Physics 7240: Advanced Statistical Mechanics

Lecture 7: Introduction to Quenched Disorder

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(Dated: August 10, 2020)

Abstract

In these lecture notes, I will discuss effects of frozen local heterogeneity ("quenched disorder") on phase transitions and on states of condensed matter phases, stability and properties.

- Introduction and motivation
- Disorder near a continuous phase transitions
  - random bond
  - random field
  - Harris criterion
- Disorder in ordered states
  - random bond
  - random field
  - Imry-Ma-Larkin stability analysis
- Physical systems
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  - vortex lattice
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  - liquid crystal with a "dirty" substrate
- Summary and conclusions

I. INTRODUCTION

A. Motivation and physical background

Initial early studies of condensed matter focused on idealized homogeneous systems, e.g., localized spins and electron liquid in ideal impurity-free crystals, phase transitions and ordered states of homogeneous matter. These by now are quite well understood.[1]

However, all realistic systems include local random heterogeneities, motivating extensive studies of random frozen "*quenched* disorder". Important traditional examples in solid-state context include localization of electrons, responsible for the existence of Anderson insulators characterized by a vanishing zero-temperature conductivity, as well as a finite zero-temperature residual resistivity of metals, pinning of vortices in superconductors (which would otherwise move and dissipate energy resulting in finite resistivity), and charge density waves exhibiting impurities-induced nonlinear current-voltage characteristics[2, 3].

Quenched disorder is frozen heterogeneity that is a background random potential for the fluctuating (thermal and/or quantum mechanical) degrees of freedom. Here, however, we will have little to say about much more challenging problems of "self-generated disorder", as in structural glasses and jammed systems, where even without background heterogeneity the degrees of freedom get kinetically arrested, falling out of equilibrium. We also distinguish *quenched* disorder from *annealed* "disorder", where the random degrees of freedom are ergodic. The latter is nothing more than an additional thermodynamic degree of freedom, and so is not qualitatively distinct from a disorder-free multi-component system. The additional "annealed disorder" degrees of freedom can in principle be traced out, obtaining a disorder-free system with modified parameters.

Impurity defects can typically easily rearrange and equilibrate (acting like annealed "disorder") inside soft matter and thus quenched disorder is less common in such soft systems. However, there are many interesting and nontrivial exceptions. These include soft matter encapsulated inside a random solid matrix or in contact with a solid rough substrate. Interesting studied examples include liquid crystals confined inside a random aerogel or aerosil matrix[5–8] or liquid crystal cells perturbed by a random substrate.[9].

Influence of ever-present quenched disorder on phase transitions and on concomitant ordered phases is another extremely developed subject of research.[2] Prominent examples include magnetism and elastic soft media randomly pinned by the defected host atomic matrix or an underlying heterogeneous substrate as realized in pinned vortex lattices, charge density waves, magnetic domain walls, contact lines, earthquake and friction phenomena. These will be extensively discussed by Pierre Le Doussal.

In general, this is an extremely challenging subject. It requires one to understand the behavior of infinite number of degrees of freedom without the usual powerful crutch of translational invariance in the presence of thermal fluctuations, divergent near a continuous phase transition. Even at zero temperature, the problem difficult because minimizing heterogeneous energy functional requires balancing two frustrated tendencies, order and disorder. Clearly, it is impossible and actually unnecessary to find solutions for a specific realization of disorder. Instead, we often only need statistical typical properties of the system. Thus, in many cases, it is sufficient to compute disorder averaged physical properties, such the average free-energy and order parameter correlators.

Even the simplest limit of weak, statistically homogeneous disorder, that we will focus on here can be quite nontrivial, though significant progress has been made. We will have little to say about the infinitely more challenging problems of strong disorder such as spin-glasses and (even more difficult) structural glasses, where there is no obvious state to perturb about, and in fact understanding the statistical properties of the highly nontrivial random ground state is the key problem itself.

B. Model

For concreteness it is important to have an explicit lattice model in mind, as for example Ising and Heisenberg model, that describe broad range of physical systems with, respectively discrete Z_2 and continuous $O(N)$ symmetries. Since the former is a special $N = 1$ case of the latter (though for some quantities the two are qualitatively very different) we consider the $O(N)$ model with a Hamiltonian

$$H = -\sum_{\mathbf{x},\mathbf{x}'} J_{\mathbf{x}\mathbf{x}'} \vec{S}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}'} - \sum_{\mathbf{x}} \vec{h}_{\mathbf{x}} \cdot \vec{S}_{\mathbf{x}}, \tag{1}$$

describing for a example a ferromagnetic, spin-aligning exchange interaction $J_{ij} > 0$ between a lattice of spins $\vec{S}_i$ on sites i, j, and under an additional influence of a local magnetic field $\vec{h}_i$. For the simplest case of nearest neighbor J_{ij} and uniform $\vec{h}_i$ clearly the state is ferromagnetic with all spins aligned along $\vec{h}$ and with each other. A quite amazing observation is that for sufficiently low $T < T_c$ and high enough dimensions, despite randomizing thermal fluctuations such state is stable even for vanishing external field. One can encode significant additional complexity through the exchange J_{ij} extending beyond nearest neighbors, that can frustrate and destabilize ferromagnetic order even for translationally invariant case and for a vanishing field.

We focus on the simplest case of nearest-neighbor ferromagnetic exchange and include the effects of quenched disorder by taking $J_{ij} > 0$ to be randomly distributed according to a distribution $P_J[J_{ij}]$. This is the ferromagnetic “random bond“ problem, for example corresponding to non-magnetic impurities and vacancies. As mentioned above, instead, positive and negative random exchanges, with a vanishing mean is the "spin-glass" problem that is infinitely more difficult as even the ground state is highly random and nontrivial. We will not consider spin-glasses, focusing on the case where disorder is a weak perturbation to an obvious ferromagnetic ground state. In addition, we can include random $\vec{h}_i$'s, characterized by a distribution $P_h[h_{ij}]$, the so-called "random field" problem, corresponding to random magnetic impurities.

With the exceptions where (rare and highly cherished) exact solutions are available, or numerical analysis is undertaken, to make progress it is helpful to work with a continuous field theory that is a long-wavelength limit approximation of the underlying lattice model. The corresponding $O(N)$ field theory is given by

$$H = \int d^d x \left[\frac{1}{2} K (\boldsymbol{\nabla} \vec{S})^2 + \frac{1}{2}(t_0 + \delta t(\mathbf{x}))|\vec{S}|^2 + \frac{\lambda}{4}|\vec{S}|^4 - \vec{h}(\mathbf{x}) \cdot \vec{S} \right], \tag{2}$$

where $\vec{S}$ is the coarse-grained local order parameter, K, λ are effective model parameters, that for weak disorder can be taken to be constants, $t_0 \sim T - T_{c0}$ is the "bare" reduced temperature, whose sign change drives the paramagnet-ferromagnet transition at T_{c0} (within mean-field approximation), spontaneously breaking $O(N)$ symmetry for $t_0 < 0$; the true transition is shifted to $t = t_c$ by thermal fluctuations and disorder. The random bond and random field disorder are respectively encoded into $\delta t(\mathbf{x})$ and $\vec{h}(\mathbf{x})$, that for simplicity we take to be characterized by zero-mean, Gaussian distribution completely specified by variance Δ_t, Δ_h

$$\overline{\delta t(\mathbf{x}) \delta t(\mathbf{x}')} = \Delta_t \delta^d(\mathbf{x} - \mathbf{x}') \ , \quad \overline{h_i(\mathbf{x}) h_i(\mathbf{x}')} = \delta_{ij} \Delta_h \delta^d(\mathbf{x} - \mathbf{x}') \ , \tag{3}$$

The problem we are then faced with is to understand the effects of random $\delta t(\mathbf{x}), \vec{h}(\mathbf{x})$ in the presence of thermal fluctuations, inside the phases and near the critical point that separates them.

C. Physical observables

1. *disorder-free criticality*

We first recall that even in the absence of disorder, this and many other by now well-understood phase transitions exhibit quite nontrivial universal phenomenology that took a few decades to sort out in a beautiful set of theoretical developments[1] in the 1970s, led by Ben Widom, Leo Kadanoff, Sasha Migdal, Michael Fisher, Sergey Pokrovsky, and Ken Wilson.

These singular effects of fluctuations only become truly important near a critical point of transition $t = 0$ (and below the upper-critical dimension $d < d_{uc} = 4$ in this case; see below), where the stabilizing $t|\vec{S}|^2$ is tuned to vanish and the nonlinearities (e.g., $\lambda|\vec{S}|^4$) must be taken into account nonperturbatively (a conventional perturbation theory in λ fails), typically using numerical analysis or renormalization group (RG) transformation. The upshot of such analysis is that fluctuations are controlled by a single correlation length $\xi(t)$, that characterizes the range of spatial correlations and diverges near the critical point. As a result all physical properties scale with $\xi(t)$, forgetting about microscopic details, and thereby exhibiting universality. More specifically for continuous transitions (couched in the language of a PM-FM transition) RG analysis predicts consistent with experiments that magnetization, magnetic susceptibility to external magnetic field, heat capacity and the correlation length scale according to:

$$M(T, B = 0) \propto |T_c - T|^{\beta}, \quad \chi(T) \propto |T - T_c|^{-\gamma}, \tag{4}$$

$$M(T = T_c, B) \propto B^{1/\delta}, \quad C(T) \propto |T - T_c|^{-\alpha}, \tag{5}$$

$$\xi(T_c, B = 0) \propto |T - T_c|^{-\nu}, \tag{6}$$

$$\tag{7}$$

where "critical exponents" $\beta, \gamma, \delta, \alpha, \nu$ are universal, in that they depend only on the symmetry and dimensionality of the continuous phase transition, namely the so-called its "universality class". These exponents satisfy a variety of exact relations:

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1), \tag{8}$$

$$2 - \alpha = d\nu, \quad \gamma = (2 - \eta)\nu, \tag{9}$$

$$\tag{10}$$

leading to only two independent exponents. In mean-field theory $\beta = 1/2, \gamma = 1, \delta = 3, \alpha = 0, \nu = 1/2$, but more generally are irrational but universal numbers. In above we defined the correlation length ξ that characterizes the range of spatial correlations that diverge at the phase transition.

This phenomenology is captured by the RG theory, where one integrates out a shell of high momenta (short-scale) modes $\vec{S}_>$ with $\Lambda/b < k < \Lambda$, for simplicity rescales the length scales and fields according to:

$$\mathbf{x} = \mathbf{x}'b, \quad \vec{S}_<(\mathbf{x}'b) = b^\zeta \vec{S}'(\mathbf{x}') \tag{11}$$

so as to bring the microscopic uv-cutoff lattice cutoff ab back to a. The resulting effective Hamiltonian H' governing the lower momenta (long-scale) modes then takes on the same form as H, but with rescaled parameters, that to 0th order (no diagrammatic corrections) are

$$K(b) = Kb^{d-2+2\zeta}, \quad t(b) = tb^{d+2\zeta}, \quad \lambda(b) = \lambda b^{d+4\zeta}. \tag{12}$$

It is convenient to choose $\zeta = (2-d-\eta)/2$ to keep K fixed i.e., $K'(b) = K$ or to equivalently look at the dimensionless couplings $\hat{t}(b) \equiv t(b)/K(b)$ and $\hat{\lambda}(b) \equiv \lambda(b)/K^2(b)$. To 0th order $\eta = 0$ and we have

$$\hat{t}(b) = \hat{t}b^2, \quad \hat{\lambda}(b) = \hat{\lambda}b^{4-d}, \tag{13}$$

showing that the Gaussian fixed point $\lambda = 0$ is unstable for $d < d_{uc} = 4$ and that $t > 0$ ($t < 0$) flows to positive (negative) infinity corresponding to the disordered paramagnetic (ordered ferromagnetic) phase.

