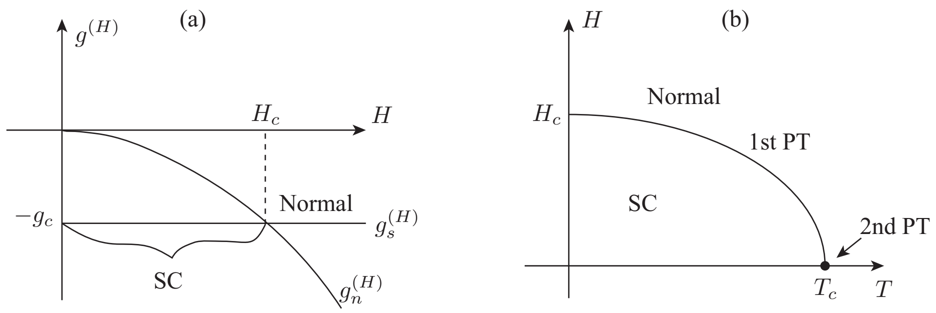


Figure 11: (a)$g_n^{(H)}$ and $g_s^{(H)}$ as a function of H. (b) Phase diagram of type I superconductor.

3.2.3 $B \neq 0$: Flux quantization

In this case, magnetic field can penetrate into superconductor. The gauge field couple to Ψ field can fluctuation and is to be determined according to the external field. Thus, in order to optimize the free energy of superconductor, we have two equations:

$$\begin{aligned}\frac{\delta g_s}{\delta \Psi^*} &= \frac{\hbar^2}{2m^*}(\vec{\nabla} + \frac{ie^*}{\hbar c}\vec{A})^2\Psi + a\Psi + b|\Psi|^2\Psi = 0 \\ \frac{\delta g_s}{\delta \vec{A}} &= \underbrace{-\frac{|\Psi|^2}{m^*c^2}|\Psi|^2(e^*)^2\vec{A}}_{\text{diamagnetic current } J_s^d/c} + \underbrace{\frac{1}{2m^*}\frac{\hbar e^*}{ic}\vec{A}(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*)}_{\text{paramagnetic current } J_s^p/c} - \frac{1}{4\pi}\nabla \times \vec{B}\end{aligned} \tag{28}$$

Then, these two equations give self-consistent solution for $\vec{A}$ and Ψ.

In the Meissner state, $\Psi = \sqrt{n_s}e^{i\theta(x)}$ where $n_s = |a|/b$. The corresponding current is

$$\vec{J}_s = -\frac{e^*\hbar}{m^*}n_s\vec{\nabla}\theta(x) - \frac{n_s(e^*)^2}{m^*c}\vec{A} = -\frac{e^*}{m^*}n_s(\hbar\vec{\nabla}\theta(x) + \frac{e^*}{c}\vec{A}) \tag{29}$$

The flux surrounding by current inside the superconducting state $B = 0$ is

$$0 = \frac{c}{4\pi}\oint \vec{\nabla} \times \vec{B} \cdot d\vec{l} = \oint \vec{J}_s \cdot d\vec{l} = -\frac{e^*n_s}{m^*}\hbar \underbrace{\oint \vec{\nabla}\theta(x) \cdot d\vec{l}}_{2\pi p} - \frac{(e^*)^2 n_s}{cm^*}\underbrace{\oint \vec{A} \cdot d\vec{l}}_{\int \vec{\nabla}\times\vec{A}\cdot d\vec{a}=\Phi} \tag{30}$$

where p is an integer. Thus, $\phi = -p(\frac{hc}{e^*}) = -p\phi_0$. ϕ_0 is a flux quantum

$$\phi_0 = \frac{hc}{2e} = 2 \times 10^{-7} \quad Gauss \times cm^2 \tag{31}$$

3.2.4 Critical T_c, H_c and J_c

The electrical current can be written as

$$J = e^*|\Psi|^2 v \tag{32}$$

where v is the velocity of current. The free energy can be written in term of this velocity as

$$g_s = |\Psi|^2(a + \frac{1}{2}b|\Psi|^2 + \frac{1}{2}mv^2) + g_s^{(H)} \tag{33}$$

In superconducting phase $a < 0$, optimizing this free energy with respect to Ψ at fixed v, we get

$$a + b|\Psi|^2 + \frac{1}{2}mv^2 = 0 \Rightarrow |\Psi|^2 = \frac{1}{b}(-a - \frac{1}{2}mv^2) > 0 \Rightarrow v < v_c = \sqrt{\frac{2|a|}{m}} \tag{34}$$

Use the superconductor order parameter $|\Psi_0|^2 = -\frac{a}{b}$, the equation for v can be written as

$$\frac{1}{2}mv^2 = b(|\Psi_0|^2 - |\Psi|^2) = b|\Psi_0|^2(1 - r) \Rightarrow v = \sqrt{\frac{2b}{m}|\Psi_0|^2}(1 - r^2)^{1/2} \tag{35}$$

where $r = |\Psi_0|^2/|\Psi|^2$ can be regarded as the fraction of condensation. The current can then be written as

$$J = e^*|\Psi|^2 v = e^*|\Psi_0|^2\sqrt{\frac{2|a|}{m}}r^2(1 - r^2)^{1/2} \tag{36}$$

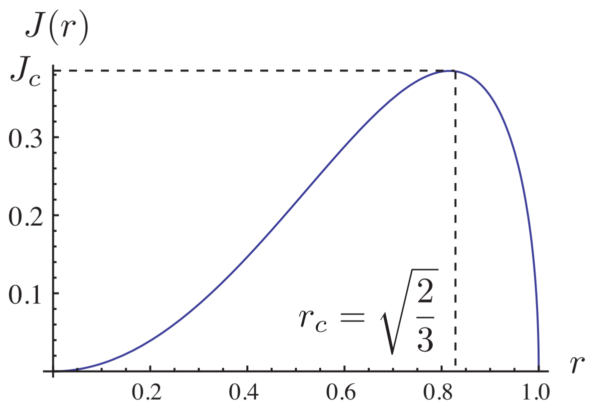


Figure 12: Current as a function of condensation fraction r.

