Physics 7240: Advanced Statistical Mechanics

Lecture 6: Thermal Stability of Ordered Phases: Goldstone modes and topological defects

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Abstract

In these lecture notes, we will analyze the stability of ordered phases to thermal fluctuations, focusing particularly on states that break continuous symmetry. Computing the size of these Goldstone-mode fluctuations as a function of temperature and system's dimensionality, we will derive the so-called Hohenberg-Mermin-Wagner-Coleman theorems that inform us when such systems become absolutely unstable to fluctuations, thereby precluding the existence of the ordered state, at least in its mean-field form. We will also discuss topological defects, e.g., vortices, dislocations, disclinations, solitons, etc., and their role in destroying ordered phases by way of Kosterlitz-Thouless and related transition. We will also explore dualities, roughening and commensurate-incommensurate phase transitions. Finally we will study the disordering of the O(N) nonlinear σ-model via $d-2$ and large N expansions.

- States that break continuous symmetries and their Goldstone mode description: $O(N)$ models, superfluids, crystals, membranes, nematics, smectics, superconductors, etc.
- Stability of phases: Hohenberg-Mermin-Wagner-Coleman theorems
- Topological defects: vortices, dislocations, disclinations, solitons
- Kosterlitz-Thouless transition via Coulomb gas and sine-Gordon duality, and roughening and commensurate-incommensurate phase transitions
- Nonlinear $O(3)$ σ-model and the FM-PM transition in $d = 2 + \epsilon$ dimensions
- Large N expansion for the $O(N)$ model FM-PM transition

I. INTRODUCTION AND MOTIVATION

Much of our focus so far has been on the phase transition between different states of matter. However, in a sense, a prerequisite to studying phase transitions and critical phenomena is the study phases of matter themselves and in particular their stability to fluctuations. On the other hand, almost by definition of the ordered phase, fluctuations inside ordered phases are typically, but not always (see the upcoming discussion of "critical phases") finite and bounded, vanishing with reduced temperature. If they were not finite the phase would not be ordered. Below we will explore the range of stability and related questions in a number of important physical systems and corresponding models, some of whose critical properties we have studied in previous lectures. These analyses will also give us an approach to phase transitions from the ordered state side, by studying how a phase disorders, that is complementary to our treatment of phase transitions by studying how a disordered phase orders.

To this end, here I will focus on ordered phases, with particular attention on states that *spontaneously* break a *continuous* symmetry and thus exhibit Goldstone modes as the low-energy excitations. I will thus derive an effective Goldstone-mode Hamiltonians characterizing excitations at low temperatures and will use them to assess the stability of variety of ordered states to low-temperature fluctuations. As a result I will derive stability theorems that are due to Landau-Peierls[16, 17] and to Hohenberg, Mermin, Wagner, and Coleman[18] and their generalizations. From these I will extract the so-called the lower-critical dimension, d_{lc}, below which the ordered phase at hand is unstable to arbitrarily weak thermal fluctuations. I will leave quantum and quenched disorder counter-part of these theorems to future lectures.

Having established the stability of phases to “small” low-energy fluctuations, we will examine nonlinear topological excitations, that appear at finite energy and will study stability of a myriad of phases to these excitations and their disordering phase transitions.

II. ORDERED PHASES OF O(N) MODEL AND THEIR STABILITY

A. Goldstone-modes Hamiltonian

We start out with a generic class of models, the $O(N)$ model of an N-component real vector field $\vec{S}$, with Landau-Ginzburg Hamiltonian,

$$H[\vec{S}(\mathbf{x})] = \int_{\mathbf{x}} \left[\frac{1}{2}J(\boldsymbol{\nabla}\vec{S})^2 + \frac{1}{2}t|\vec{S}|^2 + \frac{1}{4}u|\vec{S}|^4\right]. \tag{1}$$

that we have already explored in the earlier lectures near the critical point. Here instead our focus is on the ordered FM state, that, as we explored extensively in earlier lectures appears when $t < 0$ and is characterized by a spontaneous magnetization order parameter $\vec{S}_0 = \hat{S}_0\sqrt{-t/u}$ (see Fig.1).

There a number of equivalent representations of the fluctuations in the ordered state. The simplest form is given in terms of longitudinal ($S_l\hat{S}_0$) and transverse (Goldstone modes $\vec{S}_t$) fluctuations

$$\delta\vec{S} \equiv S_l\hat{S}_0 + \vec{S}_t = \vec{S} - \vec{S}_0, \tag{2}$$

governed by a Hamiltonian in the ordered state,

$$\begin{aligned} H[\delta\vec{S}(\mathbf{x})] &= \int_{\mathbf{x}} \left[\frac{1}{2}J(\boldsymbol{\nabla}\delta\vec{S})^2 + \frac{1}{4}u\left(|\vec{S}|^2 - S_0^2\right)^2\right], \\ &= \int_{\mathbf{x}} \left[\frac{1}{2}J(\boldsymbol{\nabla}\delta\vec{S})^2 + \frac{1}{4}u\left(2\vec{S}_0\cdot\delta\vec{S} + |\delta\vec{S}|^2\right)^2\right], \quad (3) \\ &= \int_{\mathbf{x}} \left[\frac{1}{2}J(\boldsymbol{\nabla}\delta\vec{S})^2 + u(\vec{S}_0\cdot\delta\vec{S})^2 + u(\vec{S}_0\cdot\delta\vec{S})|\delta\vec{S}|^2 + \frac{1}{4}u|\delta\vec{S}|^4\right], \quad (4) \\ &\approx \int_{\mathbf{x}} \left[\frac{1}{2}J(\boldsymbol{\nabla}S_l)^2 + \frac{1}{2}(2|t|)S_l^2 + \frac{1}{2}J(\boldsymbol{\nabla}\vec{S}_t)^2\right], \quad (5) \\ &\approx \frac{1}{2}J\int_{\mathbf{x}}(\boldsymbol{\nabla}\vec{S}_t)^2, \quad (6) \end{aligned}$$

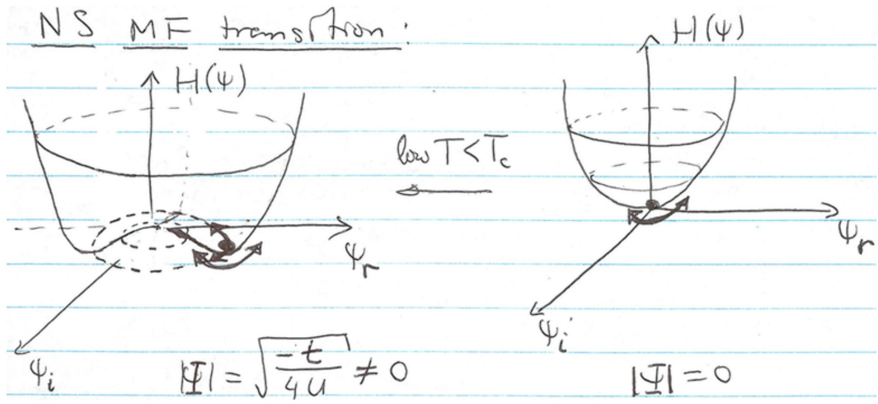

FIG. 1: A depiction of the "Normal to Superfluid" (and more generally PM - FM) transition in the $O(N)$ model. The "Mexican-hat" potential illustrates one ($N-1$ for O(N) model) Goldstone mode, and one Higgs gapped mode, respectively corresponding to fluctuations transverse and longitudinal with respect to the spontaneous order parameter $\vec{S}_0$.

where in the penultimate line we neglected the nonlinearities, focusing on the harmonic component. Given that, as anticipated, the longitudinal Higgs mode component is "massive" (gapped) deep below T_c, (with "mass" $\sqrt{2|t|}$) S_l can be safely and inconsequentially integrated out, leading in the last line to the Hamiltonian for the $N-1$ gapless Goldstone-modes.

1. *Nonlinear σ-model*

Alternatively and more symmetrically we can use the magnitude-orientation (spherical) representation $\vec{S} = S\hat{n}$, where $\hat{n}(\mathbf{x})$ is a unit vector field that characterizes the orientation of the magnetization inside the FM state (fluctuating about a spontaneously chosen direction) and $S(\mathbf{x}) = S_0 + S_l(\mathbf{x})$, with S_0 the average spontaneous magnitude of the uniform non-fluctuating magnetization and $S_l(\mathbf{x})$ the fluctuating longitudinal part. Neglecting the massive longitudinal fluctuations (i.e., $S(\mathbf{x}) \approx S_0$), the Goldstone-mode Hamiltonian is given by,

$$H_{n\sigma m}[\hat{n}(\mathbf{x})] = \frac{1}{2}K\int d^dx(\boldsymbol{\nabla}\hat{n})^2, \tag{7}$$

the so called Nonlinear Sigma Model (NSM = nσm) in the $S_{N-1} \equiv O(N)/O(N-1)$ universality class, with the effective stiffness $K = S_0^2 J$.