The result of this procedure is summarized by RG flow equations for the dimensionless couplings , $\hat{t}(b)$, and $\hat{\lambda}(b)$, as a function of the rescaling factor b, that for an infinitesimal rescaling $b = e^{\delta\ell}$ take a differential equations form. To first order in $\hat{\lambda}$, near the critical point (small $\hat{t}$) these are illustrated in Fig.1, and are given by

$$\frac{d\hat{t}}{d\ell} = 2\hat{t} + c_0\hat{\lambda} - c_1\hat{t}\hat{\lambda}, \quad \frac{d\hat{\lambda}}{d\ell} = (4-d)\hat{\lambda} - c_2\hat{\lambda}^2 \tag{14}$$

where c_i are universal constants that are functions of d, N. These determine the phenomenology at long scales, $\ell \to \infty$ and its relation to the microscopic model as one coarse-grains by scale $b = e^\ell$. Namely, we observe that for a positive $\lambda(b)$, the so-called Gaussian fixed point (characterized by harmonic Hamiltonian, i.e., $\lambda = 0$) is unstable and on a critical manifold $t = -\frac{1}{2}c_0\lambda$ the model flows to the infrared attractive Wilson-Fisher fixed point at a finite $\lambda^* = (4-d)/c_2$, which controls the critical behavior sufficiently close to the transition.[1] On the left (right) sides of the critical separatrix, the flow are to the large negative (positive)

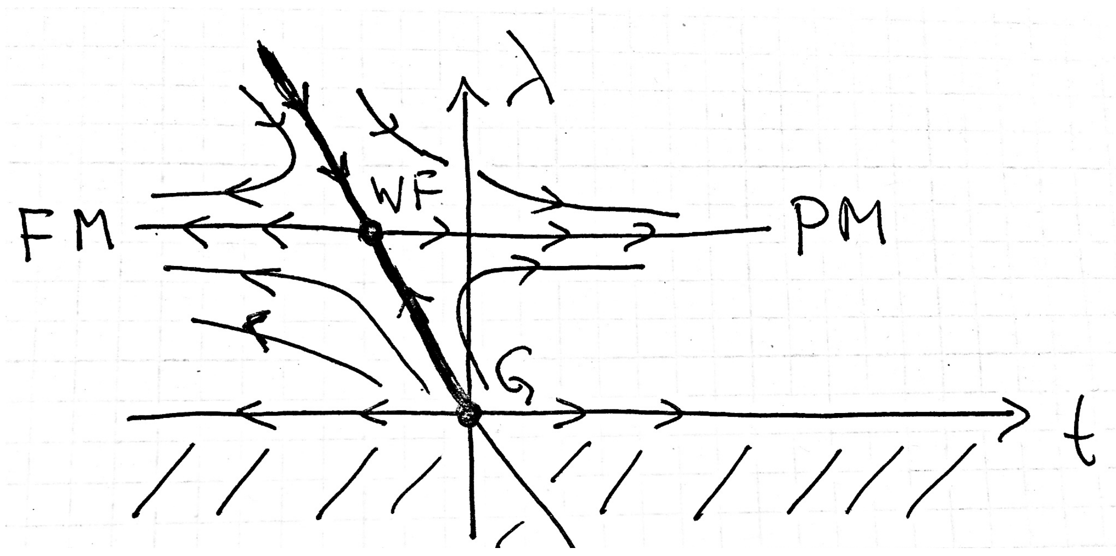


FIG. 1: Renormalization-group flow in the $O(N)$ field theory, illustrating the nontrivial infrared attractive Wilson-Fisher critical point, λ^* controlling the nature of the PM-FM phase transition sufficiently close to T_c. The thick line is the critical surface separating the PM and FM phases.

reduced temperature t, respectively, corresponding to the ordered FM and disordered PM phases, encoding the universal dependence on reduced temperature t.

We recall that thermodynamics that reflects above critical behavior is fully described by the free energy $F[\vec{h}] = -T \ln Z[\vec{h}]$ (with $k_B = 1$), derived from the partition function

$$Z[\vec{h}] \;=\; Tr\left[e^{-H/T}\right] = \int [d\vec{S}] e^{-H[\vec{S},\vec{h}]/T}. \tag{15}$$

When computed in the presence of a local external field, $\vec{h}(\mathbf{x})$ (and other fields coupling to physical observables of interest), $F[\vec{h}]$ also gives connected correlation functions, through differentiation of the free energy with respect to external field. For example, simple analysis shows

$$\begin{aligned} C_{ij}(\mathbf{x}-\mathbf{x}') \;&=\; \langle S_i(\mathbf{x})S_j(\mathbf{x}')\rangle - \langle S_i(\mathbf{x})\rangle\langle S_j(\mathbf{x}')\rangle, \qquad (16)\\ &= Z^{-1}\int [d\vec{S}] S_i(\mathbf{x})S_j(\mathbf{x}')e^{-H[\vec{S},\vec{h}]/T} - Z^{-2}\int [d\vec{S}] S_i(\mathbf{x})e^{-H[\vec{S},\vec{h}]/T}\int [d\vec{S}] S_j(\mathbf{x}')e^{-H[\vec{S},\vec{h}]/T},\\ &= \frac{\partial^2 F[\vec{h}(\mathbf{x})]}{\partial h_i(\mathbf{x})\partial h_j(\mathbf{x}')}. \qquad (17) \end{aligned}$$

At the critical point and vanishing $\vec{h}$, the latter provides the definition of the η exponent according to $C_{ij}(\mathbf{x}) \sim 1/x^{d-2+\eta}$.

Problem 1:

Explicitly demonstrate the last relation between $C(\mathbf{x})$ and $\vec{h}$ derivatives of F.

Problem 2:

Demonstrate the Gaussian integral identities. Namely, that for a Gaussian random variable $\mathbf{x}$ obeying Gaussian statistics, with variance $\mathbf{A}_{ij}^{-1}$, we have

$$\langle \mathbf{x}_i \mathbf{x}_j \rangle_0 \equiv G_{ij}^0 = \frac{1}{Z_0}\int_{-\infty}^{\infty} [d\mathbf{x}] x_i x_j e^{-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}} = \mathbf{A}_{ij}^{-1}, \tag{18a}$$

$$\langle e^{\mathbf{h}^T\cdot\mathbf{x}} \rangle_0 = e^{\frac{1}{2}\langle(\mathbf{h}^T\cdot\mathbf{x})^2\rangle_0} = e^{\frac{1}{2}\mathbf{h}^T\cdot\mathbf{G}\cdot\mathbf{h}}, \tag{18b}$$

with second identity the relative of the Wick's theorem, which is extremely important for computation of various correlators, e.g., those associated with x-ray and neutron scattering structure function. Also show that the first Gaussian propagator is reproduced by differentiating twice respect to $\vec{h}$ the second identity for the generating function.

Problem 3:

Using above generating function or equivalently the Wick's theorem that arises from it (namely $\langle x_{i_1} x_{i_2} \ldots x_{i_{2n}} \rangle_0 = \sum_{all\ permutations\ P} \langle x_{i_{P1}} x_{i_{P2}} \rangle_0 \ldots \langle x_{i_{P(2n-1)}} x_{i_{P2n}} \rangle_0$), compute the Gaussian average $\langle x^{2n} \rangle_0$.

In the harmonic theory, $\lambda = 0$, the correlator (for a vanishing $\langle \vec{S} \rangle$ is equivalent to the propagator) G, is straightforwardly computed using equipartition or Gaussian integration (the work-horse of statistical physics; see the appendix), giving in momentum and coordinate spaces

$$\langle S_{\mathbf{k}}^i S_{\mathbf{k}'}^j \rangle_0 = \frac{T}{Kk^2 + t_0}\delta_{ij}(2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}'), \tag{19a}$$

$$\langle S_i(\mathbf{x}) S_j(\mathbf{x}') \rangle_0 = \delta_{ij} \int \frac{d^d k}{(2\pi)^d} \frac{T}{Kk^2 + t_0} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \sim \delta_{ij} \frac{T}{|\mathbf{x} - \mathbf{x}'|^{d-2}}, \tag{19b}$$

giving $\eta = 0$.

2. *disordered observables*

For disordered systems, it is very difficult and not very useful to compute physical observables (the free energy and correlation functions) for a specific realization of random disorder, as this requires an analysis of an arbitrary non-translationally invariant field theory. Instead, we are interested in statistical properties, such as the mean, typical, or perhaps the full distribution function of physical observables, as these are the quantities typically measured in a macroscopically large sample.

In cases of physical observables that are narrowly distributed (referred to as self-averaging) experimentally measured spatially averaged quantities can be replaced by the average over disorder realizations, $\delta t(\mathbf{x}), \vec{h}(\mathbf{x})$ (not unlike statistical ensemble average replaces time average in conventional statistical mechanics). The free energy F is often one of such self-averaging observable and we are faced with computing

$$\overline{F} = -T\overline{\ln Z} = -T\overline{\ln Tr\,[e^{-H/T}]}, \tag{20}$$

where in the double average $\overline{\langle \ldots \rangle}$ the angular brackets indicate thermodynamic average for a particular realization of disorder and overline the disorder-average. This is quite a challenging quantity to compute as it is not formulated in terms of a standard high-dimensional trace over statistical degrees of freedom. This is what distinguishes *quenched* (difficult) and *annealed* (a multi-component homogeneous system) disordered problems. In principle the problem can be handled by averaging over disorder order by order perturbatively in disorder, simplified by noting that $\ln Z$ eliminates all disconnected graphical (Feynman diagram) contributions before disorder-averaging.

Similarly, in disordered systems we compute (quenched) disorder-averaged correlation functions, such as

$$\begin{aligned} C_{ij}(\mathbf{x}-\mathbf{x}') &= \overline{\langle S_i(\mathbf{x}) S_j(\mathbf{x}')\rangle} = \overline{\left[Z^{-1}\int [d\vec{S}] S_i(\mathbf{x}) S_j(\mathbf{x}') e^{-H_{\delta t(\mathbf{x}),\vec{h}(\mathbf{x})}[\vec{S}]/T}\right]}, & (21a)\\ &= \overline{\langle (S_i(\mathbf{x}) - \langle S_i(\mathbf{x})\rangle)\,(S_j(\mathbf{x}') - \langle S_j(\mathbf{x}')\rangle)\rangle} + \overline{\langle S_i(\mathbf{x})\rangle \langle S_j(\mathbf{x}')\rangle}, & (21b)\\ &\equiv C^T_{ij}(\mathbf{x}-\mathbf{x}') + C^\Delta_{ij}(\mathbf{x}-\mathbf{x}'), & (21c)\end{aligned}$$

where by the last two lines we defined two qualitatively distinct contributions to the disorder- and thermally-averaged two-point correlator. The first contribution, C^T is the disorder-averaged connected correlator of fluctuations of $\vec{S}$ around the thermally-averaged magnetization $\langle \vec{S} \rangle$, which in the simplest situation (as we will see below) quantifies thermal fluctuations about a random background. In contrast, the C^Δ piece quantifies zero-temperature correlations of the disorder-induced ground-state background $\langle \vec{S}(\mathbf{x}) \rangle$ itself.

The key difficulty in computing above disorder-averaged correlators is associated with the $1/Z_{\delta t(\mathbf{x}),\vec{h}(\mathbf{x})}$ normalization factor, that (like the disorder-averaged $\ln Z_{\delta t(\mathbf{x}),\vec{h}(\mathbf{x})}$ in the free energy above) does not have the conventional annealed field theory form. These quantities can nevertheless be computed by disorder averaging term by term in a perturbative expansion.

D. Replica trick

1. free energy

However, a more efficient but formal approach of handling the logarithm in the free energy and $1/Z$ in the correlation functions is available through the ingenious "replica trick" introduced by Edwards and Anderson[11, 12]. It relies on a simple mathematical identity

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n} \;, \tag{22}$$

which when applied to the free energy reduces to computing a disorder-averaged nth *power* (rather than of a logarithm) of the partition function over a random field $g(\mathbf{x})$ (referring to $\delta t(\mathbf{x}), \vec{h}(\mathbf{x})$ collectively)

$$\overline{Z^n} \;=\; \overline{\prod_{a=1}^{n} Tr\,[e^{-H[S_a,g(\mathbf{x})]/T}]} = \overline{\prod_{a}^{n} \left[\int dS_a\right] e^{-\sum_a^n H[S_a,g(\mathbf{x})]/T}}, \tag{23a}$$

$$\equiv \int [dg(\mathbf{x})] P[g(\mathbf{x})] \left[\int [dS_a] e^{-\sum_a^n H[S_a,g(\mathbf{x})]/T}\right]. \tag{23b}$$

The key point of this last form is that after averaging over disorder $g(\mathbf{x})$ (that can be done exactly if $g(\mathbf{x})$ enters linearly and its distribution is Gaussian, but by universality the result typically holds more generally) the problem reduces to that of a *homogeneous* disorder-free problem, at the expense of introducing n species of replicated (annealed) fields S_a. However, this not much of a complication as it can be easily handled by standard methods.

We now carry out this procedure for the random bond, random field theory introduced above. For simplicity we take bond and field disorder to be independent Gaussian fields defined by Eq.(3). Applying the disorder average in (23b), above and utilizing Gaussian integration in (132), we find

$$\overline{Z^n} \;=\; \int [d\vec{S}_a] e^{-H_r[\vec{S}_a]/T}, \tag{24a}$$

where the replicated Hamiltonian is given by

$$H_r[\vec{S}_a] = \int d^d x \left[\sum_a^n \left(\frac{1}{2}K(\boldsymbol{\nabla}\vec{S}_a)^2 + \frac{1}{2}t_0|\vec{S}_a|^2 + \frac{\lambda}{4}|\vec{S}_a|^4\right) - \frac{1}{T}\sum_{a,b}^n \left(\frac{1}{2}\Delta_h \vec{S}_a\cdot\vec{S}_b + \frac{1}{8}\Delta_t|\vec{S}_a|^2|\vec{S}_b|^2\right)\right]. \tag{25}$$

To analyze the effects of disorder, we thus need to study critical fluctuations of $\vec{S}_a$ governed by the above effective Hamiltonian functional. The added complexity beyond the disorder-free problem is the need to handle the random-bond nonlinearity Δ_t (off-diagonal in the replica index a, b) in addition to the conventional interaction λ (diagonal in a) and do this using a replicated matrix propagator G_{ab} arising from off-diagonal quadratic random-field Δ_h term, taking the $n \to 0$ limit at the end of calculation. This can be handled using a standard renormalization-group analysis[1] generalized to above functional. Due to its somewhat technical nature, we will not pursue this full analysis here. Instead, we will build up further technical tools and discuss the associated physics using more direct physical approach.