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The relation between J and the ratio r is shown in Fig. (12).

Since $v < v_c$, i.e. once $J > J_c$, the condition in Eq. (34) cannot be true. Thus, we must have $\Psi = 0$. A phase transition from superconducting to normal state happens. As current increases when $|\Psi|$ decreases, as long as J reaches J_c at $r = \sqrt{\frac{2}{3}}$, r immediately jumps to be zero if applied magnetic field is going on increasing.

3.2.5 Two important length scales

The first length scale is coherence length $\xi(T)$. As we know, deep inside the superconductor, $|\Psi| \sim |\Psi_0|$. While outside it, $|\Psi| = 0$. Thus, ξ is a length scale of $\Psi(\vec{r})$ varies spatially.

Starting from free energy of superconductor, below we use e and m instead of e^* and m^* to represent the charge and mass of charge carrier in superconductor for simplicity.

$$g_s = \frac{1}{2m}|(\frac{\hbar}{i}\vec{\nabla} + \frac{e}{c}\vec{A})\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{(\vec{\nabla} \times \vec{A})^2}{8\pi} \tag{37}$$

Set $\vec{A} = 0$, and rescale the free energy above, we have

$$\tilde{g}_s = \frac{\hbar^2}{2m|a|}|\vec{\nabla}\Psi|^2 - |\Psi|^2 + \frac{1}{2}|\Psi|^4 \tag{38}$$

From the first term, we can get the coherence length from dimensional analysis, which is

$$\xi^2 = \frac{\hbar^2}{2m|a|} \propto \frac{1}{|T - T_c|} \tag{39}$$

This length is divergent at critical point as shown in Fig. (13). When $T \to 0$, $\xi(T \to 0) \to \xi_0 = \frac{\hbar v_F}{\pi \Delta}$, where Δ is the gap of superconductor. The schematic representation of superconductor at its boundary is shown in Fig. (14(a)).

Suppose our system has one 1 dimension and $\Psi^* = \Psi = |\Psi|$, the saddle point

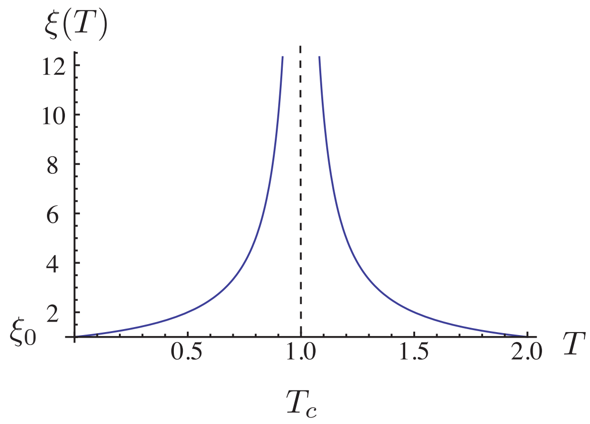
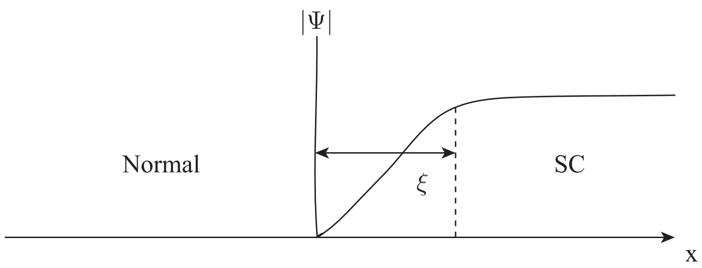
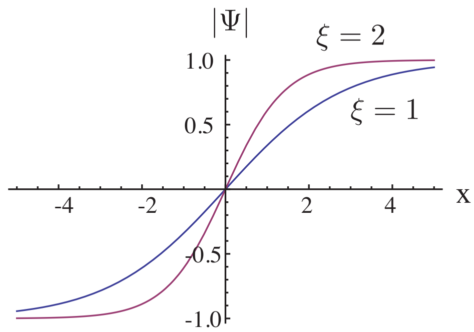


Figure 13: Coherence length as a function of temperature.



(a) Schematic plot



(b) Interface of 1d superconductor

Figure 14: The interface between superconductor and normal phase.

equation for free energy is

$$-\xi^2 \frac{d^2|\Psi|}{dx^2} - |\Psi| + |\Psi|^3 = 0 \tag{40}$$

After integration of x, it becomes

$$-\xi^2 (\frac{d|\Psi|}{dx})^2 - |\Psi|^2 + |\Psi|^4 = C \tag{41}$$

Since we have boundary condition: at $x = 0$, $|\Psi| = 0$ and at $x \to \infty$, $|\Psi| = 1$, thus constant $C = 1/2$. Finally,

$$|\Psi| = \tanh(\frac{x}{\sqrt{2}\xi}) \tag{42}$$

and the plot is in Fig. (14(b)).

The second length scale is London penetration length $\lambda_L(T)$. This is the length scale for varying $\vec{B}$, i.e. $\vec{A}$ and $\vec{J}$. Inside superconducting state, $|\Psi|^2 = n_s$. The free energy has relative parts

$$g = \frac{e^2 n_s}{2mc^2}\vec{A}^2 + \frac{1}{8\pi}(\vec{\nabla}\times\vec{A})^2 \tag{43}$$

Similarly, dimensional analysis gives us

$$\lambda_L = \sqrt{\frac{mc^2}{4\pi n_s e^2}} \sim \frac{1}{\sqrt{n_s}} \sim \frac{1}{|T-T_c|^{1/2}} \tag{44}$$

It is also divergent at critical point with the same critical exponent.

The comparison between coherence length and penetration length in type I and II superconductor is shown in Fig. (15).