Note that although this Hamiltonian appears to be quadratic in $\hat{n}$, the nontrivial constraint $|\hat{n}(\mathbf{x})|^2 = 1$ (that implicitly supplements above Hamiltonian) makes this theory non-trivial, i.e., effectively interacting. It should be clear that in fact this unit constraint is the extreme limit of interactions, where $t \to -\infty, u \to \infty$ with the ratio $S_0^2 = -t/u$ fixed.

Using a Cartesian representation $\hat{n}(\mathbf{x}) = (\vec{n}_\perp, \sqrt{1-n_\perp^2})$, to quadratic order in $\vec{n}_\perp$, the Hamiltonian reduces to the last line in (6), with $\vec{S}_t = S_0 \vec{n}_\perp$.

Using the spherical representation for $N = 3$, in terms of the Euler's angles, with

$$\hat{n} \; = \; (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{8}$$

the Hamiltonian reduces to,

$$H_{n\sigma m}^{sph}[\theta(\mathbf{x}), \phi(\mathbf{x})] \; = \; \frac{1}{2}K \int_{\mathbf{x}} \left[\sin^2\theta(\nabla\phi)^2 + (\nabla\theta)^2\right]. \tag{9}$$

I note that it is indeed explicitly nonlinear in this representation.

2. *Schwinger boson CP_{N-1} representation*

An alternative convenient form for the $O(3)$ nonlinear σ-model is the Schwinger boson (also called the CP_2) representation

$$\hat{n} \; = \; z^* \vec{\sigma} z, \tag{10}$$

with $\vec{\sigma}$ the three Pauli matrices and constraint $|z_\uparrow|^2 + |z_\downarrow|^2 = |z|^2 = 1$ on two complex component fields, forming a complex spinor, $z_a = (z_\uparrow(\mathbf{x}), z_\downarrow(\mathbf{x}))$. Substituting (10) into the nσm Hamiltonian (118), and using identities

$$\vec{\sigma}_{ab} \cdot \vec{\sigma}_{cd} \; = \; 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}, \tag{11}$$

$$(\boldsymbol{\nabla} z^*)z \; = \; -z^* \boldsymbol{\nabla} z, \tag{12}$$

with latter following by differentiating the constraint $|z|^2 = 1$, we obtain the corresponding CP_2 Hamiltonian, given by

$$H_{n\sigma m}^{CP_2}[z_\sigma(\mathbf{x})] \; = \; K \int_{\mathbf{x}} \left[|\boldsymbol{\nabla} z|^2 + (z^* \boldsymbol{\nabla} z)^2\right], \tag{13}$$

$$= \; K \int_{\mathbf{x}} |\left(-i\boldsymbol{\nabla} - \mathbf{A}\right) z|^2. \tag{14}$$

Above, an effective gauge vector potential $\mathbf{A}(\mathbf{x})$, a real field, has emerged and is given by

$$\mathbf{A}(\mathbf{x}) \; = \; z^*(-i\boldsymbol{\nabla} z). \tag{15}$$

Note, a dot product between the two-component spinors $z^* z \equiv z^* \cdot z = \sum_{a=\uparrow,\downarrow} z_a^* z_a$ is implied.

Observe that because of the constraint $|z|^2 = 1$, the "diamagnetic" term $A^2|z|^2 = A^2$ (i.e., independent of the "matter" field z_a) is quadratic in $\mathbf{A}$ and therefore in above this pseudo field vector potential can be treated as an independent fluctuating gauge field, rather than constrained to z_a as given in Eq. (15). As we will see, such model describes charged superconductors, but also emerges in the context of more complex correlated states of matter, where the gauge field $\mathbf{A}$ is not of electromagnetic origin.

I further note that z_a contains three real degrees of freedom (2 complex numbers with 1 real constraint). However, in the ordered FM phase, where $z_a \neq 0$ a Higgs mechanism takes place, that gaps out the phase of z_a, leaving only 2 real degrees of freedom corresponding to Goldstone modes of $\hat{n} \in S_2$. On the other hand, the disordered PM phase exhibits 3 gapped degrees of freedom. In terms of $\hat{n}$ (when the constraint is softened) the system is expected to be described by the $O(3)$ model, which (consistently with above gauge theory) undergoes a transition into the $O(3)/O(2) = S_2$ ordered phase. Consistent with this, z_a then contains 3 real degrees of freedom, as required.

As a check, the Schwinger bosons can be represented in terms of three Euler angles according to:

$$z = e^{i\chi} \begin{pmatrix} e^{i\phi/2} \cos(\theta/2) \\ e^{-i\phi/2} \sin(\theta/2) \end{pmatrix}. \tag{16}$$

Plugging this representation into $H_{n\sigma m}^{CP_2}[z_\sigma(\mathbf{x})]$, we find that it reduces to the previously derived spherical coordinates form $H_{n\sigma m}^{sph}[\theta(\mathbf{x}), \phi(\mathbf{x})]$, as required.

An important CP_N generalization of above model corresponds to extension of z_a to N complex components, with $|z|^2 = 1$ normalization. We will return to further discussion of this model later in the lectures.

3. *The O(2) XY model*

Now let us consider an important special case, the XY model, with $N = 2$ component spins. The corresponding model describing fluctuations in the XY FM phase has $\hat{n}(\mathbf{x})$ confined to a plane in spin-space, that can be accomplished by adding to $H_{n\sigma m}^{sph}$, (9) an easy-plane crystalline anisotropy $\frac{1}{2}\alpha n_z^2 = \frac{1}{2}\alpha \cos^2\theta \approx \frac{1}{2}\alpha(\theta - \pi/2)^2$, that pins θ at $\pi/2$ and gaps out its fluctuations. At low energies gapped $\theta(\mathbf{x})$ fluctuations can be neglected or simply integrated out of the partition function, leading to an effective XY-model Hamiltonian

$$H_{XY}[\phi(\mathbf{x})] \;=\; \frac{1}{2}K\int_{\mathbf{x}}(\nabla\phi)^2. \tag{17}$$

Alternatively, one can think about a model of a classical finite temperature normal-to-superfluid (NS) transition, described by the familiar Ginzburg-Landau theory for the scalar complex order parameter

$$\Psi(\mathbf{x}) \;=\; |\Psi|e^{i\phi(\mathbf{x})}, \tag{18}$$

$$\;=\; \sqrt{n(\mathbf{x})}e^{i\phi(\mathbf{x})} = \sqrt{n_0+\pi(\mathbf{x})}e^{i\phi(\mathbf{x})}, \tag{19}$$

with

$$H_{XY} = \int d^dx\left[\frac{\hbar^2}{2m}|\nabla\Psi|^2 + \frac{1}{2}t|\Psi|^2 + \frac{1}{4}u|\Psi|^4\right], \tag{20}$$

i.e., an XY model for $\Psi = \Psi_r + i\Psi_i$ (see Fig.(2)) isomorphic to a planar ferromagnet with $\vec{S} = S_1\hat{x} + S_2\hat{y}$, $\Psi = S_1 + iS_2$, the effective exchange constant $J = \hbar^2/m$, $\frac{1}{2}t = -\mu$ negative of the chemical potential, and $\frac{1}{4}u = \frac{4\pi\hbar^2}{2m}a_s$ set by the s-wave scattering length a_s. Above $\pi(\mathbf{x}), \phi(\mathbf{x})$ are canonically conjugate condensate number density and superfluid phase fluctuations of the order parameter, with the latter the single Goldstone mode expected in this $N = 2$ case.

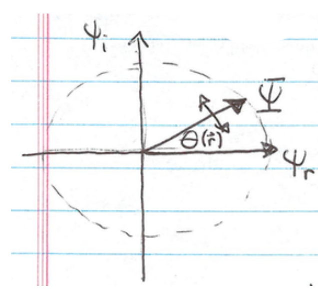

FIG. 2: XY model complex scalar order parameter.