2. *correlation functions*

Replica trick can also be used to convert the disorder-averaged correlation functions (with their challenging normalization factor $1/Z$) into replicated, effectively annealed correlator. To this end we note that $1/Z$ factor below can be eliminated by multiplying numerator and denominator by $1/Z^{n-1}$ and taking $n \to 0$ limit,

$$C(\mathbf{x}-\mathbf{x}') = \overline{\langle S(\mathbf{x})S(\mathbf{x}')\rangle} \underset{n\to 0}{=} \overline{\left[Z^{-n}\prod_{a=1}^{n}\int[dS_a]S_1(\mathbf{x})S_1(\mathbf{x}')e^{-\sum_{a=1}^{n}H_{\delta t(\mathbf{x}),\vec{h}(\mathbf{x})}[S_a]/T}\right]}, \quad (26a)$$

$$= \int [dS_a]S_b(\mathbf{x})S_b(\mathbf{x}')e^{-H_r[S_a]/T} \equiv \langle S_b(\mathbf{x})S_b(\mathbf{x}')\rangle = C^r_{bb}(\mathbf{x}-\mathbf{x}'), \quad (26b)$$

where the normalization denominator has been eliminated. In contrast,

$$C^{\Delta}(\mathbf{x}-\mathbf{x}') = \overline{\langle S(\mathbf{x})\rangle\langle S(\mathbf{x}')\rangle}, \quad (27a)$$

$$= \int [dS_c]S_a(\mathbf{x})S_b(\mathbf{x}')e^{-H_r[S_a]/T} \equiv \langle S_a(\mathbf{x})S_b(\mathbf{x}')\rangle = C^r_{ab}(\mathbf{x}-\mathbf{x}'). \quad (27b)$$

These show that the replica-diagonal component of the replicated correlator gives the full correlation function and the off-diagonal component then gives the thermally disconnected correlator averaged over disorder.

We use above results to examine the form of the correlators in the random-field model. Focussing on the quadratic part of the replicated Hamiltonian, H_r and decoupling it into Fourier modes, we find

$$H_{r0}[\vec{S}_a] = \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\sum_{a,b}^{n}\left[(Kk^2+t_0)\delta_{ab} - \frac{\Delta_h}{T}J_{ab}\right]\vec{S}_a(-\mathbf{k})\cdot\vec{S}_b(\mathbf{k}), \quad (28)$$

where $J_{ab} = 1$ is a matrix of 1s. With this, we find the harmonic propagator, $G^0_{\alpha\beta}(\mathbf{q})$, defined through

$$\langle S^i_a(\mathbf{k})S^j_b(\mathbf{k}')\rangle_0 = G^0_{ab}(\mathbf{k})\delta^d(\mathbf{k}+\mathbf{k}') \quad (29)$$

to be

$$G^0_{ab}(\mathbf{k}) = \delta_{ij}\left[\frac{T}{Kk^2+t}\delta_{ab} + \frac{\Delta_h}{(Kk^2+t)^2}J_{ab}\right] . \quad (30)$$

Above we utilized an identity for inverting matrices of the type

$$\Gamma_{ab} = A\delta_{ab} - BJ_{ab} \ , \quad (31)$$

namely:

$$\begin{aligned} \Gamma^{-1}_{ab} &= \frac{1}{A}\delta_{ab} + \frac{B}{A(A+Bn)}J_{ab} \ , \\ &\underset{n\to 0}{=} \frac{1}{A}\delta_{ab} + \frac{B}{A^2}J_{ab} \ . \end{aligned} \quad (32)$$

Problem 4:

Prove above matrix inverse identity.

II. EFFECTS OF DISORDER NEAR PHASE TRANSITION

With this background in place we now turn to examine some key questions. One is the effects of disorder near a nontrivial critical point. Does the disorder smear (eliminate) the phase transition? Alternatively, does it modify its qualitative character, or does it leave it qualitatively unaffected at long scales, perhaps only shifting its critical temperature? Below, we will address these questions for the O(N) model.

A. Random bond disorder

We first focus on the random bond disorder, setting $\Delta_h = 0$. This is a stable choice in the sense that $\Delta_h = 0$ is preserved by fluctuations, fundamentally protected by the underlying $O(N)$ rotational invariance for $\vec{h}(\mathbf{x}) = 0$. As we discussed above, for the ferromagnetic-only random (positive) bonds, this may correspond to non-magnetic vacancies and interstitials. Given this physical picture of frustration-free random bond disorder it is quite clear that the ordered FM phase and the associated PM-FM phase transition are expected to be stable to such weak heterogeneity.

1. *global RG flow*

On the technical level, one can see this by noting that in the replicated theory, near the Gaussian fixed point, indeed Δ_t scales in the same way as the λ, both, coefficients of quartic operators, becoming relevant for $d < d_c = 4$. To assess the full role of bond disorder requires full a RG treatment, treating λ and Δ_t on equal footing. The upshot of such analysis[13] is that the Wilson-Fisher critical point for $N < 4$ is unstable to weak bond-disorder Δ_t and flows to a new infrared stable fixed point with nonzero values, Δ_t^* and λ^* (see Fig.(2)). For $N > 4$, WF critical point is stable to weak Δ_t. Thus for $N < 4$, Δ_t thereby qualitatively changes the universality class of the PM-FM transition to its "random-bond" counterpart, characterized by a new set of critical exponents and crossover functions.

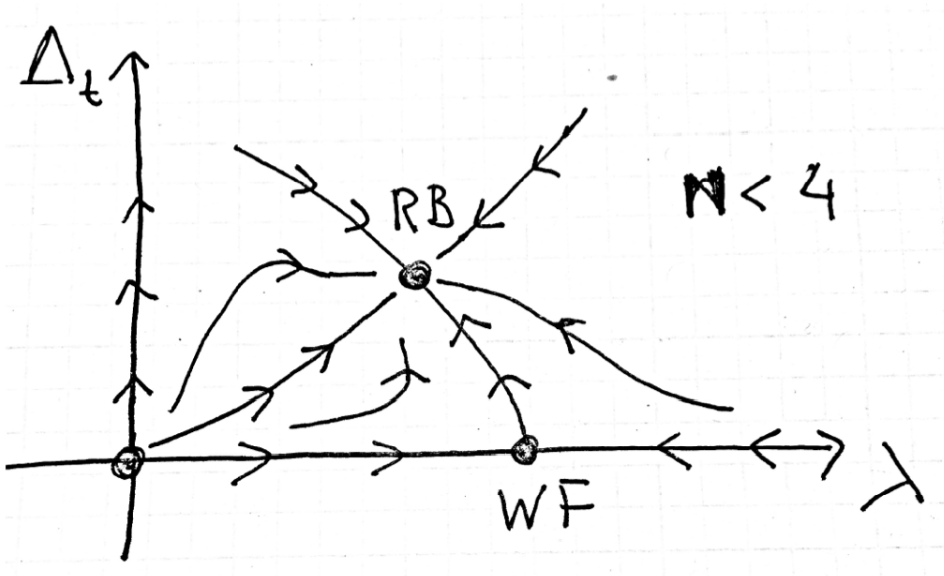


FIG. 2: Renormalization-group flow in the random-bond $O(N)$ field theory, illustrating for $N < N_c = 4$ the instability of the disorder-free Wilson-Fisher critical point to the random-bond counterpart, that controls the nature of the PM-FM phase transition in the presence of random distribution of ferromagnetic exchange bonds, sufficiently close to T_c.

2. *Harris criterion*

More generally and physically, the stability of the disorder-free critical point under weak random-bond disorder can be analyzed following a powerful argument by Harris[15]. Namely, to assess the importance of bond randomness δJ near the critical point, we look at root-mean-squared fluctuations in its average over the correlation volume, $\xi(t)^d$, that for uncorrelated random bonds J_{ij} decays as the square-root of this volume (central limit theorem),

$$J_{rms} \sim \frac{1}{\xi(t)^{d/2}} \sim |T - T_c|^{\nu d/2}, \tag{33}$$

where we used the nontrivial disorder-free correlation length exponent ν (e.g., associated with the Wilson-Fisher critical point). Since J controls T_c, these rms fluctuations translate into smearing of T_c and therefore of the "distance" $|t|$ to T_c. The transition will be qualitatively affected by disorder (whether smeared or modified cannot be assessed from this weak-disorder stability analysis), if as the mean T_c is approached, the condition

$$T_c^{rms} \gg |T - T_c^{mean}| \longleftrightarrow |T/\overline{T}_c - 1|^{\nu d/2} \gg |T/\overline{T}_c - 1|, \tag{34a}$$

$$|t|^{\nu d/2-1} \gg 1, \quad \text{in the limit of } t \to 0, \tag{34b}$$

is satisfied. This gives the famous Harris criterion

$$\alpha = 2 - \nu d > 0, \tag{35}$$

that in physical terms states, that weak random-bond disorder will qualitatively modify a disorder-free critical point if the latter is characterized by a positive heat capacity exponent.

An explicit analysis in the complementary field-theoretic description gives

$$\frac{t_{rms}}{|t|} = \frac{1}{|t|}\sqrt{\overline{\left[\frac{1}{\xi^d}\int_\xi d^dx\delta t(\mathbf{x})\right]^2}}, \tag{36a}$$

$$= \frac{1}{|t|\xi^d}\sqrt{\int_\xi d^dx\int_\xi d^dx'\overline{\delta t(\mathbf{x})\delta t(\mathbf{x}')}} = \frac{\Delta_t^{1/2}}{|t|\xi^d}\sqrt{\int_\xi d^dx\int_\xi d^dx'\delta^d(\mathbf{x}-\mathbf{x}')}, \tag{36b}$$

$$= \frac{\Delta_t^{1/2}}{\xi^{d/2}} \sim |t|^{d\nu/2-1} \sim |t|^{-\alpha}, \tag{36c}$$

an identical result.

B. Random-field disorder

We now turn to the more general problem of the random-field disorder (corresponding to magnetic impurities or an diluted anti-ferromagnet with a uniform external field) , noting that it locally breaks the $O(N)$ and upon coarse-graining also generates the random-bond disorder and thus the full problem need be treated. In contrast to the pure bond disorder, random field competes with the ferromagnetic bonds and thus the question of even the existence of a distinct FM phase and therefore of the PM-FM transition is unclear.

The full problem near the phase transition can be systematically handled using an RG analysis, keeping track of $t(b)$, $\lambda(b)$, $\Delta_h(b)$ and $\Delta_t(b)$. The dominant behavior can be understood through the replicated Hamiltonian, H_r. We first note that random field appears as a "mass"-like coupling Δ_h/T that compared to the exchange K (i.e., keeping K fixed) grows quadratically under rescaling

$$\frac{\Delta_h(b)}{T(b)} = b^2 \frac{\Delta_h}{T},$$

as does the reduced temperature $t(b)$. This can equivalently be interpreted as the flow of the effective temperature

$$T(b) = T b^{-\theta} \to 0, \quad \theta = 2 + O(\epsilon),$$

while keeping Δ_h fixed, encoding the dominance of the random-field energies over thermal fluctuations. Since the absolute temperature scales nontrivially, the free energy density now scales as $f \sim \xi^{-d+\theta}$, leading to a violation of hyperscaling, with the new relation given by

$$2 - \alpha = (d - \theta)\nu.$$

To see these predictions in a field-theoretic RG analysis, we note that in the presence of random fields Δ_h, the dominant perturbative correction to λ comes from a one-loop diagram, where two λ vertices are connected by C^T and C^{Δ_h} components of the random-field propagator. This gives

$$\delta\lambda \sim \left(\frac{\Delta_h \lambda}{T}\right) \lambda, \tag{37}$$

leading to a dimensionless coupling $w \equiv \frac{\Delta_h \lambda}{T}$, that (because of the multiplicative factor of Δ_h with eigenvalue of 2) becomes relevant for $d < d_{uc} = 6$. Working near $d \lesssim 6$ the full RG analysis leads to the flows illustrated in Fig.3. These show the instability of the Wilson-Fisher critical point toward a zero-temperature critical point at Δ_h^*, which determines the new set of critical exponents that characterize the singularities near the transition driven by the strength of the random-field disorder.[13]

The formal field-theoretic analysis also shows that the upward shift of the upper-critical dimension to $d_{uc}^{rf} = 6$ (i.e., $(d_{uc}^{rf} - 2) - 4 = 0$) for the random-field problem, is more generally