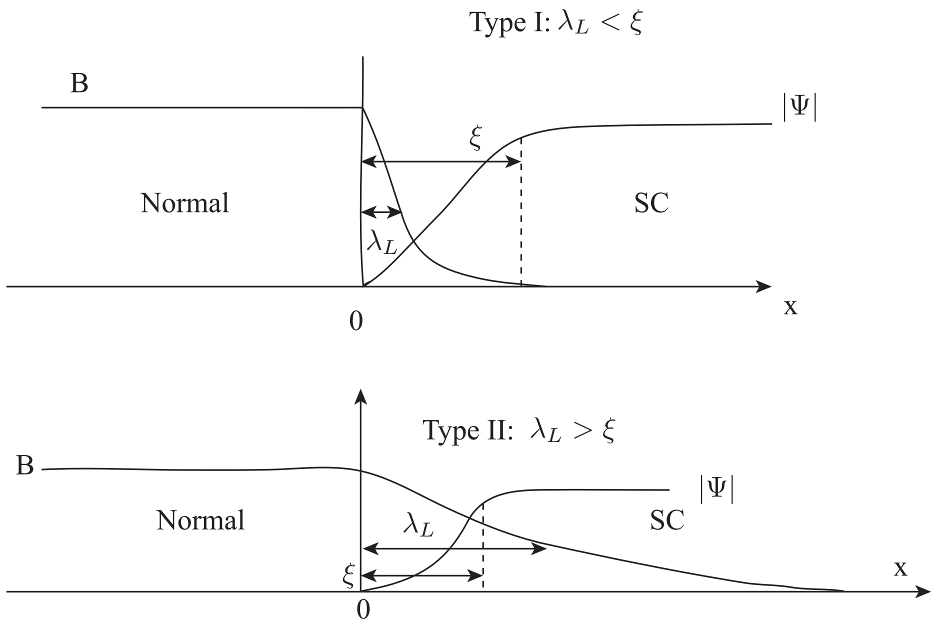


Figure 15: Coherence length and penetration length in type I and II superconductor.

According this distinction between type I and II superconductor, we can define a quality, which is called G-L ratio:

$$\kappa = \frac{\lambda_L}{\xi} = \frac{mc}{\hbar e}\sqrt{\frac{b}{2\pi}} \tag{45}$$

This quantity is nearly temperature independent. It only depends on the property of material. For type I superconductor, $\kappa < \frac{1}{\sqrt{2}}$, while for type II superconductor, $\kappa > \frac{1}{\sqrt{2}}$. The consequence of larger λ_L is the vortex state, which can be revealed by calculation of interface energy.

$$\Delta E_{interface} = L^2 \left[\int_{-\infty}^{\lambda} \underbrace{(-\frac{H_c^2}{8\pi})}_{\text{magnetic flux energy}} dx + \int_{\xi}^{\infty} \underbrace{(-\frac{H_c^2}{8\pi})}_{\text{condensation energy}} dx - \int_{-\infty}^{\infty}(-\frac{H_c^2}{8\pi})dx \right] \tag{46}$$

where L^2 is the area of interface. The first term above is magnetic flux energy

$$E_m = \frac{B^2}{8\pi} - \frac{B \cdot H}{4\pi} \tag{47}$$

for $H = H_c$. The second term is condensation energy $E_c = -\frac{a^2}{2b}$ which is also equal to the energy supplemented by critical external field H_c. Thus, $E_c = -\frac{H_c^2}{8\pi}$. The third term is the net energy without interface.

Then the interface energy per area is

$$\sigma = \frac{\Delta E_{interface}}{L^2} = \frac{H_c^2}{8\pi}(\xi - \lambda) \tag{48}$$

For $\xi > \lambda$, i.e. type I superconductor, $\sigma > 0$, no interface would like to be generated. The magnetic field is repelled outside the superconductor. On the contrary, $\xi < \lambda$, the interfaces are tended to be generated. The magnetic field can penetrate into superconductor in the form of vortices. The distinct phase diagrams of type I and type II superconductor as external magnetic and temperature change are shown in Fig. (16).

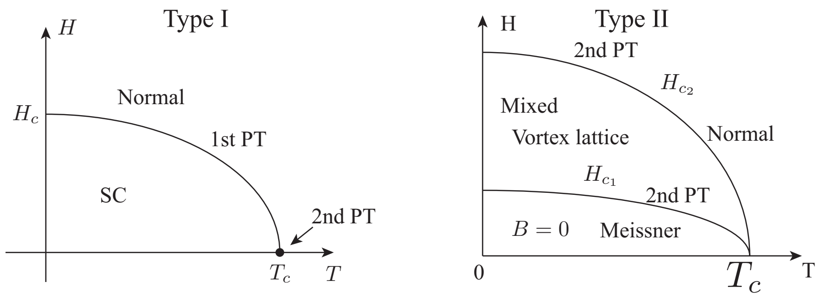


Figure 16: phase diagrams of type I and type II superconductor as functions of external magnetic and temperature.

3.2.6 Vortices

Forming vortices (Fig. (17)) is the cheapest way to create interfaces.

Recall

$$\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi \tag{49}$$

we have

$$\vec{\nabla}\theta = \frac{\hat{\phi}}{r} \tag{50}$$

and

$$\theta = \tan^{-1}\frac{y - y_0}{x - x_0} = \phi \tag{51}$$

Even though $\vec{\nabla}\theta = \frac{1}{r}\hat{\phi}$, but

$$\vec{\nabla}\theta - \frac{2\pi}{\phi_0}\vec{A} \propto \vec{v}_s \propto \vec{J}_s \approx e^{-r/\lambda} \tag{52}$$