Utilizing the polar representation for Ψ inside H_{XY}, (20) for $t < 0$, i.e., positive chemical potential at which bosons condense into a superfluid Bose-Einstein condensate, BEC, we find

$$H_{XY} \;=\; \int_{\mathbf{x}}\left[\frac{\hbar^2 n}{2m}|\boldsymbol{\nabla}\phi|^2 + \frac{\hbar^2}{8mn}|\boldsymbol{\nabla}\pi|^2 + \frac{1}{2}tn + \frac{1}{4}un^2\right], \tag{21}$$

$$\approx\; \int_{\mathbf{x}}\left[\frac{\hbar^2 n_0}{2m}|\boldsymbol{\nabla}\phi|^2 + \frac{\hbar^2}{8mn_0}|\boldsymbol{\nabla}\pi|^2 + \frac{1}{4}u\pi^2\right], \tag{22}$$

$$\approx\; \frac{1}{2}K\int_{\mathbf{x}}|\boldsymbol{\nabla}\phi|^2 = \frac{\rho_s}{2}\int_{\mathbf{r}}v_s^2, \tag{23}$$

where $n_0 = |\Psi_0|^2 = \sqrt{-t/u}$ is the condensate density, and in the last line we neglected the gapped density fluctuations $\delta n(\mathbf{x}) \equiv \pi(\mathbf{x}) = n(\mathbf{x}) - n_0$ about n_0, or equivalently integrated out these gapped Higgs mode excitations. Utilizing the well-known form of superfluid velocity $\vec{v}_s = \frac{\hbar}{m}\boldsymbol{\nabla}\phi$, we see that the effective stiffness $K = \hbar^2\rho_s/m^2 = \frac{\hbar^2 n_0}{m}$ is proportional to the superfluid mass density ρ_s (defined by the last equality in (23)). The last equality of ρ_s with mass times the condensate density, mn_0, is however violated beyond mean-field theory. In a Galilean-invariant system, at $T = 0$, $\rho_s = mn$, meaning, the superfluid stiffness is determined by the total boson density, n, rather than condensate density n_0. One important illustration of this is that even if the condensate density n_0 vanishes due to strong interactions and associated fluctuations, as it does in superfluid films (as we will see shortly), the superfluid density ρ_s remains nonzero and in fact approaches mn at $T = 0$, as required by aforementioned Galilean invariance.

Comparing (23) with the case of $N \geq 3$ (118) and, in particular with (9) with $O(3)$, the crucial difference in the $O(2)$ XY model is that its spin-wave fluctuations are exactly harmonic. Their spin-wave phase fluctuations can thus be computed exactly using Gaussian field theory calculus or equivalently the equipartition theorem (derived from the former).

B. Stability to Goldstone-modes fluctuations: Hohenberg-Mermin-Wagner-Coleman theorem

1. O(N) model

To assess the stability of the spontaneously-ordered (e.g., FM, a superfluid, etc.) state to low-temperature fluctuations, we approximate the corresponding Hamiltonian by a quadratic form in Goldstone modes (which, as noted above is exact for $N = 2$) by neglecting, or equivalently integrating out the gapped longitudinal fluctuations in (6), obtaining

$$H_{fluct} \approx \frac{1}{2} J \int_{\mathbf{x}} (\boldsymbol{\nabla}\vec{S}_t)^2, \tag{24}$$

a $N-1$-component version of the XY Hamiltonian above.

By definition, the thermodynamically stability of the state requires that the fluctuations about the ordered state are small. A one good measure of this is the root-mean-squared fluctuations of $\delta\vec{S} \approx \vec{S}_t$, given by

$$\delta S^2_{rms} = \langle \vec{S}_t(\mathbf{x}) \cdot \vec{S}_t(\mathbf{x}) \rangle = (N-1)\int_{L^{-1}}^{a^{-1}} \frac{d^dk}{(2\pi)^d}\frac{k_BT}{Jk^2} \sim \frac{k_BT}{J}\begin{cases} \frac{1}{a^{d-2}}, & \text{for } d > 2, \\ L^{2-d}, & \text{for } d < 2, \\ \ln(L/a), & \text{for } d = 2, \end{cases} \quad (25)$$

where to control fluctuations we have performed the analysis in the finite box size $L \gg a$, with a the UV lattice cutoff. We thus find that for arbitrary small temperature, fluctuations diverge with system size L for $d \leq 2$. On the other hand, a reasonable stability criterion (which in the context of melting of crystals known as the Lindemann criterion) for a breakdown of an ordered phase is

$$\delta S^2_{rms} \approx S^2_0. \quad (26)$$

Combining this with (25), shows that for $d \leq 2$ even arbitrarily small temperature, fluctuations destabilizing the $O(N)$ ordered state in the thermodynamic $L \to \infty$ limit.

Thus we find a far reaching and quite generic result, often referred to as the Hohenberg-Mermin-Wagner-Coleman theorem[18] (though it was derived in three quite physically different and specific contexts, 2d superfluids and 2d crystal, i.e., films, and in relativistic quantum field theory), that forbids a spontaneous breaking of a continuous symmetry in $d \leq 2$. In contrast, for $d > 2$ such $O(N)$ ordered phases are stable at small nonzero temperature, and the above stability criterion, (26) gives an estimate for the disordering transition temperature, $k_BT_c \approx J/a^{d-2}$.

2. *Superfluids and planar magnets: XY model*

For the simplest case of an XY ($O(2)$) model, the Goldstone mode $\phi(\mathbf{x})$ (superfluid phase or planar magnet's local azimuthal spin orientation) is furthermore characterized by a correlation function, that for $x \gg a$ is given by

$$C(\mathbf{x}) = \frac{1}{2}\langle(\phi(\mathbf{x}) - \phi(0))^2\rangle = \langle\phi(\mathbf{x})\phi(\mathbf{x}) - \phi(\mathbf{x})\phi(0)\rangle, \quad (27)$$

$$= \int \frac{d^dk}{(2\pi)^d}\frac{k_BT}{Kk^2}\left(1 - e^{i\mathbf{k}\cdot\mathbf{x}}\right), \quad (28)$$

$$\approx \frac{k_BTC_d}{K}\begin{cases} \frac{1}{a^{d-2}}, & \text{for } d > 2, \\ x^{2-d}, & \text{for } d < 2, \\ \ln(x/a), & \text{for } d = 2, \end{cases} \quad (29)$$

In more detail, $C(\mathbf{x})$ grows quadratically with x/a for $x \ll a$, and then asymptotes to above limiting forms. Thus, consistent with (25), the average difference in phase fluctuations $\phi(\mathbf{x})$ between two points separated by $x \gg a$ is finite (and small for $k_BT/K \ll 1$) for $d > 2$, but diverge for $d \leq 2$.

Utilizing above result for $\phi(\mathbf{x})$, we can now calculate the order parameter correlator, focusing on gapless phase fluctuations and neglecting gapful magnitude of the order parameter (density n in the case of the superfluid) fluctuations. Using Wick's theorem, valid for the Gaussian Goldstone mode $\phi(\mathbf{x})$, at large x we find

$$\langle\psi^*(\mathbf{x})\psi(0)\rangle \approx n\langle e^{i(\phi(\mathbf{x})-\phi(0))}\rangle \sim e^{-C(\mathbf{x})}, \tag{30}$$

$$\sim \begin{cases} e^{-(k_BTC_d/K)a^{2-d}}, & \text{for } d > 2, \text{ LRO}, \\ e^{-(k_BTC_d/K)x^{2-d}}, & \text{for } d < 2, \text{ SRO}, \\ \left(\frac{a}{x}\right)^\eta, & \text{for } d = 2, \text{ QLRO}, \end{cases} \tag{31}$$

with the order parameter correlator asymptotically approaching a nonzero value for $d > 2$, but vanishing for $d \leq 2$. Noting that by the cluster decomposition property

$$\langle\psi^*(\mathbf{x})\psi(0)\rangle \stackrel{x\to\infty}{\to} |\langle\psi(\mathbf{x})\rangle|^2 = |\Psi_0|^2 \tag{32}$$

this correlator asymptotes to the square of the Landau order parameter Ψ_0. This then shows that at an arbitrary small temperature the order parameter vanishes for $d \leq 2$, as dictated by the Hohenberg-Mermin-Wagner-Coleman theorem[18]. We then say that for the XY (as well as more generally for the $O(N)$) model, the lower-critical dimension for the stability of the phase is $d = 2$, i.e.,

$$d_{lc} = 2, \text{ for O(N) model.} \tag{33}$$

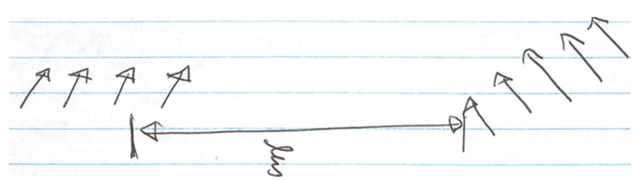

FIG. 3: Characteristic thermal correlation length ξ_T beyond which rms fluctuations are comparable to the extent of the order and thus the ordered state is destroyed.