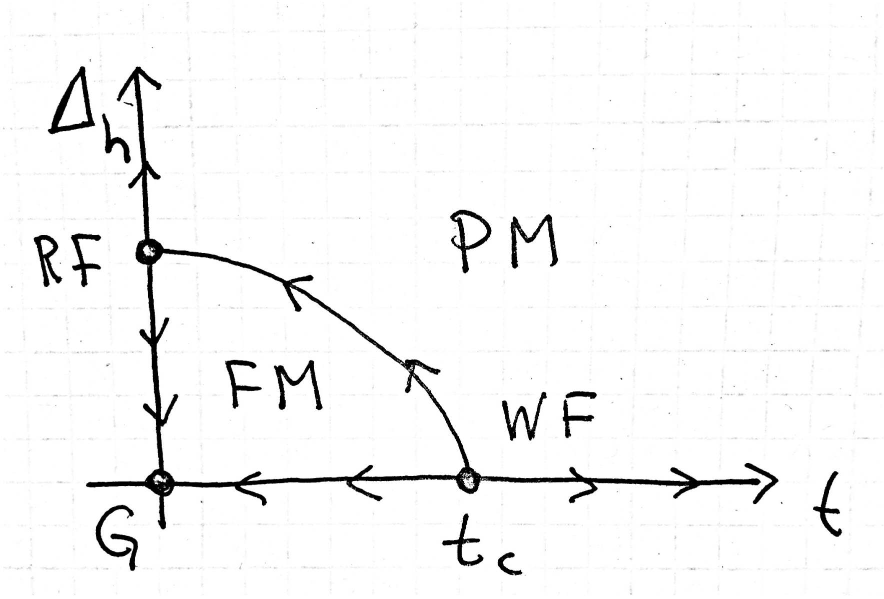


FIG. 3: Renormalization-group flow in the random-field $O(N)$ field theory, illustrating the instability of the disorder-free Wilson-Fisher critical point to the random-field zero-temperature critical point, with the transition driven by the random-field strength Δ_h at $T = 0$.

accompanied by an upward shift of the effective dimension relative to the disorder-free WF critical point. This formal prediction is known as "dimensional reduction", and if naively extended to lower dimensions would predict a lower-critical dimension (below which the phase and the associated transition are destroyed) to be $d_{lc}^{rfO(N)} = 4$ (i.e., $(d_{lc}^{rf} - 2) = 2$) for the continuous symmetry problems ($N > 1$) and $d_{lc}^{rfIM} = 3$ (i.e., $(d_{lc}^{rf} - 2) = 1$). Thus the prediction of dimensional reduction is that random field Ising model does not exhibit a transition in three dimensions, a result that was rigorously proved to be incorrect (namely RFIM does exhibit a transition in 3d)[18]. We will resolve this contradiction in the next section.

III. RENORMALIZATION GROUP NEAR A PHASE TRANSITION

Ginzburg-Landau effective Hamiltonian for n-component vector $\vec{\phi}$ random exchange magnet:

$$\overline{H} \equiv \frac{H}{T} = \int d^d x [\frac{1}{2} K |\nabla \vec{\phi}|^2 + \frac{1}{2} r |\vec{\phi}|^2 + u |\vec{\phi}|^4 + g(x) |\vec{\phi}(x)|^2 + K_4 |\nabla^2 \vec{\phi}|^2 + w |\vec{\phi}|^6 + u_2 |\vec{\phi}|^2 |\nabla \vec{\phi}|^2 + \ldots \quad (38)$$

The randomness in $g(x)$ is most important; take it to be Gaussian with mean zero and covariance

$$\overline{g(x)g(x')} = \Delta\delta^d(x - x') \tag{39}$$

(bar denotes average over quenched randomness.)
In order to make problem well-defined, we need a short-wavelength cutoff. This is best done in "momentum" space.
Define

$$\vec{\phi}(x) = \int_q e^{iq\cdot x}\vec{\phi}_q \tag{40}$$

where $\int_q \equiv \frac{1}{(2\pi)^d}\int^\Lambda d^dq$, and $\phi(x)$ only contains Fourier components with $|q| \leq \Lambda$. The randomness in Fourier space g_q has covariance:

$$\overline{g_q g_{q'}} = \Delta\delta(q + q') \tag{41}$$

where, $\delta(q) \equiv (2\pi)^d\delta^d(q)$ is short-hand notation.
We have

$$\begin{aligned}\overline{H} = &\int_q \frac{1}{2}Kq^2|\vec{\phi}_q|^2 + \frac{1}{2}r|\vec{\phi}_q|^2 + u\int_{q_1}\int_{q_2}\int_{q_3}\int_{q_4}\delta(q_1+q_2+q_3+q_4)\phi^i_{q_1}\phi^i_{q_2}\phi^i_{q_3}\phi^i_{q_4}\\ &+\int_{q_1}\int_{q_2} g_{-q_1-q_2}\phi^i_{q_1}\phi^i_{q_2} + \ldots\end{aligned} \tag{42}$$

where the complex conjugate $\vec{\phi}^*_q = \vec{\phi}_{-q}$, and the superscripts i, j run from 1 to n, labelling the components of $\vec{\phi}$ with sums over repeated "spin" indices.
The 4 terms in $\overline{H}$ above are the only important terms for a lowest order expansion in

$$\epsilon \equiv 4 - d \tag{43}$$

All the $\int_q$ are restricted to have $|q| \leq \Lambda$. [physically, $\Lambda \sim 1/\text{atomic size}$.]
To perform RG transformation, we divide up each ϕ_q as:

$$\phi_q = \phi_<(q) + \phi_>(q) \tag{44}$$

with $\phi_<(q)$ restricted to $|q| \leq \Lambda/b$, and $\phi_>(q)$ restricted to $\Lambda/b < |q| \leq \Lambda$.
Notation: $\int_q^> \equiv \int_q$, with $\Lambda/b < |q| \leq \Lambda$.
The new effective Hamiltonian $\overline{H}'$ is found from

$$e^{-\overline{H}'} = Tr_{\{\phi_>\}}e^{-\overline{H}} \tag{45}$$

lengths are then rescaled by:

$$x \to x'b, q \to q'/b \tag{46}$$

and the fields by

$$\phi_<(q) \to b^{d+\zeta}, \text{ with } q' = bq \tag{47}$$

thus, $|q'| \le \Lambda$ restorias the basic scale of the cutoff.
We can choose ζ for later convenience.[Note: the $d + \zeta$ arises from the change in scaling of $\phi(x) \to b^\zeta \phi'$ to ϕ_q coming from the $\int d^d x$.]
The temperature has been absorbed into the couplings; since we are interested in critical fixed point, it is natural to take $T' = T$ (i.e. $\theta = 0$).
To do $Tr_{\phi_>}$, we expand $e^{-\overline{H}}$ in powers of r, u and $g(x)$, do the traces and then put the resulting ($\phi_<$ dependent) quantities bach in exponential.
The $Tr_{\{\phi_>\}}$ are straightforward Gaussian integrals using the weight factor $e^{-\overline{H}_0^>}$ with

$$\overline{H}_0^> = \frac{1}{2}\int_q^> Kq^2 |\vec{\phi}_>(q)|^2 \tag{48}$$

We shall see that we need primarily renormalizations of the terms in (5) although $\overline{H}'$ will include other terms as well such as those on the second line of (1).
The terms can conveniently be represented graphically:

vertices:  $u\varphi^4$, $r\varphi^2$, and  $g_{-q_1-q_2}\phi_{q-1}\phi_{q_2}$.
internal lines: $\frac{1}{Kq^2}$
$\to$ propagator for "integrated out" $\phi_>(q)$, with $|q|$ in "momentum shell" ($\Lambda/b < |q| \le \Lambda$)
external lines: $\phi_<$ not integrated.
momentum convervation at  and and 
combinatoric factors:
- from expansion of $e^{-\overline{H}}$;
- from ways of decomposing vertices into $\phi_<$ and $\phi_>$, e.g.  ;
- from reexponentiating;
- from sums over $i\&j$ in $\phi^i\phi^i\phi^j\phi^j$.

Disconnected graphs cancelled by reexponentiating. Graphs with no external legs, and also factor from $Tr_{\{\phi_>\}}$, give constants in $\overline{H}'$ which can be ignored unless interested in actual full free energy.

Define $b = e^l$ and take l small $\to dl$.

A. Non-random case

1. $n = 1$ *(simplest)*

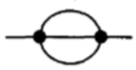
• Renormalization of r: from rescaling

$$rb^{d+2\zeta} \approx r[1 + (d+2\zeta)Inb + ...], \text{ for } Inb \to dl \text{ small} \tag{49}$$

from expanding exponential $e^{-\overline{H}}$

 : $(-u)(-2)6 \int_q^> \frac{1}{Kq^2}$,

 :$(-u)(-\frac{1}{2}r)(-2) \cdot 6 \cdot 2 \int_q^> \frac{1}{K^2q^4}$,

 :$O(u^2)$ complicated. etc.

$$\frac{dr}{dl} \equiv \frac{r' - r}{Inb} \approx \frac{r' - r}{b - 1} = (d + 2\zeta)r + 12c_d\Lambda^{d-2}u/K - 12c_d\Lambda^{d-4}ru/K^2 + O(u^2, ur^2, ...), \tag{50}$$

using

$$\int_q^> \frac{1}{q^x} = \frac{1}{(2\pi)^d} A_d[\Lambda^{d-x} - (\frac{\Lambda}{b})^{d-x}]\frac{1}{d-x} \approx C_d\Lambda^{d-x}(b-1), \tag{51}$$

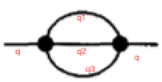
for $b - 1$ small. Here $A_d = \frac{\pi^{d/2}2}{\Gamma(d/2)}$ is area of d-dimensional unit sphere and $C_d \approx A_d/(2\pi)^d$

• Renormalization of K from rescaling: $Kb^{d-2+2\zeta}$, from expanding $e^{-\overline{H}}$

 and  are independent of q of external legs $\varphi_<$ (due to q conservation at vertices).

$\Rightarrow$ can't give q dependence of $\varphi_<(q)\varphi_<(-q)$ in $\overline{H}'$,

$\Rightarrow$ don't renormallize K.

 $$\propto u^2 \int_{q_1}^> \int_{q_2}^> \int_{q_3}^> \delta(q + q_1 + q_2 + q_3)\frac{1}{q_1^2}\frac{1}{q_2^2}\frac{1}{q_3^2} \tag{52}$$

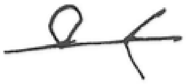
does have dependence on external $q(< \Lambda/b)$

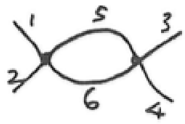
$\Rightarrow$ can renormalize T, K (and also K_4 etc.)

hard to evaluate:

$$\Rightarrow \frac{dK}{dl} = (d-2+2\zeta)K + O(u^2/K^3) \tag{53}$$

• Renormalization of u from rescaling: $ub^{d+4\zeta}$ form expanding $e^{-\overline{H}}$

 cannot have all external legs with $q^{<}$ and internal with $q^{>}$. $\Rightarrow 0$.



gives $u(q_1, q_2, q_3, q_4)$. expand in external $q's$:

$$u(q_1, q_2, q_3, q_4) = u\delta(q_1+q_2+q_3+q_4) + O[q_1^2, q_1 \cdot q_2, etc] \tag{54}$$

q^2 terms correspond to $|\varphi|^2|\nabla\varphi|^2$ terms, ignore for now (irrelevant at Gaussian fixed point)

To get u, set all external q to 0,$\Rightarrow q_6 = -q_5$

$$\frac{1}{2} \cdot (-u)^2 \cdot (-1) \cdot 72 \cdot \int_q^{>} \frac{1}{K^2 q^4} \tag{55}$$

(Note factor 72 comes from choice of $\varphi^{>}$ legs & attaching together.)

$$\Rightarrow \frac{du}{dl} = (d+4\zeta)u - 36C_d\Lambda^{d-4}\frac{u^2}{K^2} + O(u^2 r, u^3, ...) \tag{56}$$

B. Non-random, general n

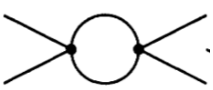
Need to do spin sums:

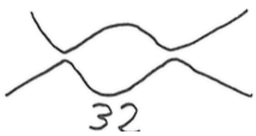
vertex  becomes 

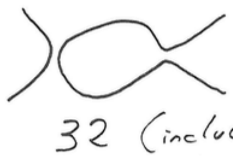
with spin index conserved along continuous lines.

$\Rightarrow$ extra combinatoric factors from choosing i or j legs & summing over i for closed internal loops.

i.e.  $\rightarrow$  (combinatoric factor: $2n$) +  (combinatoric factor: 4),

 $\rightarrow$ 

+ 

+ 

(including '2' from choosing which vertex divides which way).

Chose $K = K' = 1$ to keep basic energy scale with $q \approx \Lambda$ fixed. $\Rightarrow$ from (16),

$$\zeta = \frac{2-d}{2} + O(u^2), \tag{57}$$

$$\text{yielding, } \frac{dr}{dl} \approx 2r + 2(2n+4)C_d\Lambda^{d-2}u - 2(2n+4)C_d\Lambda^{d-4}ru + O(u^2, \ldots), \tag{58}$$

$$\frac{du}{dl} \approx (4-d)u - \frac{1}{2}(8n+64)C_d\Lambda^{d-4}u^2 + O(u^3, \ldots) \tag{59}$$

where effects of choosing ρ in (20) give rise to some of ru^2 and u^3 terms in (21) and (22) respectively.

If $\epsilon \equiv 4-d$ is small,

$$\Lambda^{d-4} \approx 1 + O(\epsilon), C_d \approx c_4 = \frac{1}{8\pi^2} \tag{60}$$

& can truncate recursion relations, (keep only terms in (20), (21), (22))

Validity of truncation ? for $\epsilon \ll 1$

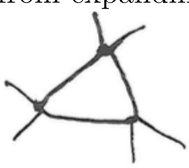
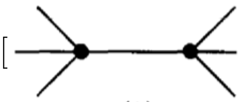
(22) $\Rightarrow u^* = O(\epsilon)$

(21) $\Rightarrow r^* = O(\epsilon)$ fixed point.