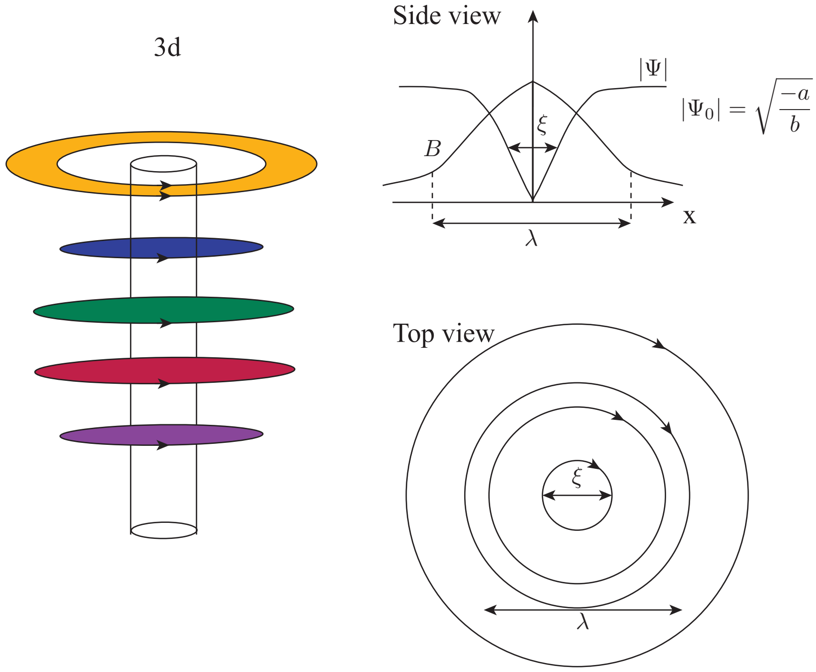


Figure 17: Pictorial illustration of vortices from different perspectives.

the physical current decays exponentially.

For $r \ll \lambda$, B is almost a constant and $B \approx H$, while $\vec{A} \approx \frac{1}{2}\vec{B} \times \vec{r} \approx 0$. Thus,

$$\vec{\nabla}\theta - \frac{2\pi}{\phi_0}\vec{A} \approx \vec{\nabla}\theta = \frac{\hat{\phi}}{r} \propto \vec{J}_s \tag{53}$$

When $r \ll \lambda$, $\vec{A}$ and $\int \frac{B^2}{8\pi} d^2r$ vanish.

Therefore, we can get the energy of single vortex line per length, which is

$$\frac{E_v}{L} = \int \left[\frac{\hbar^2}{2m} n_s (\vec{\nabla}\theta - \frac{2\pi}{\phi_0}\vec{A})^2 + \frac{B^2}{8\pi}\right] d^2r = \frac{\hbar^2}{2m} n_s \int_\xi^\lambda (\frac{1}{r})^2 d^2r = \frac{\pi\hbar^2 n_s}{2m} \ln\frac{\lambda}{\xi} = (\frac{\phi_0}{4\pi\lambda})^2 \ln\frac{\lambda}{\xi} \tag{54}$$

It is a finite value depending on λ and ξ.

3.2.7 Two critical magnetic field H_{c_1} and H_{c_2}

Starting from the free energy

$$g = \int \left[\frac{\hbar^2}{2m} n_s (\vec{\nabla}\theta - \frac{2\pi}{\phi_0}\vec{A})^2 + \frac{B^2}{8\pi} - \frac{B \cdot H}{4\pi}\right] d^2r \tag{55}$$

Use energy of single vortex to represent free energy, we get

$$g = N_v L (E_v - \frac{H}{4\pi}\phi_0) \tag{56}$$

where N_v is the number of vortices.

For $H > H_{c_1} = \frac{4\pi E_v}{\phi_0}$, $g < 0$, vortices enter superconductor. The lower critical field is

$$H_{c_1} = \frac{4\pi E_v}{\phi_0} = \frac{\phi_0}{4\pi\lambda^2}\ln(\frac{\lambda}{\xi}) \ll H_c = \frac{\phi_0}{4\pi\lambda\xi} \tag{57}$$

The higher critical field is defined when cores of vortices overlap. Since vortex core has length scale ξ. Thus, we have relation $H_{c_2}\xi^2 = \frac{\phi_0}{2\pi}$, then

$$H_{c_2} = \frac{\phi_0}{2\pi\lambda^2} = \kappa\sqrt{2}H_c > H_c \tag{58}$$

3.2.8 Movement of vortices

In type II conventional superconductor, vortices form beautiful hexagonal vortex lattice[7, 8]. In high temperature superconductor, vortices can move due to the large fluctuation and exotic interaction[9]. There are variety of phases due to the pinning and melting of vortices. Generally speaking, the phase diagram includes vortex liquid phase besides Meissner phase and vortex solid phase, which is shown in Fig. (18).

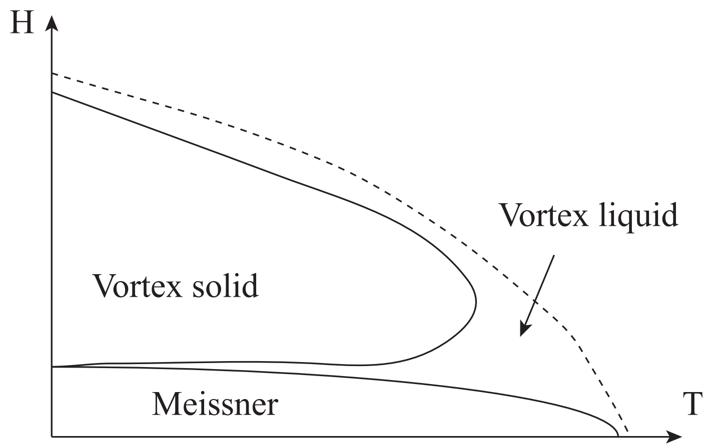


Figure 18: Phase diagram for high temperature superconductor.

Vortices determine limits on dissipationless transport. There are two ways to consider the movement of vortices. The first one is using Lorentz transformation. In the rest frame of vortices,

$$\vec{B} = n_v\phi_0\hat{z} \quad \vec{E} = 0 \tag{59}$$

Transforming to lab frame where vortices have velocity $\vec{v}_v$, then there is electric field generated

$$\vec{E} = \frac{1}{c}\vec{v}_v \times \vec{B} = \frac{n_v\phi_0}{c}\vec{v}_v \times \hat{z} \tag{60}$$

Obviously, $\vec{E} \perp \vec{v}_v$.