As illustrated in Fig.3, for $d \leq d_{lc} = 2$ we can then define a finite correlation length, ξ_T beyond which the fluctuations are large, i.e., of the order of the size of the order parameter in O(N) model or equivalently for the XY model $\phi_{rms} \sim O(2\pi)$. Imposing this condition of $S^2_{rms} = S^2_0$ in (25) we find,

$$\xi_T \approx \begin{cases} \left(\frac{K}{k_B T}\right)^{1/(2-d)}, & \text{for } d < 2, \\ a e^{K/k_B T}, & \text{for } d = 2, \end{cases}, \tag{34}$$

For $d > 2$, ordered state is stable and is said to exhibit the so-called *long-range order* (LRO). For $d < 2$, the correlator falls off (stretched-) exponentially and thus the state is unstable to fluctuations, characterized by *short-range order* (SRO). In the marginal dimension of $d = 2$, long-range order is absent, the Landau order parameter still vanishes, but the correlations fall off slower than exponentials of the fully disordered phase, as a power-law, referred to as the *quasi-long-range order* (QLRO), characterized by temperature-dependent exponent,

$$\eta = \frac{k_B T}{2\pi K}. \tag{35}$$

This was first observed by Peierls and Landau in mid 1930s[17]. It was eventually formulated into a full theory of the so-called Kosterlitz-Thouless transition[19, 20], with many extensions[21], that are the first example of a topological (non-symmetry breaking) phase transitions, that is not characterized by a Landau order parameter. These type of transitions are said to be of non-Landau type and have no direct description in terms of a Ginzburg-Landau theory.

3. Physical interpretation

A complementary (to rms fluctuations) analysis of the stability of the ordered state can be done by estimating the energy of low-energy Goldstone mode excitations. In a state that spontaneously breaks a continuous symmetry, e.g., an XY ferromagnet, the excitation that destroys the ordered state are smooth spin-waves of overturned spins at wavelength of the size of the system L, illustrated in Fig.4

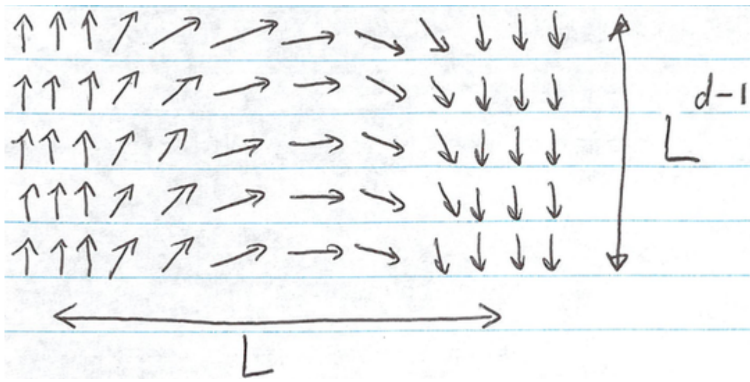

FIG. 4: Depiction of low-energy thermal spin-wave fluctuations in an XY ferromagnet, a superfluid and more generally an O(N) FM. Because it is a continuous symmetry that is broken, the lowest energy distortion of the ordered state can spread smoothly across the size of the system, L and can thus vanish (or more importantly, becomes much smaller than $k_B T$) in the thermodynamic limit for low dimension d.

The corresponding energy is easily estimated from (23) and is given by

$$E_{excitSW} = \frac{1}{2}K\int(\nabla\phi)^2 d^d x, \tag{36}$$

$$\simeq \frac{1}{2}KLL^{d-1}\left(\frac{2\pi}{L}\right)^2, \tag{37}$$

$$\simeq KL^{d-2} \overset{L\to\infty}{\simeq} \begin{cases} \to 0 \ll k_B T, & \text{for } d < 2, \\ \to \infty \gg k_B T, & \text{for } d > 2. \end{cases} \tag{38}$$

Thus, consistent with our finding based on rms fluctuations, above, for $d > 2$ low-energy Goldstone mode excitations diverge and will therefore not appear in the thermodynamic limit at low T. On the other hand, for $d < 2$ excitation energy vanishes in the thermodynamic limit and thus will appear even at low temperatures. The energy of excitations will match $k_B T$ at the scale of the thermal correlation length, that, as is easy to check is given by ξ_T in (34), above. Again, consistent with our finding, this predicts the lower critical dimension to be $d_{lc} = 2$.

It is instructive to compare this analysis of the $O(N > 1)$ model with that of the $N = 1$ Ising model. The qualitative difference is made apparent by comparing the Fig.5 for the Ising model to that for the XY model, Fig.4. The key distinction is the absence of Goldstone modes in the Ising model. As a result, the low-energy excitations are domain wall of finite-width $\xi_0 \sim \sqrt{K/|t|}$, that connect two Z_2 degenerate states, corresponding to a

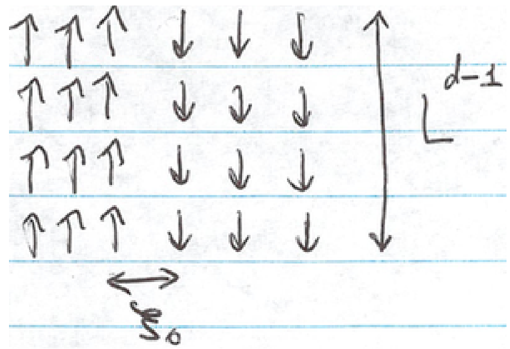

FIG. 5: Depiction of low-energy thermal fluctuations in an Ising ferromagnet. Because it breaks a discrete Z_2 symmetry the fluctuation is a domain wall confined to a finite width $\xi_0 \approx \sqrt{K/|t|}$.

gapped field excitation (microscopically a domain of flipped spins) with energy

$$E_{excitDW} = \frac{1}{2}K\int(\nabla\phi)^2 d^d x, \tag{39}$$

$$\simeq \frac{1}{2}K\xi_0 L^{d-1}\left(\frac{2\pi}{\xi_0}\right)^2, \tag{40}$$

$$\simeq K\xi^{-1}L^{d-1} = \sqrt{K|t|}L^{d-1} \overset{L\to\infty}{\simeq} \begin{cases} \to 0 \ll k_B T, & \text{for } d<1, \\ \to \infty \gg k_B T, & \text{for } d>1. \end{cases} \tag{41}$$

Since this domain wall energy is finite for $d = 1$ and vanishes for $d < 1$, the lower-critical dimension for the Ising model is $d_{lc} = 1$ as argued in earlier lectures.

III. GENERALIZED ELASTICITY

As we have seen and discussed above, low-energy fluctuations of states that spontaneously break a continuous symmetry are Goldstone modes. Because by its very nature the state's energy must vanish for a spatially uniform Goldstone mode (which just transforms the state to its energetically equivalent symmetry broken one), the governing low-energy Hamiltonian is a low-order power-series in gradients of the Goldstone modes, with harmonic approximation typically (but not always; see below) sufficient. While details will very depending on the physical system, symmetry broken, etc., the gradient expansion property of such Goldstone-modes Hamiltonians is generic. Because one can visualize this energetics in a mechanical analogy of a spatial distortion of Goldstone modes, we refer to such Hamiltonians are *generalized elasticity* models. Below we discuss a variety of such systems and their corresponding generalized elasticity.

A. Crystals

An appropriate natural example is that of the elasticity of a crystal, whose stability we will explore, finding an important realization of the Hohenberg-Mermin-Wagner-Coleman theorem[18]. A crystal a state characterized by a density $n(\mathbf{x})$ that is periodic in all d dimensions Complementary to this, is a reciprocal, momentum (Fourier) space description, in which the distinguishing feature of a crystal state is the appearance of nontrivial Fourier coefficients $n_{\mathbf{G}}$ of the number density,

$$n(\mathbf{x}) = \sum_{\mathbf{G}} n_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{x}}, \tag{42}$$

where $\mathbf{G}$ span the reciprocal lattice of the crystal. $n_{\mathbf{G}}$ are thus a set of order parameters for crystallization of the liquid. A crystal spontaneously breaks d translational and $\frac{1}{2}d(d-1)$ rotational symmetries of the isotropic and homogeneous liquid state.

The corresponding Landau Hamiltonian that describes the L-Cr transition must be translationally invariant and is thus given by

$$\mathcal{H}[n_{\mathbf{G}}] = \frac{1}{2}t|n_{\mathbf{G}}|^2 - \frac{1}{3}w \sum_{\{Gv_i\}}' n_{\mathbf{G}_1} n_{\mathbf{G}_2} n_{\mathbf{G}_3} + \frac{1}{4}u \sum_{\{Gv_i\}}' n_{\mathbf{G}_1} n_{\mathbf{G}_2} n_{\mathbf{G}_3} n_{\mathbf{G}_4} + \ldots, \tag{43}$$

where the sums can be limited to a set of fundamental reciprocal lattice vectors, and prime denotes a constraint, $\sum_i \mathbf{G}_i = 0$ of momentum conservation. Because of the cubic invariant, allowed in three dimensions (but not in 2d, nor in a superfluid Hamiltonian), a generic crystallization transition is first-order.