Other operators from (1)

$w\varphi^6$ rescaling $b^{d+6\zeta} = b^{6-2d}$

from expanding

 $\rightarrow$ u^3, [ cannot contribute with external $q = 0$, gives $w_2|\nabla\varphi|^2|\varphi|^6$.]

$$\frac{dw}{dl} \approx -(2d-6)w + O(u^3), \tag{61}$$

$$\Rightarrow w^* = O(\epsilon^3) \tag{62}$$

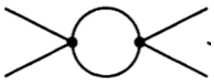
but feedback

 $\rightarrow \delta u \sim w$

$\Rightarrow$ get extra $O(w) = O(\epsilon^3)$ term in $\frac{du}{dl}$. Negligible for ϵ small (need for next order in ϵ.)

$u_2|\nabla\varphi|^2\varphi^2$ from (1)

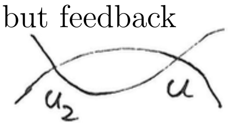
rescaling $b^{d-2+4\zeta} = b^{2-d}$

expanding  $\rightarrow \delta u_2 \sim u^2$

$$\frac{du_2}{dl} = -(d-2)u_2 + O(u^2), \tag{63}$$

$$\Rightarrow u_2^* = O(\epsilon^2) \tag{64}$$

but feedback



$\rightarrow \delta u \sim u u_2$

$\Rightarrow$ extra $O(uu_2) = O(\epsilon^3)$ term in $\frac{du}{dl}$. Negligible for ϵ small.

Can check other operators only give $O(\epsilon^4)$ in $\frac{du}{dl}$.

$\Rightarrow$ To $O(\epsilon)$, Need only truncated equations (20)-(22).

Fixed point & exponents to $O(\epsilon)$

From (21) & (22), have Gaussian fixed point with $u^* = r^* = 0$, but $\lambda_r > 0$ and $\lambda_u > 0$, two relevant eigenvalues $\Rightarrow$ cannot be usual critical point.

Non-trivial (Wilson-Fisher) fixed point

$$u^* = \frac{\epsilon}{4(n+8)C_4} + O(\epsilon^2), \tag{65}$$

$$r^* = -\frac{(n+2)\Lambda^2\epsilon}{2(n+8)} + O(\epsilon^2) \tag{66}$$

eigenvalues:

expand $r = r^* + \delta, u = u^* + \gamma$ with δ, γ small.

$$\frac{d\delta}{dl} = (2 - 4(n+2)C_4u^*)\delta + [4(n+2)C_4\Lambda^2 + O(u^*)]\gamma, \tag{67}$$

$$\frac{d\gamma}{dl} = [\epsilon - 8(n+8)C_4u^*]\gamma + O(u*^2)\delta \tag{68}$$

yielding

$$\frac{d}{dl}\begin{pmatrix}\delta \\ \gamma\end{pmatrix} = \begin{pmatrix} 2 - \frac{n+2}{n+8}\epsilon + O(\epsilon^2) & 4(n+2)C_4\Lambda^2 + O(\epsilon) \\ O(\epsilon^2) & -\epsilon + O(\epsilon^2)\end{pmatrix}\begin{pmatrix}\delta \\ \gamma\end{pmatrix} \tag{69}$$

which has eigenvalues to $O(\epsilon)$

$$\lambda_\delta = 2 - \frac{n+2}{n+8}\epsilon, \tag{70}$$

$$\lambda_\gamma = -\epsilon \tag{71}$$

So, associate $\delta \sim T - T_c, \& \gamma$ irrelevant operator (other operators are irrelevant with eigenvalues $\lambda \leq -2$).

Note: needed ru term in $\frac{dr}{dl}$, but not u^2 term.

Flows: with other operators small [or near to fixed point values $O(\epsilon^{n\geq 2})$]

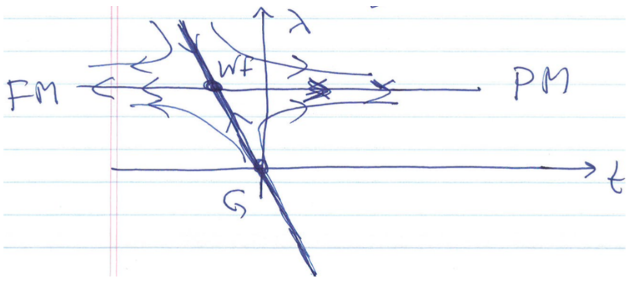


FIG. 4: RG flows

critical manifold is $r_c(u)$, which flows into W-F fixed point.

$$\delta = r - r_c(u) \sim T - T_c(u) \tag{72}$$

Exponents:

$$\nu = \frac{1}{\lambda_\delta} \approx \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + O(\epsilon^2), \tag{73}$$

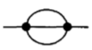
$$\eta = 2 - d - 2\zeta = 0 + O(\epsilon^2) \tag{74}$$

Others from scaling

$$\alpha = 2 - d\nu, \text{ specific heat} \tag{75}$$

$$\beta = \left(\frac{d-2+\eta}{2}\right)\nu, \text{ spontaneous magnetization} \tag{76}$$

$$\gamma = (2-\eta)\nu, \text{ susceptibility} \tag{77}$$

From  renormalization of K

$$\rightarrow \eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + O(\epsilon^3) \tag{78}$$

1. *Effects of magnetic field*

Add to $\overline{H} := -\vec{H} \cdot \int \vec{\varphi} d^d x$ (absorbed T^{-1} into H),

rescaling: $\vec{H} b^{d+\zeta}$,

no operators symmetric in $\vec{\varphi} \to -\vec{\varphi}$ can generate $\vec{H}$

$$\Rightarrow \frac{\vec{H}}{dl} = (d+\zeta)\vec{H} + O(H^3 u, ...), \tag{79}$$

$$\lambda_H = d + \zeta \text{ exactly} \tag{80}$$

Get γ and β from λ_H and λ_δ.

C. Random Exchange

Add $\int d^d x g(x)|\vec{\varphi}(x)|^2$ to $\overline{H}$.

First look at rescaling near Gaussian fixed point:

$$\int d^d x g(x)|\vec{\varphi}(\vec{x})|^2 \to b^{d+2\zeta} \int d^d x' g(bx')|\vec{\varphi}'(x')|^2, \text{ with } x \to bx', \tag{81}$$

$$\text{so, } g'(x') = b^{d+2\zeta} g(bx') \tag{82}$$

covariance

$$\overline{g'(x_1')g'(x_2')} = b^{2d+4\zeta}\overline{g(bx_1')g(bx_2')} = b^{2d+4\zeta}\delta^d(bx_1' - bx_2')\Delta = b^{d+4\zeta}\delta^d(x_1' - x_2')\Delta \tag{83}$$

using $\delta^d(bx') = \frac{1}{b^d}\delta(x')$

So, have covariance of g' of similar form to g, but with

$$\Delta' = b^{d+4\zeta}\Delta = b^{4-d}\Delta \tag{84}$$

$\Rightarrow$ randomness irrelevant for $d > 4$.

Note: non-Gaussian cumulants irrelevant near Gaussian fixed point near $d = 4$.

e.g. skewness

$$\begin{aligned} \overline{g'(x_1')g'(x_2')g'(x_3')} &= b^{3d+6\zeta}\overline{g(bx_1')g(bx_2')g(bx_3')} \\ &= b^{3d+6\zeta}\mu_3\delta^d(bx_1' - bx_2')\delta^d(bx_1' - bx_3') \\ &= b^{d+6\zeta}\mu_3\delta^d(x_1' - x_2')\delta^d(x_1' - x_3'), \end{aligned} \tag{85}$$

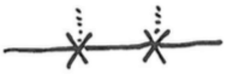
$$\Rightarrow \mu_3' = b^{d+6\zeta}\mu_3 = b^{6-2d}\mu_3 \tag{86}$$

Also randomness in $|\nabla\varphi|^2$ or $|\varphi|^4$ coefficients also irrelevant.

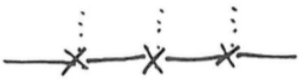
For $d \lesssim 4$ i.e. ϵ small, keep only covariance of random coefficient of $|\varphi|^2$.

ϵ-expansion

• Renormalization of g_q vertex 

 $\frac{1}{2}(-g)^2 \cdot 4 \cdot (-1)$;  same with extra $(-r)$.

 $(-u)(-g)(-1) \cdot 12$ for $n = 1$ case.

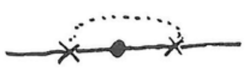
 $\frac{1}{6}(-g)^3(-1) \cdot 24$

average of $g'_{q'}$ gives contribution to r'

$$\overline{g'_{q'}} = \frac{\delta r'}{2}\delta(q') \tag{87}$$

averaging for Gaussian g can be done by joining all possible pairs of dotted lines which (after averaging) can carry momentum.

• Renormalization of r

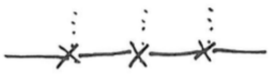
 $2 \cdot (-2\Delta) \int^{>} \frac{1}{Kq^2}$  $\cdot\, 2 \cdot (2\Delta r) \int_q^{>} \frac{1}{K^2q^2}$,

with eqn (),

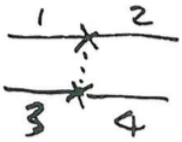
$$\frac{dr}{dl} = 2r + [4(n+2)u - 4\Delta]C_d\Lambda^{d-2} + [-4(n+2)ur + 4\Delta r]C_d\Lambda^{d-4} + O(u^2, \Delta^2, ...) \tag{88}$$

• Renormalization of Δ

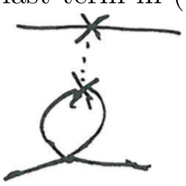
calculate $\overline{g'g'}$ with

$g' =$  +  +  +  ...

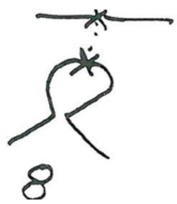
need even number of vertex 's, gives:

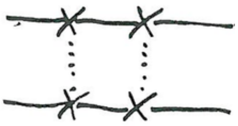


orginal Δ (no internal lines), averaging $\Rightarrow \delta(q_1 + q_2 + q_3 + q_4)$ from (4) and last term in (5).

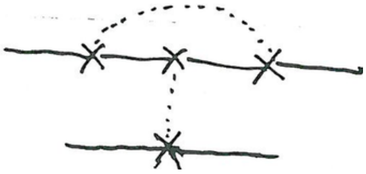
 : $2 \cdot (-12u) \cdot \Delta \cdot \int_q^{>} \frac{1}{K^2q^4}$, for $n = 1$;

general n: $12 \rightarrow$ 

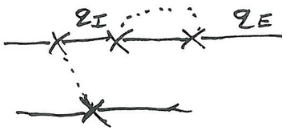
+ 



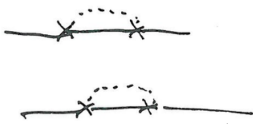
: $(-2)^2 \cdot 2 \cdot \Delta^2 \cdot \int_q^> \frac{1}{K^2 q^4}$



: $2 \cdot 4 \cdot \Delta^2 \cdot \int_q^> \frac{1}{K^2 q^4}$



average forces $q_I = q_E \Rightarrow$ no contribution with $q_I > \frac{n}{b} \& q_E < \frac{n}{b}$

disconnected graphs  gives $\overline{g'}^2$

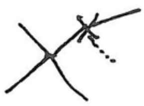
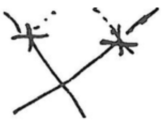
Variance of $g' \to$ renormalization of Δ:

$$\overline{[g'(x_1') - \overline{g'}][g'(x_2') - \overline{g'}]} = \Delta' \delta(x_1' - x_2') \tag{89}$$

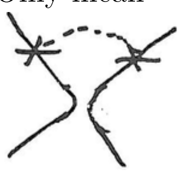
disconnected graphs cancel.

$$\frac{d\Delta}{dl} = (4-d)\Delta + [16\Delta^2 - 8(n+2)u\Delta] C_d \Lambda^{d-4} + O(\Delta^3, u^2\Delta, ru\Delta, etc.) \tag{90}$$

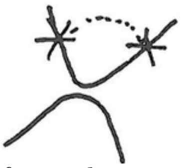
• Renormalization of u

get random $|\varphi|^4$ term from:  ,  , etc.

Only mean $\to u$ (other random terms are irrelevant near Gaussian fixed point)



: $(-u) \cdot \frac{1}{2} \cdot (-1)^2 \cdot \overline{gg} \cdot (-1) \cdot 32 \cdot \int_q^> \frac{1}{K^2 q^4}$,



: $(-u) \cdot \frac{1}{2}(-1)^2 \cdot \overline{gg} \cdot (-1) \cdot 16 \cdot \int_q^> \frac{1}{K^2 q^4}$

from these and Eq (22), get:

$$\frac{du}{dl} = (4-d)u + [24\Delta u - 4(n+8)u^2] C_d \Lambda^{d-4} + O(u^3, u\Delta^2, u^2\Delta, etc.) \tag{91}$$

Can check other operators and non-Gaussian randomness, do not feedback in dangerous ways to $O(\epsilon)$. $\Rightarrow$ Use Eq's (51), (53) & (54).