The second way is using Josephson relation. Suppose we have system size $w \times L$, with velocity of vortices along w direction. Then, the voltage cross L is generated by phase difference arising by moving vortices:

$$N_v\frac{2\pi|\vec{v}_v|}{w} = \frac{d\phi}{dt} = \frac{eV}{\hbar} = \frac{e}{\hbar}|\vec{E}|L \tag{61}$$

where ϕ is the phase difference. $\vec{v}_v$ is along y direction, i.e. the direction where system has w size, and $\vec{E}$ is along x direction which is perpendicular to the velocity. From this equation, we can obtain

$$E_x = \frac{1}{c}\frac{N_v}{wL}(\frac{hc}{e})v_v^y = \frac{1}{c}n_v\phi_0 v_v^y \tag{62}$$

Super current exerts force on a vortex and cause it to move. The currents can be stable by the balance between friction and Lorentz force exerting on vortices.

$$\gamma\vec{v}_v - \phi_0 n_s \hat{z}\times(\vec{v}_v - \vec{v}_s) = 0 \tag{63}$$

The total current $\vec{J} = n_s(\vec{v}_v - \vec{v}_s)$ can have relation with magnetic field generated by moving vortices:

$$E_x = (\frac{n_v\phi_0^2}{\gamma c})J_x \tag{64}$$

The conductivity thus is a finite value $\frac{\gamma c}{n_v\phi_0^2}$. Therefore, if we want to have true superconductor, we must pin the vortices by impurities.

4 Microscopics of superconductivity

4.1 Electron-phonon interaction

Assume ion on site $\vec{l}$ interacting with electron at $\vec{r}_j$ through interaction $V(\vec{r}_j - \vec{l})$. If ion has displacement $\vec{u}_{\vec{l}}(t)$, the interaction becomes $V(\vec{r}_j - \vec{l} - \vec{u}_{\vec{l}})$. Thus, the interaction between electron and vibration of lattice is

$$H_{ep} = \sum_{j,l}\left[V(\vec{r}_j - \vec{l} - \vec{u}_{\vec{l}}) - V(\vec{r}_j - \vec{l})\right] = -\sum_{j,l}\vec{u}_{\vec{l}}\cdot\vec{\nabla}V(\vec{r}_j - \vec{l}) \tag{65}$$

In second quantization notation, we can have wave function of electron is

$$\Psi(\vec{r}) = \sum_k c_k\psi_k(\vec{r}) \quad \Psi^\dagger(\vec{r}) = \sum_k c_k^\dagger\psi_k^\dagger(\vec{r}) \tag{66}$$

where c_k and $c_k^\dagger$ are fermionic annihilation and creation operator. And

$$\psi_k(\vec{r}) = u_k(\vec{r})e^{i\vec{k}\cdot\vec{r}} \quad u_k(\vec{r}) = u_k(\vec{r}+\vec{l}) \tag{67}$$

are Bloch functions. Then, the Hamiltonian becomes

$$\begin{aligned} H_{ep} &= -\int\Psi^\dagger(\vec{r})(\sum_l\vec{u}_{\vec{l}}\cdot\vec{\nabla}V(\vec{r}_j - \vec{l}))\Psi(\vec{r})d\vec{r} \\ &= -\sum_l\sum_{k,k'}\vec{u}_{\vec{l}}\cdot\int d\vec{r}u_{k'}^*(\vec{r})u_k(\vec{r})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\vec{\nabla}V(\vec{r}-\vec{l})c_{k'}^\dagger c_k \end{aligned} \tag{68}$$

Use phonon operator

$$
\begin{aligned}
\vec{u}_{\vec{l}} &= \frac{1}{\sqrt{NM}} \sum_{q,s} \vec{e}_{qs} Q_{qs} e^{i\vec{q}\cdot\vec{l}} \\
Q_{qs} &= \sqrt{\frac{\hbar}{2\omega_{qs}}} (a_{qs} + a^\dagger_{-qs}) \\
\vec{e}_{qs} &= \vec{e}_{-qs}, \quad |\vec{e}_{qs}| = 1
\end{aligned}
\tag{69}
$$

the above Hamiltonian becomes

$$
H_{ep} = -\sum_l \sum_{k,k'} \sum_{qs} (\frac{\hbar}{2NM\omega_{qs}})^{1/2} (a_{qs} + a^\dagger_{-qs}) c^\dagger_{k'} c_k e^{i\vec{q}\cdot\vec{l}} \int d\vec{r} u^*_{k'}(\vec{r}) u_k(\vec{r}) e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} \vec{\nabla} V(\vec{r}-\vec{l})
\tag{70}
$$

where N is the number of cell and M is the mass of ion. s is the index of lattice wave, suppose there are more than one wave of lattice vibration.

For the expression, the integral part can be simplified as

$$
\begin{aligned}
I &= \int d\vec{r} u^*_{k'}(\vec{r}) u_k(\vec{r}) e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} \vec{\nabla} V(\vec{r}-\vec{l}) \\
&= i \sum_p \vec{p} V_p \int d\vec{r} u^*_{k'}(\vec{r}) u_k(\vec{r}) e^{i(\vec{k}+\vec{p}-\vec{k}')\cdot\vec{r}} e^{i(\vec{k}-\vec{k}')\cdot\vec{l}}
\end{aligned}
\tag{71}
$$

For simplicity but without loss any generality, $u_k(\vec{r}) = (N\Omega)^{-1/2}$. Then

$$
I = i \sum_p \vec{p} V_p \delta_{\vec{p},\vec{k}'-\vec{k}} e^{i(\vec{k}-\vec{k}')\cdot\vec{l}}
\tag{72}
$$