The crystal's complex order parameter $n_{\mathbf{G}} = e^{i\mathbf{G}\cdot\mathbf{u}(\mathbf{x})}$ gives the complex amplitudes of the reciprocal lattice $\mathbf{G}$ appearing in x-ray scattering, with $\mathbf{u}(\mathbf{x})$ the lattice phonon displacement field, which is crystal's Goldstone modes corresponding to spontaneous breaking of translational symmetries. In analogy to a superfluid order parameter $\Psi(\mathbf{x}) = e^{i\phi(\mathbf{x})}$ there is a natural correspondence between the phases $\mathbf{G}\cdot\mathbf{u}(\mathbf{x})$ and $\phi(\mathbf{x})$.

Because $\mathbf{u}(\mathbf{x})$ is a d-component vector, the corresponding Goldstone mode elastic Hamiltonian is a bit richer, written in terms of the elastic symmetric strain tensor

$$u_{ij} = \frac{1}{2}(\partial_i \mathbf{R}\cdot\partial_j \mathbf{R} - \delta_{ij}), \tag{44}$$

$$= \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i \mathbf{u}\cdot\partial_j \mathbf{u}), \tag{45}$$

$$\tag{46}$$

where in the first line we used the fact that the strain is the difference between the distortion-induced metric g_{ij} and the the undistorted metric δ_{ij} and expressed the position of an atom $\mathbf{x}$ that has been moved to $\mathbf{R}(\mathbf{x}) = \mathbf{x}+\mathbf{u}(\mathbf{x})$ in a deformed crystal in terms of the phonon field $\mathbf{u}(\mathbf{x})$. To quadratic order in the strains and the phonons, the governing elastic Hamiltonian is given by

$$H_{el}[\mathbf{u}(\mathbf{x})] \;=\; \frac{1}{2}\int d^d x C_{ij,kl}\; u_{ij}u_{kl} = \frac{1}{2}\int d^d x C_{ij,kl}\;(\partial_i u_j)(\partial_k u_l), \tag{47}$$

where $C_{ij,kl}$ is the tensor encoding crystal's symmetry and elastic constants, symmetric in i, j and in k, l and in $(ij), (kl)$. Except for the multi-component index on the phonon phase u_i, $H_{el}[\mathbf{u}(\mathbf{x})]$ is of the same "gradient elasticity" structure as the XY model of a superfluid, (20). It can be shown[7] that the translational and rotational invariances of the underlying liquid from which a crystal spontaneously emerges guarantee that the strain is a *symmetric* tensor of the gradients of the phonon displacements $\mathbf{u}(\mathbf{x})$, with the antisymmetric part of $\partial_i u_j$ corresponding to bond rotations that cannot appear in the Hamiltonian. While the above elastic energy is quadratic in u_{ij}, the strain tensor itself is a nonlinear function of the phonon field u_i, and is responsible for thermal expansion of a crystal upon warming. While these nonlinearities can be neglected in a conventional crystal above, their counter-parts in smectic liquid crystals and in polymerized membranes will play a crucial role.

In the case of an isotropic (noncrystalline) solid (which also applies to some crystals such as e.g., the hexagonal lattice in 2d) the elasticity is characterized by only two elastic constants traditionally called the Lame' constants, μ and λ, with

$$C_{ij,kl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{48}$$

In this notation the shear modulus $G = \mu$, the bulk modulus $B = \lambda + 2\mu/d$, the inverse of the compressibility κ, Poisson ratio $\nu = \frac{\lambda}{2(\mu+\lambda)}$, Young's modulus $E = \frac{\mu(2\mu+d\lambda)}{\mu+\lambda}$, and the elastic Hamiltonian, expressed in terms of the phonon fields $u_i(\mathbf{x})$ is given by

$$\begin{aligned} H_{el} \;&=\; \int_{\mathbf{x}}\left[\mu u_{ij}u_{ij} + \frac{1}{2}\lambda u_{ii}u_{jj}\right], && (49)\\ &\approx\; \frac{1}{2}\int_{\mathbf{x}} u_i(\mathbf{x})\left[\mu(-\nabla^2)P^T_{ij} + (2\mu+\lambda)(-\nabla^2)P^L_{ij}\right]u_j(\mathbf{x}), && (50)\\ &=\; \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}u_i(-\mathbf{k})\left[\mu k^2 P^T_{ij}(\mathbf{k}) + (2\mu+\lambda)k^2 P^L_{ij}(\mathbf{k})\right]u_j(\mathbf{k}), && (51)\\ &\equiv\; \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}u_i(-\mathbf{k})D_{ij}(\mathbf{k})u_j(\mathbf{k}), && (52)\end{aligned}$$

where we decoupled the Hamiltonian in terms of the Fourier modes $\mathbf{k}$, and defined transverse and longitudinal projection operators, transverse and along $\mathbf{k}$, respectively,

$$P_{ij}^T(\mathbf{k}) \;=\; \delta_{ij} - \frac{k_i k_j}{k^2}, \quad P_{ij}^L(\mathbf{k}) = \frac{k_i k_j}{k^2}. \tag{53}$$

Because the projection operators are independent, the inverse of the dynamic matrix $D_{ij}(\mathbf{k})$ (that we will need for study of thermodynamics and correlation functions) is easily obtained

$$V^{-1}\langle u_i(-\mathbf{k})(\mathbf{k})u_j(\mathbf{k})\rangle = k_B T D_{ij}^{-1}(\mathbf{k}) = \frac{1}{\mu k^2}P_{ij}^T(\mathbf{k}) + \frac{1}{(2\mu+\lambda)k^2}P_{ij}^L(\mathbf{k}), \tag{54}$$

as can be straightforwardly verified using $P_{ik}^T P_{kj}^T = P_{ij}^T$, $P_{ik}^L P_{kj}^L = P_{ij}^L$, $P_{ik}^T P_{kj}^L = 0$.

Because the propagator scales as $1/k^2$, independent of the index complexity of crystalline elasticity with vector phonons, the qualitative behavior of the correlation functions of $u_i(\mathbf{x})$ and of the order parameter $\rho_{\mathbf{G}}(\mathbf{x})$ mimic those of the scalar XY model (superfluid) in (25), (29) and (31). Namely, for $x \gg a$ we find

$$u_{rms}^2 \;=\; \langle \mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\rangle = k_B T\int_{L^{-1}}^{a^{-1}}\frac{d^d k}{(2\pi)^d}D_{ii}^{-1}(\mathbf{k}), \tag{55}$$

$$= k_B T\left(\frac{d-1}{\mu}+\frac{1}{2\mu+\lambda}\right)\int_{L^{-1}}^{a^{-1}}\frac{d^d k}{(2\pi)^d}\frac{1}{k^2} \sim \frac{k_B T}{\tilde{\mu}}\begin{cases}\frac{1}{a^{d-2}}, & \text{for } d>2,\\ L^{2-d}, & \text{for } d<2,\\ \ln(L/a), & \text{for } d=2,\end{cases} \tag{56}$$

that in the thermodynamic limit diverges for $d \le 2$, and $\tilde{\mu}$ was defined by the above equality and will be used as the effective elastic coupling.

The spatial phonon correlation function is given by

$$C_{ij}(\mathbf{x}) \;=\; \frac{1}{2}\langle (u_i(\mathbf{x})-u_i(0))(u_j(\mathbf{x})-u_j(0))\rangle = \langle u_i(0)u_j(0) - u_i(\mathbf{x})u_j(0)\rangle, \tag{57}$$

$$= k_B T\int \frac{d^d k}{(2\pi)^d}\left[\frac{1}{\mu}P_{ij}^T(\mathbf{k}) + \frac{1}{(2\mu+\lambda)}P_{ij}^L(\mathbf{k})\right]\frac{1}{k^2}\left(1-e^{i\mathbf{k}\cdot\mathbf{x}}\right), \tag{58}$$

$$\sim \frac{k_B T}{\tilde{\mu}_{ij}}\begin{cases}\frac{1}{a^{d-2}}, & \text{for } d>2,\\ x^{2-d}, & \text{for } d<2,\\ \ln(x/a), & \text{for } d=2,\end{cases} \tag{59}$$

Although anisotropic, as in a scalar superfluid, $C_{ij}(\mathbf{x})$ grows quadratically with x/a for $x \ll a$, and then asymptotes to above limiting forms. $\tilde{\mu}_{ij}$ is the effective elastic constant that depends on the axes choice i, j. Thus, consistent with (56), the average difference in phonon fluctuations $\mathbf{u}(\mathbf{x})$ between two points separated by $x \gg a$ is finite (and small for $k_B T/\mu \ll 1$, $k_B T/(2\mu + \lambda) \ll 1$) for $d > 2$, but diverge for $d \leq 2$.