1. *Random Exchange Fixed Points*

For small ϵ and $n > 1$, there are 3 fixed points with $u^*, \Delta^*, r^* = O(\epsilon)$.
Gaussian fixed point:

$$u^* = \Delta^* = r^* = 0 \tag{92}$$

Non-random fixed point:

$$\Delta^* = 0, u^* > 0, r^* < 0, \text{ as in Eq (......)} \tag{93}$$

if $n > n_c = 4, \lambda_\Delta < 0$, randomness irrelevant;
if $n < n_c, \lambda_\Delta > 0$, randomness relevant.

Random-exchange fixed point $n < n_c$ only.

$$\Delta^* = \frac{\epsilon}{32C_4}\left(\frac{4-n}{n-1}\right), \text{ [unphysical for } n > 4 \text{ (since } \Delta^* < 0)], \tag{94}$$

$$u^* = \frac{\epsilon}{16C_4(n-1)}, \tag{95}$$

$$r^* = -\frac{3\Lambda^2\epsilon}{16(n-1)} \tag{96}$$

one unstable relevant direction with $\lambda_\delta^R > 0, \Rightarrow \nu_R = 1/\lambda_\delta^R$;
two stable irrelevant directions, $u - u^*, \Delta - \Delta^*$ linear combinations.

2. *Random Exchange in $d = 4 - \epsilon$*

For $n = 1\&$ small ϵ, randomness relevant, need higher order terms in $u\&\epsilon$, find fixed point with $\Delta^*\&u^* = O(\sqrt{\epsilon})$.

For $1 < n < 4\&$ small ϵ, randomness relevant, critical behavior controlled by random fixed point with $\eta_R = O(\epsilon^2)$ and $\nu_R = \frac{1}{2} + a(n)\epsilon + O(\epsilon^2)$.

For $n > 4\&$ small ϵ, randomness irrelevant, critical behavior same as pure system (with singular corrections to scaling due to randomness).

IV. EFFECTS OF DISORDER INSIDE ORDERED PHASE

Although we have examined the effects of disorder on the phase transition, a prerequisite to that question is the analysis of the stability of the ordered phase itself ("Know where you are going *before* knowing how you get there."). This is what we turn to below.

A. Stability for disorder-free thermal fluctuations

Before looking at stability in a disordered system, it is instructive to recall such question for a disorder-free system. To this end, this can be addressed by an estimate of the size of low-energy fluctuations about the ordered state. These are dominated by Goldstone modes governed by the Hamiltonian

$$H_{GM} = \int d^d x \frac{1}{2} K(\boldsymbol{\nabla}\vec{S})^2. \tag{97}$$

A simple analysis of the mean-squared fluctuations $\delta\vec{S}(\mathbf{x})$ reduces to a Gaussian integral with H_{GM}, that for the case of $N > 1$ is dominated by the Goldstone modes and gives

$$\langle \delta S^2 \rangle \ = \ \int \frac{d^d k}{(2\pi)^d} \frac{T}{K k^2} \sim \frac{T}{K} L^{2-d}, \text{ for } d < 2 \tag{98}$$

which thus diverges for $d \leq 2$, thereby destabilizing the ordered phase in these lower dimensions.

Problem:

Show that broken FM phase of the $O(N)$ model exhibits a gapped (with gap $2|t|$) longitudinal mode corresponding to fluctuations along the spontaneous magnetization $\vec{S}_0$ and gapless transverse (to $\vec{S}_0$) Goldstone modes, that dominate the spin-wave fluctuations that destroy the ordered phase. Show that the latter is characterized by H_{GM} in (97).

The above result can also be understood more physically by considering the energetics of the low-energy Goldstone mode fluctuations about the ordered state. Consider a spin-wave with a smooth 2π variation of $\delta\vec{S}$ over a length L. It is clear from H_{GM} above that the corresponding excitation energy is given by

$$E_{ex} \sim K L^{d-2}.$$

Clearly the ordered phase can only be stable if this excitation energy is much larger than $k_B T$, which is length independent. Thus, consistent with (98), we arrive at the seminal result of the

Hohenberg-Mermin-Wagner theorem[24–26], that a phase that breaks continuous symmetry, at finite T can only be stable for $d > 2$.

Naively one may assume that this result, (98) also extends to the Ising $N = 1$ case. However, in this case, it is a *discrete* Z_2 Ising symmetry that is broken, and there are no Goldstone massless modes in the ordered FM state. Some reflection shows that the low energy excitation is a domain wall of "area" L^{d-1} and width $\xi = \sqrt{K/(2|t|)}$, separating regions of positive and negative magnetizations. The corresponding energy is clearly

$$E_{dw} = \frac{1}{2}\int d^d x K(\nabla\vec{S}_{dw})^2 \approx \frac{K}{\xi}S_0^2 L^{d-1} \sim |t|S_0^2\xi L^{d-1} \approx \sqrt{K|t|}S_0^2 L^{d-1}. \quad (99)$$

Such domain-wall energy dominates over a constant $k_B T$ only for $d > 1$, showing that Ising model's lower-critical dimension is $d_{lc}^{Ising} = 1$.

B. Stability to random-bond disorder

For weak random-bond disorder the FM phase is stable as argued earlier and so the $d_{lc}^{rbO(N)} = 2$ and $d_{lc}^{rbIM} = 1$ is the same as that of the thermal state, limited by percolation transition when the lattice breaks up into disconnected pieces that clearly cannot order ferromagnetically.

C. Stability to random-field disorder

To assess the stability of the ordered state to weak random field (analogous to thermal fluctuations above) we examine rms fluctuations $\delta\vec{S}_{rms}$ around the uniform FM state. These are dominated by Goldstone modes $\vec{S}_\perp$ governed by

$$H_{GM-RF} = \int d^d x \left[\frac{1}{2}K(\nabla\delta\vec{S}_\perp)^2 - \vec{h}(\mathbf{x})\cdot\vec{S}_\perp\right] \quad (100)$$

As we have seen above, the positive thermal exponent $\theta > 0$ corresponds to subdominant role of thermal fluctuations relative to the random pinning potential. Thus, (although the result can be computed using a field theoretic analysis with H_{GM-RF}), at long scales we can focus on the $T = 0$ ground state and simply minimize the above random-field Hamiltonian, obtaining

$$\vec{S}_\perp(\mathbf{x}) = \int d^d x' G(\mathbf{x}-\mathbf{x}')\vec{h}_\perp(\mathbf{x}'), \quad (101)$$

that decouples in Fourier space, giving (as expected) a much stronger than thermal fluctuations

$$
\begin{aligned}
S^2_{rms} &= \overline{\langle \vec{S}_\perp(\mathbf{x})^2\rangle} = \int \frac{d^dk}{(2\pi)^d}\frac{\Delta_h}{K^2k^4}, && (102a)\\
&\sim \frac{\Delta_h}{K^2}L^{4-d},\ \text{for } d<4, && (102b)\\
&\sim \frac{\Delta_h}{K^2}a^{4-d},\ \text{for } d>4, && (102c)
\end{aligned}
$$

Since for $d \leq 4$ these random-field driven fluctuations in the ground state grow without bound (logarithmically in $d = 4$), we conclude that $d_{lc}^{rfO(N)} = 4$ is the lower-critical dimension, below which the FM ordered state is unstable even at zero temperature. Setting these distortion to be of the order of the disorder-free magnetization, S_0, we extract the size ξ_{IML} of the ordered domains,

$$
\begin{aligned}
\xi^{rfO(N)}_{IML} &\sim \left(\frac{K^2S_0^2}{\Delta_h}\right)^{1/(4-d)},\ \text{for } d<4, && (103a)\\
&\sim a e^{K^2S_0^2/\Delta_h},\ \text{for } d=4, && (103b)
\end{aligned}
$$

the so-called Imry-Ma-Larkin correlation length[16, 17] beyond which long-range order is lost.

Alternatively, we can extract $\xi^{rfO(N)}_{IML}$ for a Goldstone modes system by examining the competition between the ordering elastic energy

$$E_{elO(N)} = \frac{1}{2}K\int d^dx(\boldsymbol{\nabla}\vec{S}_\perp)^2 \sim KS_0^2L^{d-2},$$

and of the random-field energy

$$E_{rfO(N)} = -\int d^dx\vec{h}(\mathbf{x})\cdot\vec{S}_\perp \sim -\Delta_h^{1/2}S_0L^{d/2}.$$

In estimating $E_{rfO(N)}$ we used the central limit theorem to conclude that random-field $\vec{h}(\mathbf{x})$, averaged over a volume L^d scales like $\pm\sqrt{L^d}$. For $d < 4$, it is clear that the elastic energy dominates at short scales $L < \xi_{IML}$ and random field at long scales $L > \xi^{rfO(N)}_{IML}$, with $\xi^{rfO(N)}_{IML}$ given by (103).

For an Ising system that breaks a discrete symmetry above analysis needs to be modified in its estimate of the excitation energy, that is the same as for our earlier estimate of stability to thermal fluctuations, (99). Using this excitation energy of the Ising domain-wall and the estimate of the random-field energy, we find

$$E_{RFIM} \approx \sqrt{K|t|}S_0^2L^{d-1} - \Delta_h^{1/2}S_0L^{d/2}. \qquad (104)$$

where for $d < 2$ the random-field energy always dominates at sufficiently long scales

$$L > \xi_{IML}^{RFIM} = \left(\frac{K|t|}{\Delta_h} S_0^2\right)^{1/(2-d)} .$$

Thus, the lower-critical dimension for the random-field Ising model is $d_{lc}^{RFIM} = 2$, and as alluded to above, this Imry-Ma analysis[16] demonstrates a breakdown of the formal dimensional reduction, that would incorrectly suggest it to be 3, rigorously proved to be incorrect by John Imbrie[18]. Namely, for $T = 0, d > 2$ and weak disorder, Imbrie proved that there is broken symmetry with nonzero magnetization in the random-field Ising model.

V. PHYSICAL REALIZATIONS

In addition to the ferromagnet used as a paradigm system above, there is a large variety of physical systems to which above analysis applies. The most interesting of these are systems that break continuous symmetry and therefore are characterized by Goldstone modes that respond strongly and richly to random fields, as illustrated in Figs.5, 6.

A ubiquitous subclass of these is the elastic periodic media that spontaneously break continuous translational symmetry and are pinned by random pinning impurities that explicitly break translational symmetry and (as we will show), couple to the underlying phonon fields like a random field. Some examples of these pinned periodic elastic media include charge density waves (CDW) common in anisotropic conductors, Wigner crystals of strongly interacting 2d electrons, Abrikosov vortex lattices in type II superconductors, smectic and nematic liquid crystals[4] confined to a fractal aerogel matrix, ordinary crystals pinned by a random substrate as in friction and tectonic plates motion.[2, 3, 8]

Another interesting class is that of elastic media of co-dimension 1 (that allows it to buckle into the third dimension) with internal disorder as in membranes with random protein inclusions, randomly polymerized elastic membrane e.g., cytoskeleton, and graphene and other crystalline monolayers.[10]

A distinct fascinating class of random-field problems is an ordered states where pinning disorder appears only on the surface, like a pristine smectic liquid crystal pinned by a random substrate.[9]