Besides,

$$
\frac{1}{N} \sum_l e^{i(\vec{k}+\vec{q}-\vec{k}')\cdot\vec{l}} = \delta_{\vec{k}',\vec{k}+\vec{q}+\vec{K}_n}
\tag{73}
$$

where $\vec{K}_n = \sum_{i=1}^{3} n_i \vec{b}_i$ is the reciprocal vectors. For situation at long wavelength and low temperature, $\vec{K}_n = 0$. In other words, we only consider scattering within first Brillouin zone. Finally, the Hamiltonian describing electron and phonon interaction is

$$
H_{ep} = \sum_{k,q} D_q (a_q + a^\dagger_{-q}) c^\dagger_{k+q} c_k
\tag{74}
$$

where

$$
D_q = -i(\frac{N\hbar}{2M\omega_q})^{-1/2} V_q
\tag{75}
$$

and V_q is Coulomb interaction

$$
V_q = -\frac{4\pi e^2}{q^2}
\tag{76}
$$

The effective interaction between electrons and phonons can be derived based on this Hamiltonian. Consider scattering matrix element by second order perturbation theory:

$$
\begin{aligned}
&\sum_m \frac{\langle f|H_{ep}|m\rangle\langle m|H_{ep}|i\rangle}{E_i - E_m} \\
&= \sum_q |D_q|^2 \left[\frac{1}{\epsilon_k - \epsilon_{k+q} - \hbar\omega_{-q}} - \frac{1}{\epsilon_k - \epsilon_{k+q} + \hbar\omega_q}\right] \langle f| c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c_{k\sigma} |i\rangle
\end{aligned}
\tag{77}
$$

where

$$
\begin{aligned}
|i\rangle &= |\ldots, n_{k,\sigma}, \ldots, n_{k',\sigma'}, \ldots, n_{k'-q,\sigma'}, \ldots, n_{k+q,\sigma}, \ldots; 0\rangle \\
|f\rangle &= |\ldots, (n_{k,\sigma}-1), \ldots, (n_{k',\sigma'}-1), \ldots, (n_{k'-q,\sigma'}+1), \ldots, (n_{k+q,\sigma}+1), \ldots; 0\rangle
\end{aligned}
\tag{78}
$$

are both vacuum for phonon. ϵ_k is the energy for electron with momentum k. The effective interaction now turns into

$$
\begin{aligned}
H_{eff} &= \frac{1}{2}\sum_{\substack{k',k,q\\ \sigma,\sigma'}} \frac{2|D_q|^2\hbar\omega_q}{(\epsilon_k-\epsilon_{k+q})^2-(\hbar\omega_q)^2} c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma} \\
&= \frac{1}{2}\sum_{\substack{k',k,q\\ \sigma,\sigma'}} V_{kq} c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}
\end{aligned}
\tag{79}
$$

When $|\epsilon_k - \epsilon_{k+q}| < \hbar\omega_q$, $V_{kq} < 0$. Thus, near fermi surface within energy difference $\hbar\omega_q$, the effective interaction is attractive. When this interaction is larger than effective Coulomb repulsion, electrons would form Cooper pairs.

4.2 Cooper pair

The interaction

$$
V_{kq} = \frac{2|D_q|^2\hbar\omega_q}{(\epsilon_k-\epsilon_{k+q})^2-(\hbar\omega_q)^2}
\tag{80}
$$

incorporating with Coulomb interaction

$$
H_{coul} = \frac{1}{2}\sum_{\substack{k',k,q\\ \sigma,\sigma'}} \frac{4\pi e^2}{q^2+\lambda^2} c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}
\tag{81}
$$

can be replaced by a constant. Then the Hamiltonian can be written as

$$
H' = -\frac{V}{2}\sum_{\substack{k',k,q\\ \sigma,\sigma'}} c^\dagger_{k,\sigma} c^\dagger_{q-k,\sigma'} c_{q-k'\sigma'} c_{k'\sigma}
\tag{82}
$$

The above scattering process largely happens at $q = 0$. The total Hamiltonian is then

$$
H = \sum_k E_k (c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) - V \sum_{k,k'} c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow}
\tag{83}
$$

Below, $(k\uparrow)$ and $(-k\downarrow)$ are replaced by k and $-k$ for simplicity. It is easy to see that the eigenstates of H should include all possible scattering pairs:

$$
|\psi\rangle = \sum_{k>k_F} a(k) c^\dagger_k c^\dagger_{-k} |F\rangle
\tag{84}
$$

where $|F\rangle$ is the ground state of electron gas.

For this state, the energy is

$$E = 2\sum_{k>k_f} \epsilon_k |a(k)|^2 - V \sum_{k,k'>k_F} a^*(k')a(k) \tag{85}$$

Together with condition

$$\langle\psi|\psi\rangle = \sum_k |a(k)|^2 = 1 \tag{86}$$

We can have eigenvalue equation

$$(2\epsilon_k - \lambda)a(k) = V \sum_{k'>k_F} a(k') \tag{87}$$

with eigenvalue λ. After changing summation into integration, the solution is

$$\lambda = -2\hbar\omega_D \left[\exp(\frac{2}{g(0)V}) - 1\right] \tag{88}$$

where $g(0)$ is the density of state at fermi surface. This energy is always negative, which means as long as the fermi surface exists, the two electron with attractive interaction and opposite spin would always form a bound state – Cooper pair[10].

4.3 BCS theory[11]

Above we saw Cooper pairs form ground state of superconductor, thus following operators are non-zero.

$$\begin{aligned} \langle c_k^\dagger c_{-k}^\dagger\rangle &= \langle 0|c_k^\dagger c_{-k}^\dagger|0\rangle \neq 0 \\ \langle c_{-k}c_k\rangle &= \langle 0|c_{-k}c_k|0\rangle \neq 0 \end{aligned} \tag{89}$$

where the ground state is

$$|0\rangle = \prod_k (u_k + v_k c_k^\dagger c_{-k}^\dagger)|vac\rangle \tag{90}$$

and u_k and v_k are parameters to be determined below. According to the normalization condition $\langle 0|0\rangle = 1$, we have the first constraint on u_k and v_k:

$$u_k^2 + v_k^2 = 1 \tag{91}$$

In this section, we will use mean field method to solve the Hamiltonian for superconductor and obtain its ground state. The pair operators can be written in terms of their average value.