Using Wick's theorem, valid for the Gaussian Goldstone mode $\mathbf{u}(\mathbf{x})$, at large x we find

$$\langle n^*_{\mathbf{G}}(\mathbf{x}) n_{\mathbf{G}}(0)\rangle \approx \langle e^{i\mathbf{G}\cdot(\mathbf{u}(\mathbf{x})-\mathbf{u}(0))}\rangle \sim e^{-G_i G_j C_{ij}(\mathbf{x})}, \tag{60}$$

$$\sim \begin{cases} e^{-(k_B T G^2/\tilde{\mu}) a^{2-d}}, & \text{for } d > 2, \text{ LRO}, \\ e^{-(k_B T G^2/\tilde{\mu}) x^{2-d}}, & \text{for } d < 2, \text{ SRO}, \\ \left(\frac{a}{x}\right)^{\eta_G}, & \text{for } d = 2, \text{ QLRO}, \end{cases} \tag{61}$$

where

$$\eta_G = \frac{k_B T G^2}{2\pi\tilde{\mu}}. \tag{62}$$

That is a $d > 2$ dimensional crystal is stable to small thermal fluctuations (melting at high T, that can be determined by the Lindemann criterion, $k_B T_m \approx \tilde{\mu} a^d$), short-range ordered i.e., melts into a fluid for $d < 2$ and exhibits QLRO for $d = 2$. Thus, a two-dimensional crystal is strictly speaking unstable to thermal fluctuations, and therefore, in the presence of fluctuations is strictly speaking not periodic.

A standard measure of this is via a static structure function (measured in x-ray and neutron scattering), a Fourier transform of the two-point density correlator function

$$S(\mathbf{q}) = \frac{1}{N}\sum_{n,m}\langle e^{-i\mathbf{q}\cdot(\mathbf{R}(\mathbf{x}_n)-\mathbf{R}(\mathbf{x}_m))}\rangle, \tag{63}$$

$$= \langle n_{-\mathbf{q}} n_{\mathbf{q}}\rangle \tag{64}$$

where $\mathbf{R}(\mathbf{x}_n)$ labels positions of n-th particle and N is the total number of particles. In the absence of fluctuations atomic positions line on a periodic lattice $\mathbf{R}(\mathbf{x}_n) = \mathbf{x}_n$, and, utilizing Poisson summation formula we find that ideal crystals structure function is given by

$$S_{T=0}(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{x}_n,\mathbf{x}_m} e^{-i\mathbf{q}\cdot(\mathbf{x}_n-\mathbf{x}_m)}, \tag{65}$$

$$= \frac{1}{N}\sum_{\mathbf{x}_n}\sum_{\mathbf{x}_n-\mathbf{x}_m} e^{-i\mathbf{q}\cdot(\mathbf{x}_n-\mathbf{x}_m)}, \tag{66}$$

$$= \sum_{\mathbf{G}_p} N\delta_{\mathbf{q},\mathbf{G}_p} = (2\pi)^d v \sum_{\mathbf{G}_p}\delta^d(\mathbf{q}-\mathbf{G}_p). \tag{67}$$

This is an iconic result of lattice Bragg (δ-function) peaks appearing at the reciprocal lattice points $\mathbf{G}_p$ (defined by $e^{i\mathbf{G}_p\cdot\mathbf{x}_n} = 1$), characterizing a perfect a crystalline order. It is this key property that makes scattering such an effective tool for analyzing the crystal structure and deviation from it. The latter comes from lattice defects and fluctuations that are incorporated into phonons $\mathbf{u}(x_n)$.

To incorporate phonon fluctuations, we take $\mathbf{R}(\mathbf{x}_n) = \mathbf{x}_n + \mathbf{u}(\mathbf{x}_n)$ and repeat above calculation,

$$S(\mathbf{q}) = \frac{1}{N}\sum_{\mathbf{x}_n,\mathbf{x}_m} e^{-i\mathbf{q}\cdot(\mathbf{x}_n-\mathbf{x}_m)}\langle e^{-i\mathbf{G}\cdot(\mathbf{u}(\mathbf{x}_n)-\mathbf{u}(\mathbf{x}_m))}\rangle = \sum_{\mathbf{x}_n} e^{-i\mathbf{q}\cdot\mathbf{x}_n}\langle e^{i\mathbf{G}\cdot(\mathbf{u}(\mathbf{x}_n)-\mathbf{u}(0))}\rangle, \tag{68}$$

$$\sim \sum_{\mathbf{x}_n} e^{-i\mathbf{q}\cdot\mathbf{x}_n} e^{-G_iG_jC_{ij}(\mathbf{x}_n)}, \tag{69}$$

$$\sim \begin{cases} \sum_{\mathbf{G}_p} \delta^d(\mathbf{q}-\mathbf{G}_p) e^{-(k_BTG^2/\tilde{\mu})a^{2-d}}, & \text{for } d>2, \text{ LRO}, \\ \sum_{\mathbf{G}_p} \frac{1}{(\mathbf{q}-\mathbf{G}_p)^2+\xi_T^{-2}}, & \text{for } d<2, \text{ SRO}, \\ \sum_{\mathbf{G}_p} \frac{1}{|\mathbf{q}-\mathbf{G}_p|^{2-\eta_G}}, & \text{for } d=2, \text{ QLRO}. \end{cases} \tag{70}$$

This shows that even with phonon fluctuations for $d > 2$, inside a crystalline phase, Bragg peaks remain, though appear with a Debye-Waller factor amplitude, reduced by *finite* phonon fluctuations. In contrast for $d < 2$ the δ-function Bragg peaks are rounded into an analytic function, well approximated by a Lorentzian, whose width is controlled by the thermal correlation length ξ_T. Exotic QLR order appears in the marginal, two dimensions, displaying divergent *power-law* peaks at Bragg vectors $\mathbf{G}_p$, with temperature-dependent exponent η_G.

B. Nematic liquid crystal

As we discussed in the earlier lectures, liquid crystals are fascinating systems of anisotropic constituents (typically rode- or disk-like, though there are quantum liquid crystals of even point-like electrons, driven by strong frustrated interactions), that exhibit a rich variety of phases intermediate between a fully-disordered isotropic fluid and fully-ordered crystalline solid. Classical liquid crystals[7, 12] are typically driven by competing orientational and positional entropies, with some most common phases illustrated in Fig.6 Liquid crystals offer a playground for exploration of rich variety of quite interesting phase transitions and associated ordered phases.

The simplest and least ordered of this is the uniaxial nematic liquid crystal that breaks $O(3)$ rotational symmetry of an isotropic fluid down to $O(2) \times Z_2$ symmetry, picking out a uniaxial axis in the 3d space. As we discussed in earlier lectures such uniaxial nematic state

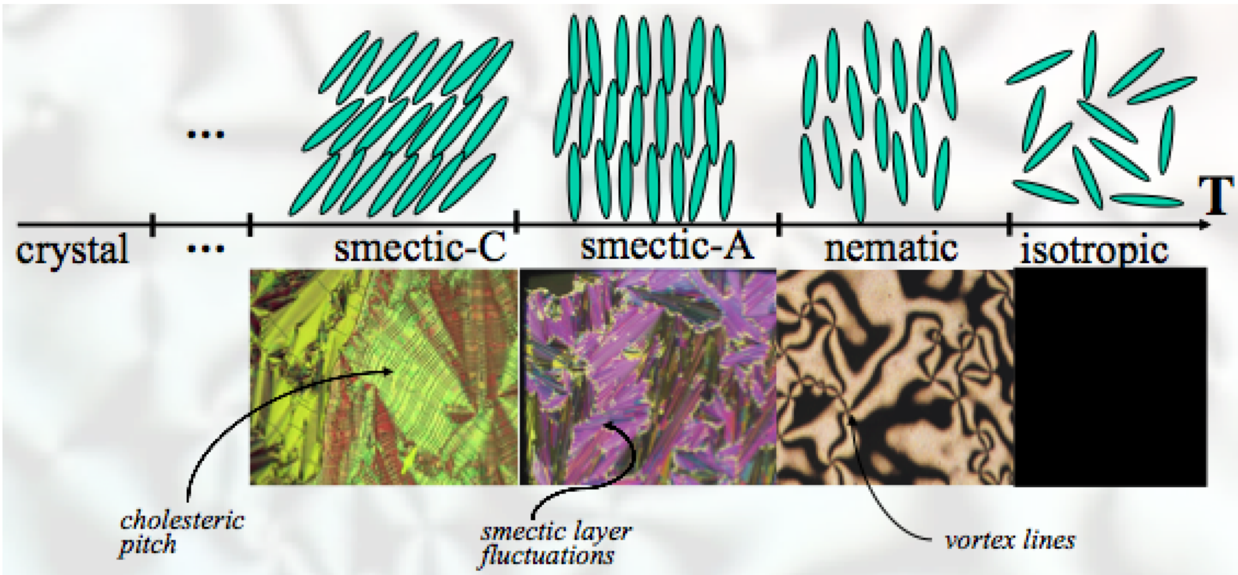

FIG. 6: Most ubiquitous nematic (orientationally ordered uniaxial fluid), smectic-A and smectic-C (one-dimensional density wave with, respectively isotropic and polar in-plane fluid orders) liquid crystal phases and their associated textures in cross-polarized microscopy (N.A. Clark laboratory).