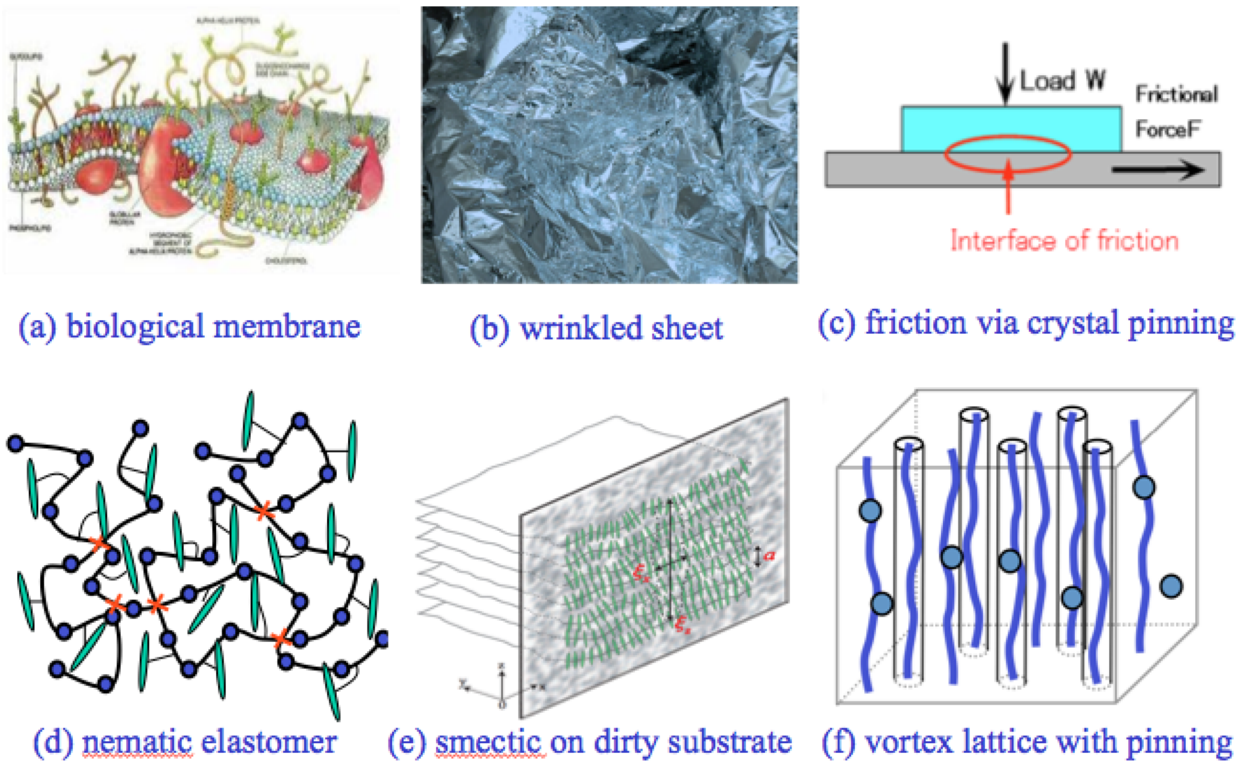


FIG. 5: Examples of physical realizations random-field elastic media

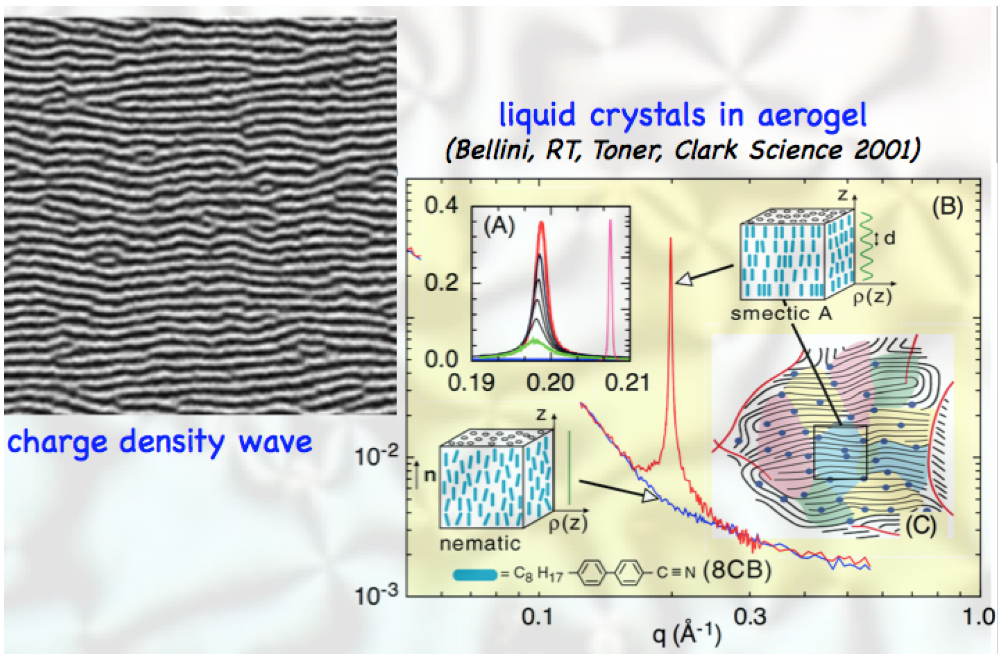


FIG. 6: Examples of physical realizations random-field elastic media, such as a charge-density wave (left) and nematic and smectic liquid crystals confined inside aerogel.

A. Pinned *scalar periodic* elastic media: CDW

Probably the most studied and simplest example of a pinned elastic medium is a charge-density wave in a "dirty" quasi-1d metal. A CDW order parameter is the complex amplitude ψ of the periodic charge density $\rho(\mathbf{x})$

$$\rho(\mathbf{x}) \;=\; \mathrm{Re}[\overline{\rho}_0 + e^{iq_0 z}\psi(\mathbf{x})] = \overline{\rho}_0 + |\psi| \cos[q_0 z + \phi(\mathbf{x})], \tag{105}$$

that is the amplitude of the lowest Fourier component of the density, and ρ_0 is the uniform background electron density. The phase $\phi(\mathbf{x})$ is the phonon Goldstone mode of the CDW associated with the spontaneous breaking of the uniaxial translational symmetry. The corresponding Goldstone mode Hamiltonian encodes the stiffness of inhomogeneous distortions of the CDW via the xy-model ($N = 2$)

$$H_{xy} = \frac{1}{2}K \int d^d x (\boldsymbol{\nabla}\phi)^2. \tag{106}$$

The ever-present random pinning potential $U(\mathbf{x})$ (corresponding to e.g., crystal defects and impurities) couple to the CDW density and therefore to the order parameter ψ according to

$$H_{pin}[\psi] \;=\; \int d^d x \frac{1}{2}\left[\delta t(\mathbf{x})|\psi|^2 + U(\mathbf{x})\rho\right], \tag{107a}$$

$$\approx \int d^d x \left[U(\mathbf{x}) e^{iq_0 z}\psi + U(\mathbf{x}) e^{-iq_0 z}\psi^*\right] \approx \int d^d x \left[V(\mathbf{x})\psi + V^*(\mathbf{x})\psi^*\right], \tag{107b}$$

$$\approx \; 2\int d^d x U(\mathbf{x})|\psi| \cos\left[q_0 z + \phi(\mathbf{x})\right], \tag{107c}$$

where we have dropped the subdominant random-bond disorder in favor of the *complex* random potential $V(\mathbf{x}) = U(\mathbf{x})e^{iq_0 z}$, which acts on the CDW order parameter ψ like a random field $\vec{h}(\mathbf{x}) = (V_{real}(\mathbf{x}), V_{imaginary}(\mathbf{x}))$ acts on a spin. We take this xy random field to be zero-mean, Gaussian characterized by the correlator

$$\overline{V(\mathbf{x})V^*(\mathbf{x}')} = \Delta_h \delta^d(\mathbf{x} - \mathbf{x}') \; . \tag{108}$$

As you will see in great detail from Pierre Le Doussal's lectures, (first demonstrated by Daniel Fisher[14]), it turns out that generically it is insufficient to only keep track of a single lowest harmonic q_0. At long scales all harmonics become equally important in their contribution to pinning ϕ. Thus, one is forced to treat a more general form of random-field disorder, namely a general random *periodic* function of ϕ, $V(\phi, \mathbf{x})$, characterized by

$$\overline{V(\phi, \mathbf{x}) V^*(\phi', \mathbf{x}')} = \Delta(\phi - \phi') \delta^d(\mathbf{x} - \mathbf{x}') \; . \tag{109}$$

and with the overall random-field xy-model Hamiltonian for the CDW given by

$$H_{rfxy} = \int d^d x \left[\frac{1}{2} K (\boldsymbol{\nabla} \phi)^2 + V(\phi, \mathbf{x}) \right] . \tag{110}$$

We leave the detailed technical analysis of this model to lectures by Pierre Le Doussal, limiting ourselves here to a linearized analysis due to Larkin[17], valid only on short-scale below the Imry-Ma-Larkin length scale ξ_{IML}.[16, 17] To this end, we note that at short scales, ϕ is small, justifying a linear in ϕ random force approximation,

$$V(\phi, \mathbf{x}) \approx const. + F(\mathbf{x}) \phi,$$

whose effects on ϕ can be straightforwardly calculated as in the previous section for a FM. The corresponding ϕ_{rms} is given by

$$\phi_{rms}^2 = \langle \phi(\mathbf{x})^2 \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{\Delta_F}{K^2 k^4}, \tag{111a}$$

$$\sim \frac{\Delta_F}{K^2} L^{4-d}, \text{ for } d < 4, \tag{111b}$$

$$\sim \frac{\Delta_F}{K^2} a^{4-d}, \text{ for } d > 4. \tag{111c}$$

Consistent with the analysis of a FM, we thus find $d_{lc}^{rfxy} = 4$ and the Larkin length $\xi_{IML} = (2\pi K^2/\Delta_F)^{1/(4-d)}$. On length beyond ξ_{IML}, multiple minima of $V(\phi, \mathbf{x})$ become important and a fully *nonlinear* treatment is required.[14] (In 2d, it is sufficient to retain only the lowest random-field harmonic, q_0, leading to the finite temperature Cardy-Ostlund glass[19, 20]). The upshot of this analysis is that within a fully elastic treatment (i.e., neglecting topological vortex defects valid for $d \geq 3$[23]) the random-field xy model (CDW phase fronts) remain topologically ordered, displaying logarithmic roughness of ϕ, with

$$\overline{\langle (\phi(\mathbf{x}) - \phi(\mathbf{x}'))^2 \rangle} = A_d \ln |\mathbf{x} - \mathbf{x}'|, \tag{112}$$

with a universal amplitude A_d found by Giamarchi and LeDoussal [21]. This low-temperature phase is thus appropriately referred to as an xy-glass, topologically ordered and elastically disordered, qualitatively distinct from the fully topologically disordered state, and thus separated from it by a sharp phase transition.

Even though we discussed above model within the context of a CDW, it is clear that it is equally appropriate to any physical system characterized by a scalar and compact (periodic) Goldstone mode with $O(2) \sim U(1)$ symmetry, such as for example an xy ferromagnet.

B. Pinned *vector periodic* elastic media: vortex lattice, Wigner crystal

Only a slight generalization of the random-field xy-model (as applied to CDW) applies to an even broader class of systems, characterized by a vector Goldstone mode. Obvious examples are generalized crystals (e.g., vortex lattice in type-II superconductor or a Wigner crystal), that spontaneously break translational symmetry along more than one axis and thus exhibit a vector phonon, $\mathbf{u}$.

Typically such periodic states occur inside a crystal (e.g., a superconductor), that in any physical system admits a finite density of impurities and lattice defects (often these are introduced intentionally to enhance pinning), that act like a random potential that couples to the periodic component of the density as discussed in the scalar case of CDW above.

A resulting Hamiltonian for a d-dimensional (2d as the physical case) Wigner crystal is then given by a sum of elastic (taken to be isotropic below) and random pinning energies

$$H_{crystal} = \int d^d x \left[\mu u_{\alpha\beta}^2 + \frac{1}{2}\lambda u_{\alpha\alpha}^2 + V(\mathbf{u}, \mathbf{x}) \right], \tag{113}$$

where $u_{\alpha\beta} \approx \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$ is the linearized strain tensor, sufficient for our treatment here.

This can be generalized in an obvious way to a $d+1$ dimensional (3d in the physical case) vortex lattice, with a d-dimensional in-plane phonon field transverse to the vortex lines, and Hamiltonian given by

$$H_{vortex\ lattice} = \int d^d x dz \left[\frac{1}{2}\epsilon(\partial_z u)^2 + \mu u_{\alpha\beta}^2 + \frac{1}{2}\lambda u_{\alpha\alpha}^2 + V(\mathbf{u}, \mathbf{x}) \right]. \tag{114}$$

The treatment of these systems is quite similar to that of their scalar "cousin" xy-model for the CDW discussed above. The distinction is in the nature of the topological defects, dislocations and disclinations here[1] and a simple generalization to d phonon components.

C. "Dirty" smectic liquid crystals

1. *Smectic liquid crystals confined in aerogel*

Another rich and qualitatively distinct randomly pinned elastic medium is liquid crystals confined to a random porous matrix as e.g., smectic in aerogel.[5–7] The key new qualitative feature of the smectic is its "soft" elasticity controlled by the Laplacian curvature energy.[1, 4]

As discussed in my Critical Phases lecture notes, this seriously enriches the phenomenology because even in the absence of quenched disorder, the treatment of smectics requires inclusion of the fully nonlinear strain tensor. Furthermore, in addition to the random positional pinning, there is a strong effect arising to orientational pinning of the layer normals, entering through the coupling of the aerogel strands to nematogens

$$H_{dn} = \frac{1}{2}\int d^dx\big(\mathbf{g}(\mathbf{x})\cdot\hat{\mathbf{n}}\big)^2 \approx \int d^dx\mathbf{h}(\mathbf{x})\cdot\delta\hat{\mathbf{n}}, \tag{115}$$

where we have defined a quenched random tilt field

$$\mathbf{h}(\mathbf{x}) \equiv g_z(\mathbf{x})\mathbf{g}(\mathbf{x}) \ . \tag{116}$$

$$\overline{h_i(\mathbf{x})h_j(\mathbf{x}')} \ = \ \Delta_h\delta^d(\mathbf{x}-\mathbf{x}')\ \delta_{ij} \ , \tag{117}$$

which is *short–ranged* and characterized by the tilt field-disorder variance Δ_h.