$$\begin{aligned} c_{-k}c_k &= \langle c_{-k}c_k\rangle + (c_{-k}c_k - \langle c_{-k}c_k\rangle) \\ c_k^\dagger c_{-k}^\dagger &= \langle c_k^\dagger c_{-k}^\dagger\rangle + (c_k^\dagger c_{-k}^\dagger - \langle c_k^\dagger c_{-k}^\dagger\rangle) \end{aligned} \tag{92}$$

Therefore, the Hamiltonian can be approximately

$$H_{mf} = \sum_k \epsilon_k (c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow}) - V\sum_{k,k'} \left[c_k^\dagger c_{-k}^\dagger \langle c_{-k}c_k\rangle + c_{-k}c_k\langle c_k^\dagger c_{-k}^\dagger\rangle - \langle c_k^\dagger c_{-k}^\dagger\rangle\langle c_{-k}c_k\rangle\right] \tag{93}$$

Define order parameter

$$\Delta = V \sum_k \langle c_{-k} c_k \rangle \quad \Delta^* = V \sum_k \langle c_k^\dagger c_{-k}^\dagger \rangle \tag{94}$$

Only consider Δ as a real number, consequently,

$$H_{mf} = \sum_k \epsilon_k (c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow}) - \Delta \sum_k (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) + \frac{\Delta^2}{V} \tag{95}$$

Then, using Bogoliubov transformation, we can diagonalize this quadratic Hamiltonian. The final result is

$$H_{mf} = E(0) + \sum_k \sqrt{\epsilon_k^2 + \Delta^2} (\alpha_k^\dagger \alpha_k + \alpha_{-k}^\dagger \alpha_{-k}) \tag{96}$$

where new fermions and parameters are

$$\begin{aligned} \alpha_k^\dagger &= u_k c_k^\dagger - v_k c_{-k} \\ \alpha_k &= u_k c_k - v_k c_{-k}^\dagger \\ u_k^2 &= \frac{1}{2}(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^2}}) \\ v_k^2 &= \frac{1}{2}(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^2}}) \end{aligned} \tag{97}$$

and

$$E(0) = 2 \sum_k \epsilon_k v_k^2 - 2\Delta \sum_k u_k v_k + \frac{\Delta^2}{V} \tag{98}$$

is the ground state energy.

We can know the meaning of each parameter clearly. While v_k^2 is the probability to generate two holes within fermi surface and excite two electrons outside the fermi surface – form a Cooper pair, u_k^2 is the probability that Cooper pair would not form.

The ground state should be the vacuum of quasiparticle.

$$\alpha_k |0\rangle = \alpha_{-k} |0\rangle = 0 \tag{99}$$

Thus, the ground state is

$$\begin{aligned} |\psi\rangle &= \prod_k \alpha_k \alpha_{-k} |vac\rangle \\ &= \prod_k (u_k c_k - v_k c_{-k}^\dagger)(u_k c_{-k} + v_k c_k^\dagger) |vac\rangle \\ &= \prod_k (u_k v_k + v_k^2 c_k^\dagger c_{-k}^\dagger) |vac\rangle \end{aligned} \tag{100}$$

The normalization factor is $(\prod_k v_k)^2$. Thus, the normalized ground state wave function is

$$|0\rangle = \prod_k (u_k + v_k c_k^\dagger c_{-k}^\dagger) |vac\rangle \tag{101}$$

4.4 Tunneling between superconductor and normal metal

Assume we have two material which are described by Hamiltonian H_1 and H_2 with eigenstates $|\lambda_1\rangle$ and $|\mu_2\rangle$:

$$H_1|\lambda_1\rangle = \epsilon_\lambda|\lambda_1\rangle \quad H_2|\mu_2\rangle = \epsilon_\mu|\mu_2\rangle \tag{102}$$

The Hamiltonian for tunneling is

$$H_T = \sum_{k,k'}(\mathcal{M}_{k,k'} c_{k'}^{(2)\dagger} c_k^{(1)}) \tag{103}$$

When there is bias voltage V. The tunneling probability is

$$\beth_{12} = \frac{2\pi}{\hbar}\sum_{\lambda,\mu}|\langle\lambda_1|\langle\mu_2|\sum_{k,k'}\mathcal{M}_{k,k'}c_{k'}^{(2)\dagger}c_k^{(1)}|0_2\rangle|0_1\rangle|^2\delta(\epsilon_\lambda - \epsilon_\mu + eV) \tag{104}$$

where $|0_i\rangle$ is the ground state for material i. Since $\mathcal{M}_{k,k'}$ is independent of momentum, the above formula for tunneling can be simplified as

$$\beth_{12} = \frac{2\pi}{\hbar}|\mathcal{M}|^2\int_{-\infty}^{\infty} d\epsilon N_1(\epsilon)N_2(\epsilon + eV) \tag{105}$$

where

$$\begin{aligned} N_1(\epsilon) &= \sum_{k,\lambda}|\langle\lambda_1|c_k^{(1)}|0_1\rangle|^2\delta(\epsilon - \epsilon_\lambda) \\ N_2(\epsilon + eV) &= \sum_{k',\mu}|\langle\mu_2|c_{k'}^{(2)}|0_2\rangle|^2\delta(\epsilon + eV - \epsilon_\mu) \end{aligned} \tag{106}$$

Assume 2 is normal metal, then

$$N_2(\epsilon + eV) = \sum_{k'}\theta(\epsilon_{k'})\delta(\epsilon + eV - \epsilon_{k'}) = g_2(\epsilon + eV)\theta(\epsilon + eV) \tag{107}$$

where g_2 is the density of state.