is characterized by a symmetric traceless tensor

$$Q_{ij} = S\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) \tag{71}$$

with a uniaxial anisotropy along $\hat{n}$. We note that this quadrapole (rather than polar) order parameter is quadratic in $\hat{n}$, "living" in the $RP_2 = O(3)/O(2)/Z_2$ (a sphere with antipodal points identified). The corresponding Landau Hamiltonian density must be rotationally invariant, with all the indices contracted, and in 3d is given by

$$\mathcal{H}[Q_{ij}] = \frac{1}{2}tQ_{ij}Q_{ji} - \frac{1}{3}wQ_{ij}Q_{jk}Q_{ki} + \frac{1}{4}u(Q_{ij}Q_{ji})^2, \tag{72}$$

$$= \frac{1}{2}t\mathrm{Tr}(\mathbf{Q}^2) - \frac{1}{3}w\mathrm{Tr}(\mathbf{Q}^3) + \frac{1}{4}u[\mathrm{Tr}(\mathbf{Q}^2)]^2, \tag{73}$$

$$= \frac{1}{2}\tilde{t}S^2 - \frac{1}{3}\tilde{w}S^3 + \frac{1}{4}\tilde{u}S^4. \tag{74}$$

The appearance of the cubic invariant guarantees that the 3d Isotropic-Nematic transition is generically first order.

With our focus here on the ordered uniaxial nematic state, adding generic gradient terms to $\mathcal{H}[Q_{\alpha\beta}]$, taking S to be a constant and focusing on $\hat{n}$, we obtain the Goldstone modes Hamiltonian, the so-called Frank-Oseen elastic Hamiltonian density,

$$\mathcal{H}_F = \frac{1}{2}K_s(\nabla\cdot\hat{n})^2 + \frac{1}{2}K_b(\hat{n}\times\nabla\times\hat{n})^2 + \frac{1}{2}K_t(\hat{n}\cdot\nabla\times\hat{n})^2, \tag{75}$$

where the three terms and associated elastic moduli correspond to splay, bend and twist deformations, illustrated in Fig.??.

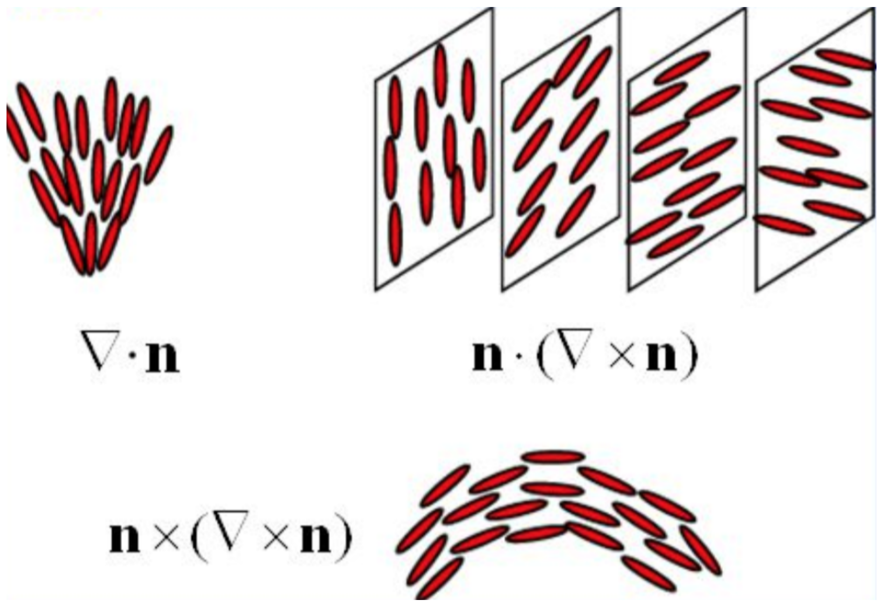

FIG. 7: Three distinct type of distortions of a uniaxial nematic liquid crystal, splay, bend, twist.

It is crucial to note that because $\hat{n}$ is a vector in real physical space, in contrast to quantum mechanical spin (without spin-orbit interaction), here "space-spin" coupling (analog of spin-orbit interaction) allows a contraction of director $\hat{n}$ and spatial indices.

I note that for equal elastic constants, $K = K_s = K_b = K_t$, up to a boundary term, above Frank elasticity reduces to that of the nonlinear σ-model,

$$\mathcal{H}_F^{(I)} = \frac{1}{2}K(\partial_i \hat{n}_j)^2, \tag{76}$$

utilizing an identity,

$$(\partial_i \hat{n}_j)^2 = (\nabla \cdot \hat{n})^2 + (\nabla \times \hat{n})^2 + \nabla \cdot \left[(\hat{n} \cdot \nabla)\hat{n} - \hat{n}\nabla \cdot \hat{n}\right] \tag{77}$$

For a chiral nematic liquid crystal (one with chiral molecules, lacking mirror symmetry, most commonly utilized for lc display applications), also referred to as a cholesteric, an additional chiral term is allowed, giving:

$$\mathcal{H}_F^* = \frac{1}{2}K_s(\nabla \cdot \hat{n})^2 + \frac{1}{2}K_b(\hat{n} \times \nabla \times \hat{n})^2 + \frac{1}{2}K_t(\hat{n} \cdot \nabla \times \hat{n} + q_0)^2, \tag{78}$$

For small q_0 the ground state of a cholosteric is actually a uniaxial spiral with pitch $2pi/q_0$, utilized in most display applications.

Focusing on the simplest achiral Frank elastic Hamiltonian it is clear (particular in the equal elastic constants approximation, (76)) that at low temperatures, it is closely related to the $O(3)$ nonlinear σ-model. It thus is governed by the same $k_BT/(Kk^2)$ correlator, with the uniaxial nematic stability constrained by the standard Hohenberg-Mermin-Wagner-Coleman theorem[18] and the lower-critical dimension $d_{lc} = 2$.

C. Smectic liquid crystal

Another important liquid crystal phase is a smectic, with two most prominent varieties, smectic-A and tilted smectic-C illustrated in Fig.6. Smectic phases (that often occurs in conventional liquid crystals, but can be realized in many other physical systems that exhibit periodic stripe-like order) is a one-dimensional crystal, in that its density is modulations along a single direction and is uniform along the $d-1$ transverse axis. Focusing on the simplest smectic-A phase (whose in-plane order is isotropic $d-1$-dimensional liquids), the elastic energy and fluctuations of this layered state are described by a single Goldstone mode, a scalar phonon $u(\mathbf{x})$, corresponding to deformations along the periodic axis that (without loss of generality) we will take to be $\hat{\mathbf{z}}$.