Combining this pinning aerogel energy with the smectic elastic energy[1, 4], we obtain

$$\begin{aligned} H[u] \ &= \ \int d^dx\Bigg[\frac{B}{2}(\partial_z u - \frac{1}{2}(\boldsymbol{\nabla}u)^2)^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 + \mathbf{h}(\mathbf{x})\cdot\boldsymbol{\nabla}_\perp u \\ &- \ |\psi_0|U(\mathbf{x})\cos[q_0\big(z+u(\mathbf{x})\big)]\Bigg] \ , \end{aligned} \tag{118}$$

The analysis of above model is quite involved as it requires nonlinearities associated with elasticity, random-field disorder and topological defects. The upshot of this analysis is that the smectic state is replaced by a new, partially ordered "smectic glass" phase, that exhibits anomalous, length scale dependent, glassy elasticity and is elastically disordered, distinct from its fully positionally disordered nematic state (that itself is converted into "nematic glass").[6, 7]

2. Smectic liquid crystals with a "dirty" substrate

Another very interesting realization of a "dirty" liquid crystal and in particular a smectic phase liquid crystal with a random substrate as in a liquid crystal cell. The governing Hamiltonian is quite close to that of the bulk smectic in aerogel, (118), but with the main difference that the pinning disorder is confined only to the $d-1$ dimensional (2d) substrate. The analysis and resulting phenomenology is very rich as discussed by Zhang and Radzihovsky[9].

D. Disordered polymerized membranes

Polymerized membranes with random local inclusions and defects contitute another qualitatively distinct class of random elastic objects (taken to be D-dimensional). The new feature is the co-dimension of d_c that allows the membrane to embed nontrivially inside the $d = D + d_c$ (3d physically) dimensional space.

We first recall that in the absence of disorder, an elastic membrane is described by a combination of in-plane elastic energy with Lamé parameters μ, λ and phonons $\mathbf{u}$, and a bending energy with curvature modulus κ and "height" function, $\vec{f}$, (nott characterized by an elastic Hamiltonian:

$$H_{flat}[\vec{f},\mathbf{u}] = \int d^D x \left[\frac{\kappa}{2}(\nabla^2\vec{f})^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2}u_{\alpha\alpha}^2\right] , \tag{119}$$

where the strain tensor is

$$u_{\alpha\beta} \;=\; \frac{1}{2}(\partial_\alpha\vec{r}\cdot\partial_\beta\vec{r} - \delta_{\alpha\beta}) \approx \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha\vec{f}\cdot\partial_\beta\vec{f}), \tag{120}$$

where we defined the strain tensor in terms of deviation of the embedding-induced metric $g_{\alpha\beta}$ from the flat metric, $\delta_{\alpha\beta}$ and in the second form neglected in-plane elastic nonlinearities that are subdominant at long scales.

In this form the effects of in-plane disorder is straightforwardly incorporated by replacing the flat background metric $\delta_{\alpha\beta}$ by a nontrivial quenched random reference metric, $g^0_{\alpha\beta}(\mathbf{x})$. This acts like an external local stress, $\sigma^0_{\alpha\beta}(\mathbf{x})$ and couples to the fully nonlinear strain tensor in the usual way, $H_{stress} \approx -\int d^D x \sigma^0_{\alpha\beta}(\mathbf{x}) u_{\alpha\beta}$. [29] Because such disorder is even in the height $\vec{f}$ undulations, it does not break $\vec{f}\to-\vec{f}$ symmetry. It is thus quite analogous to the random-bond disorder.

In addition, inclusions that distinguish top and bottom of the membrane and therefore break $\vec{f} \to -\vec{f}$ symmetry can also be naturally included by adding the random local mean curvature $\vec{c}(\mathbf{x})$. Because it couples linearly to $\vec{f}$ such disorder is the analog of the random-field disorder. The full generic disordered membrane Hamiltonian given by

$$H_{flat}[\vec{f}, \mathbf{u}] = \int d^D x \left[\frac{\kappa}{2}(\nabla^2 \vec{f} - \vec{c}(\mathbf{x}))^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2 - 2\mu\sigma_{\alpha\beta}^0(\mathbf{x}) u_{\alpha\beta} - \lambda\sigma_{\alpha\alpha}^0(\mathbf{x}) u_{\beta\beta} \right] . \tag{121}$$

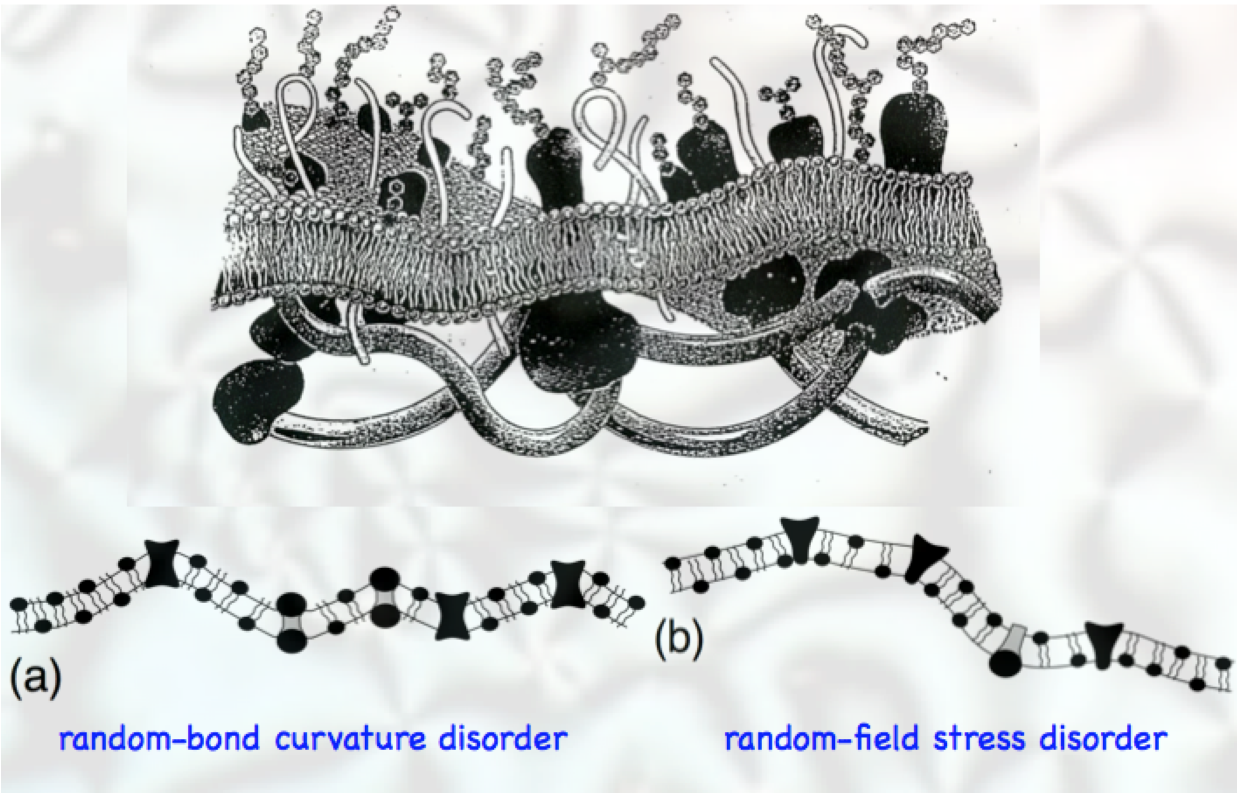


FIG. 7: An illustration of a bilayer membrane with protein and other inclusions that lead to two qualitatively distinct types of disorder. In (a) symmetric (even in $\vec{f}$), random-bond, and (b) asymmetric (odd in $\vec{f}$) random-field inclusions correspond to the stress and random mean curvature disorders, respectively.

The properties of such membranes have been extensively studies and predict that interplay of quenched internal disorder, thermal fluctuations and geometry lead to rich phenomenology. It consists of disorder-driven glassy wrinkling, power-law roughness, phase transitions and anomalous (length scale-dependent) elasticity.[10, 30–32]

VI. SUMMARY AND CONCLUSIONS

These lectures are a gentle introduction to quench-disordered systems, aimed at understanding effects of random heterogeneity near critical point and inside ordered phases. To this end we discussed lattice formulation of disorder in simplest random bond and random field cases and their long-scale field-theoretic description.

After a review of some technology, such as functional integrals and the replica trick, we discuss the effects of disorder near continuous phase transitions and inside ordered phases. For the former we derive a Harris criterion for the importance of random bond disorder near a critical point. For the latter we outline the results of RG analysis that leads to a zero-temperature disorder-driven PM-FM phase transition, replacing the disorder-free critical point.

We then turn to phase stability for states that break discrete (Ising) and continuous ($O(N)$) symmetries. We analyze the stability using field theoretic and more careful physical arguments a la Imry-Ma-Larkin,[16, 17], later demonstrating the breakdown of dimensional reduction in the random-field Ising model.

We finish with a cursory presentation of a variety physical applications of these ideas to pinned periodic media such charge-density wave, vortex lattices, as well as smectics in aerogel, polymerized membranes with quenched internal disorder, and liquid crystals with a dirty substrate.

VII. ACKNOWLEDGMENTS

The material presented in these lectures is based on research done and discussions with a number of wonderful colleagues, most notably David Nelson, John Toner, Pierre Le Doussal, Daniel Fisher, Leon Balents, and Xiangjun Xing. I am indebted to these colleagues for much of my insight into the material presented here. This work was supported by the National Science Foundation through grants DMR-1001240 and DMR-0969083 as well by the Simons Investigator award from the Simons Foundation.

VIII. APPENDIX

Let us start out slowly with standard scalar Gaussian integrals

$$Z_0(a) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}, \tag{122}$$

$$Z_1(a) = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{1}{2}ax^2} = -2\frac{\partial}{\partial a} Z_0(a) = \frac{1}{a}\sqrt{\frac{2\pi}{a}} = \frac{1}{a}Z_0, \tag{123}$$

$$Z_n(a) = \int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2} = \frac{(2n-1)!!}{a^n} Z_0, \tag{124}$$

that can be deduced from dimensional analysis, relation to the first basic integral $Z_0(a)$ (that can in turn be computed by a standard trick of squaring it and integrating in polar coordinates) or another generating function and Γ-functions

$$Z(a,h) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+hx} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}a(x-h/a)^2} e^{\frac{1}{2}h^2/a} = Z_0(a) e^{\frac{1}{2}h^2/a}, \tag{125}$$

$$= \sum_{n=0} \frac{h^{2n}}{(2n)!} Z_n(a). \tag{126}$$

Quite clearly, odd powers of x vanish by symmetry.

A useful generalization of above Gaussian integral calculus is to integrals over complex numbers. Namely, from above we have

$$I_0(a) = \int_{-\infty}^{\infty} \frac{dxdy}{\pi} e^{-a(x^2+y^2)} = \frac{1}{a} = \int \frac{d\bar{z}dz}{2\pi i} e^{-a\bar{z}z}, \tag{127}$$

where in above we treat $\bar{z}, z$ as independent complex fields and the normalization is determined by the Jacobian of the transformation from x, y pair. This integral will be invaluable for path integral quantization and analysis of bosonic systems described by complex fields, $\overline{\psi}, \psi$.

1. *d-dimensions*

This calculus can be straightforwardly generalized to multi-variable Gaussian integrals characterized by an $N \times N$ matrix $(\mathbf{A})_{ij}$,

$$Z_0(\mathbf{A}) = \int_{-\infty}^{\infty} [d\mathbf{x}] e^{-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}} = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{a_i}} = \sqrt{\frac{(2\pi)^N}{\det \mathbf{A}}}, \tag{128}$$

$$Z_1^{ij}(\mathbf{A}) = \int_{-\infty}^{\infty} [d\mathbf{x}] x_i x_j e^{-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}} = Z_0 \mathbf{A}_{ij}^{-1}, \tag{129}$$

$$Z(\mathbf{A},\mathbf{h}) = \int_{-\infty}^{\infty} [d\mathbf{x}] e^{-\frac{1}{2}\mathbf{x}^T\cdot\mathbf{A}\cdot\mathbf{x}+\mathbf{h}^T\cdot\mathbf{x}} = Z_0(\mathbf{A}) e^{\frac{1}{2}\mathbf{h}^T\cdot\mathbf{A}^{-1}\cdot\mathbf{h}}, \tag{130}$$

computed by diagonalizing the symmetric matrix $\mathbf{A}$ and thereby decoupling the N-dimensional integral into a product of N independent scalar Gaussian integrals (124), each characterized by eigenvalue a_i.

As a corollary of these Gaussian integral identities we have two more very important results, namely, that for a Gaussian random variable $\mathbf{x}$ obeying Gaussian statistics, with variance $\mathbf{A}_{ij}^{-1}$, we have

$$\langle \mathbf{x}_i \mathbf{x}_j \rangle \equiv G_{ij} = \frac{1}{Z_0} \int_{-\infty}^{\infty} [d\mathbf{x}] x_i x_j e^{-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}} = \mathbf{A}_{ij}^{-1}, \quad (131)$$

$$\langle e^{\mathbf{h}^T \cdot \mathbf{x}} \rangle = e^{\frac{1}{2}\langle (\mathbf{h}^T \cdot \mathbf{x})^2 \rangle} = e^{\frac{1}{2}\mathbf{h}^T \cdot \mathbf{G} \cdot \mathbf{h}}, \quad (132)$$

with second identity the relative of the Wick's theorem, which will be extremely important for computation of x-ray and neutron scattering structure function.

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