While 1 is superconductor, then

$$N_1(\epsilon) = \sum_k v_k^2\delta(\epsilon - \sqrt{\epsilon_k^2 + \Delta^2}) = g_1(0)Re\left[\frac{|\epsilon|}{\sqrt{\epsilon^2 - \Delta^2}}\right] \tag{108}$$

where $g_1(0)$ is the density of state at fermi energy, since

$$\sum_\lambda|\langle\lambda_1|c_k^{(1)}|0_1\rangle|^2 = \langle 0_1|c_k^{(1)\dagger}c_k^{(1)}|0_1\rangle = v_k^2 \tag{109}$$

The current is thus

$$I_{SN} = \frac{2\pi e}{\hbar}|\mathcal{M}|^2 g_2(0)\int_{-\infty}^{\infty} d\epsilon g_1(\epsilon)\theta(\epsilon + eV) \tag{110}$$

And the differential conductance is

$$\begin{aligned}
G_{SN} = (\frac{dI_{SN}}{dV})_{T=0} &= \frac{2\pi e^2}{\hbar}|\mathcal{M}|^2 g_2(0)\int_{-\infty}^{\infty} d\epsilon g_1(\epsilon)\delta(\epsilon + eV) \\
&= \frac{2\pi e^2}{\hbar}|\mathcal{M}|^2 g_1(0)g_2(0)\int_{-\infty}^{\infty} d\epsilon \frac{g_1(\epsilon)}{g_1(0)}\delta(\epsilon + eV) \\
&= G_{NN} Re\left[\frac{eV}{\sqrt{(eV)^2 - \Delta^2}}\right]
\end{aligned} \tag{111}$$

The ratio $\frac{G_{SN}}{G_{NN}}$ is the density of state of quasiparticle in superconductor. And the above derivation also gives us the tunneling current as a function of bias voltage at zero temperature.

$$I_{SN}(V) = G_{NN} Re\left[\sqrt{(eV)^2 - \Delta^2}\right] \tag{112}$$

We can conclude that when $eV < \Delta$, there is no current at all. Once $eV > \Delta$, there is current from 1 to 2 material. This agrees with Giaever's experimental result.

5 Josephson Effect

The Josephson effect is an example of a macroscopic quantum phenomenon. It is named after the British physicist Brian David Josephson, who predicted in 1962 the mathematical relationships for the current and voltage across the weak link[13, ?]. There are two type of Josephson effect.

The DC Josephson effect states that if we have a insulator whose thickness satisfies $d < \xi$, where ξ is the coherence length, two superconductors on the two sides of this insulator would have current between them.

$$I = I_0 \sin\gamma \tag{113}$$

where $\gamma = \phi_2 - \phi_1$ is the phase difference of these two superconductor.

AC Josephson effect states that when we applied a voltage between this junction, then the tunneling current satisfies

$$I(t) = I_0 \sin(\omega_0 t + \gamma) \tag{114}$$

Below, we use G-L phenomenological theory to derive above two relations.

5.1 DC Josephson effect

In this case, $A = 0$. Assume junction is along z direction. The supercurrent is

$$j_s = j_z = -\frac{i\hbar e^*}{2m^*}(\psi_1^*\frac{\partial\psi_1}{\partial z} - \psi_1\frac{\partial\psi_1^*}{\partial z}) \tag{115}$$

If $d > \xi$, $j_s = 0$. If $d < \xi$, then Cooper pairs in both side of superconductor can penetrate into each other. Phenomenologically, the Cooper pair in superconductor 2 is generated by super current of superconductor 1.

$$-i\hbar\frac{\partial\psi_1}{\partial z} = -i\frac{\hbar}{b}\psi_2 \tag{116}$$

Then the tunneling current is

$$j_s = (\frac{e^*\hbar}{m^*b})|\psi_1||\psi_2|\sin(\phi_2 - \phi_1) = j_0\sin(\phi_2 - \phi_1) \tag{117}$$

Thus, this is exactly the DC Josephson effect.

5.2 AC Josephson effect

When $A \neq 0$, the super current can be expressed as

$$j_s = -i(\frac{e^*\hbar}{2m})(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{(e^*)^2}{m^*c}|\psi|^2A = \frac{n_se^*}{m^*}(\hbar\nabla\phi - \frac{e^*}{c}A) = \frac{n_se^*}{m^*}v_s \tag{118}$$

where

$$v_s = \hbar\nabla\phi - \frac{e^*}{c}A \tag{119}$$

When we do gauge transformation to A:

$$A \to A' + \nabla\chi \tag{120}$$

the phase would be changed correspondently:

$$\phi \to \phi' + \frac{e^*}{\hbar c}\chi \tag{121}$$

In order to make j_z gauge invariant, we define a gauge invariant phase difference

$$\gamma = \phi_2 - \phi_1 - \frac{e^*}{\hbar c}\int_1^2 A_z dz \tag{122}$$

Consider the magnetic field generated by current is along y direction, we can choose a gauge

$$A_x = 0 \quad A_y = 0 \quad A_z = A_z(x) \tag{123}$$

Then

$$B_y(x) = -\frac{\partial A_x}{\partial x} = B_0 \Rightarrow A_z(x) = -B_0x \tag{124}$$

which is a constant. In this case,

$$\gamma = \phi_2 - \phi_1 - \frac{2e}{\hbar c}\int_{-\frac{d}{2}-\lambda}^{\frac{d}{2}+\lambda} A_z(x)dz = \phi_2 - \phi_1 + \frac{2e}{\hbar c}B_0(d + 2\lambda)x \tag{125}$$

where λ is the penetration length.

Therefore, the current is

$$j_s(x) = j_0 \sin\left[\phi_2 - \phi_1 + \frac{2e}{\hbar c} B_0(d + 2\lambda)x\right] \tag{126}$$

Besides,

$$\frac{\partial \gamma}{\partial t} = -\frac{2e}{\hbar c}\int_1^2 \frac{\partial A_z}{\partial t} dz = \frac{2e}{\hbar}\int_1^2 E_z dz = \frac{2eV}{\hbar} \tag{127}$$

Thus, actually, γ is a function of time:

$$\gamma = (\frac{2eV}{\hbar})t + \gamma_0 \tag{128}$$

where $\gamma_0 = \phi_2 - \phi_1$.

Consequently, we can get the AC Josephson relation:

$$j_z = j_0 \sin\gamma = j_0 \sin(\omega_0 t + \gamma_0) \tag{129}$$

where $\omega_0 = \frac{2eV}{\hbar}$

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