1. *Smectic nonlinear elasticity*

The smectic elastic Goldstone mode Hamiltonian can be derived from a number of approaches, but can also be simply "guessed" (deduced) based on its underlying layered symmetry, and is given by

$$H_{sm} = \int dz d^{d-1}x \left[\frac{1}{2}K(\nabla_\perp^2 u)^2 + \frac{1}{2}B\big(\partial_z u + \frac{1}{2}(\nabla u)^2\big)^2\right]. \tag{79}$$

This smectic-A Hamiltonian can also be derived by starting with a isotropic, homogeneous liquid and spontaneously developing a unaxial density modulation. I begin with a generic energy density functional that captures system's tendency to develop a unidirectional wave at wavevector $\mathbf{q}_0$, whose magnitude q_0 is fixed but not its direction,

$$\mathcal{H}_{sm} = \frac{1}{2}J\left[(\nabla^2\rho)^2 - 2q_0^2(\nabla\rho)^2\right] + \frac{1}{2}t\rho^2 - w\rho^3 + v\rho^4 + \ldots, \tag{80}$$

where J, q_0, t, w, v are parameters. From the first term, clearly dominant fluctuations are at a finite wavevector with magnitude q_0. Thus let's focus on the density at a finite wavevector

$\mathbf{q}$ that for now we will take to be unrelated to q_0

$$\rho(\mathbf{x}) = \mathrm{Re}\left[n_q(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\right], \tag{81}$$

where $\rho_q(\mathbf{x})$ is a complex scalar and Re is a real part. Without loss of generality we take n_q have a (constant) magnitude ρ_0 and phase $qu(\mathbf{x})$

$$n_q(\mathbf{x}) = \rho_0 e^{iqu}. \tag{82}$$

Clearly $u(\mathbf{x})$ is just a phonon displacement along $\mathbf{q}$. Gradients of ρ are easy to workout

$$\nabla\rho = \rho_0\mathrm{Re}\left[i(\mathbf{q}+q\nabla u)e^{i(\mathbf{q}\cdot\mathbf{x}+qu)}\right], \tag{83}$$

$$\nabla^2\rho = \rho_0\mathrm{Re}\left[\left\{-(\mathbf{q}+q\nabla u)^2 + iq\nabla^2 u\right\}e^{i(\mathbf{q}\cdot\mathbf{x}+qu)}\right]. \tag{84}$$

Substituting this form of ρ and its gradients into $\mathcal{H}_{sm}$ we find

$$\mathcal{H}_{sm} = \frac{1}{4}J\rho_0^2\left[(\mathbf{q}+q\nabla u)^4 + q^2(\nabla^2 u)^2 - 2q_0^2(\mathbf{q}+q\nabla u)^2\right] + \frac{1}{4}t\rho_0^2 + \frac{1}{4}v\rho_0^4 + \ldots, \tag{85}$$

$$= \frac{1}{4}J\rho_0^2\left[\left((\mathbf{q}+q\nabla u)^2 - q^2\right)^2 + q^2(\nabla^2 u)^2 + 2(q^2-q_0^2)(\mathbf{q}+q\nabla u)^2\right] + \frac{1}{4}t\rho_0^2 + \ldots, \tag{86}$$

$$= J\rho_0^2\left[\frac{1}{4}q^2(\nabla^2 u)^2 + \left(q\mathbf{q}\cdot\nabla u + \frac{1}{2}q^2(\nabla u)^2\right)^2 + 4(q^2-q_0^2)\left(q\mathbf{q}\cdot\nabla u + \frac{1}{2}q^2(\nabla u)^2\right)\right] + \ldots, \tag{87}$$

where we dropped constant parts as well as fast oscillating pieces as they will average away after spatial integration of the above energy density. Note that then only even parts in ρ_0 appear. We observe that (as discussed on general grounds above) linear gradient elasticity in u only appears for gradients *along* $\mathbf{q}$, namely $\mathbf{q}\cdot\nabla$, with elasticity transverse to $\mathbf{q}$ starting with a Laplacian type. Secondly the elastic energy is an expansion in a rotationally-invariant strain tensor combination

$$u_{qq} = \hat{\mathbf{q}}\cdot\nabla u + \frac{1}{2}(\nabla u)^2 \tag{88}$$

whose nonlinearities in u ensure that it is fully rotationally invariant even for large rotations. Thirdly, the last term vanishes when $|\mathbf{q}|$ is picked to equal q_0.

Looking ahead, as one includes effects of fluctuations, the "bare" condition $q = q_0$ needs to be adjusted so as to eliminate the linear term in u_{qq} order by order, which amounts to an expansion in the nonlinear strain u_{qq} around the correct (fluctuation-corrected) ground state. Finally, we note that the relation between the Laplacian (first) term and the gradient (second) term is not generic and can be relaxed to have distinct elastic constants, as can be seen if higher order gradient terms are included in the original energy density, Eq. (80).

Choosing the coordinate system such that $\hat{\mathbf{z}}$ is aligned along $\mathbf{q}$, we find that for $q = q_0$, Eq. (87) reduces to a more standard elastic energy

$$\mathcal{H}_{sm} = \frac{1}{2}K(\nabla^2 u)^2 + \frac{1}{2}B\big(\partial_z u + \frac{1}{2}(\nabla u)^2\big)^2, \tag{89}$$

familiar from studies of smectic liquid crystals and with all the fascinating consequences[**?**]

To see that the nonlinear elastic strain u_{zz} is rotationally invariant note that under rotation of $\mathbf{q}$ ($q_0\hat{\mathbf{z}} \to \mathbf{q} = q_0(\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{x})$), gives $u(\mathbf{x}) = z(\cos\theta - 1) + x\sin\theta$ even for a vanishing physical distortion. It can be easily seen that such u displacement gives a vanishing strain and thus is of vanishing energy as required by rotational invariance.

2. *Smectic harmonic fluctuations*

By neglecting the nonlinear terms in u, i.e., approximating H_{sm} by a quadratic part only, the harmonic smectic Hamiltonian is given by

$$\begin{aligned} H^0_{sm} &= \int d^{d-1}x_\perp dz \left[\frac{1}{2}K(\nabla^2_\perp u)^2 + \frac{1}{2}B(\partial_z u)^2.\right], && (90)\\ &= \int \frac{d^{d-1}k_\perp dk_z}{(2\pi)^d}\left(\frac{1}{2}Kk_\perp^4 + \frac{1}{2}Bk_z^2\right)|u_{\mathbf{k}}|^2. && (91)\end{aligned}$$

To assess the role of fluctuations, as above we calculate u_{rms} using Gaussian integrals calculus (or equivalently the equipartition theorem),

$$\begin{aligned} \langle u(\mathbf{x})^2\rangle_0 &= \int_{L_\perp^{-1}}^{a^{-1}}\int_{\infty}^{\infty}\frac{d^{d-1}k_\perp dk_z}{(2\pi)^d}\frac{1}{Bk_z^2 + Kk_\perp^4} && (92)\\ &\approx \begin{cases} \frac{1}{2(3-d)\sqrt{BK}}C_d L^{3-d}, & d < 3,\\ \frac{1}{4\pi\sqrt{BK}}\ln(L/a), & d = 3.\end{cases} && (93)\end{aligned}$$

I note that for $d \leq 3$ and in particular for the case of physical interest, $d = 3$, harmonic fluctuations diverge with system size, suggesting an instability of the truly ordered smectic phase and perhaps a qualitative importance of nonlinearities. Above result shows that the lower-critical dimension for a smectic phase is

$$d_{lc}^{Sm} = 3. \tag{94}$$

The corresponding connected harmonic correlation function

$$C(\mathbf{x}_\perp, z) = \frac{1}{2}\langle[u(\mathbf{x}_\perp, z) - u(0,0)]^2\rangle_0 \tag{95}$$

is also straightforwardly worked out and in 3d is given by the logarithmic Caillé form[**?**]

$$\begin{aligned} C^{3d}(\mathbf{x}_\perp, z) &= \int \frac{d^2k_\perp dk_z}{(2\pi)^3}\frac{1-e^{i\mathbf{k}\cdot\mathbf{x}}}{Kk_\perp^4 + Bk_z^2} \\ &= \frac{1}{4\pi\sqrt{KB}}\left[\ln\left(\frac{x_\perp}{a}\right) - \frac{1}{2}\mathrm{Ei}\left(\frac{-x_\perp^2}{4\lambda|z|}\right)\right], \qquad (96) \\ &\approx \frac{1}{8\pi\sqrt{KB}}\begin{cases} \ln\left(x_\perp^2/a^2\right) &, \ x_\perp \gg \sqrt{\lambda|z|}\ , \\ \ln\left(4\lambda|z|/a^2\right) \ , & x_\perp \ll \sqrt{\lambda|z|}\ , \end{cases} \qquad (97) \end{aligned}$$

where $\mathrm{Ei}(x)$ is the exponential-integral function and $\lambda = \sqrt{K/B}$. As indicated in the last form, in the asymptotic limits of $x_\perp \gg \sqrt{\lambda z}$ and $x_\perp \ll \sqrt{\lambda z}$ this 3d correlation function reduces to a logarithmic growth with x and τ, respectively.

In the case of $d = 2$, physical interest we instead have[27]

$$\begin{aligned} C^{2d}(x_\perp, z) &= \int \frac{dk_\perp dk_z}{(2\pi)^2}\frac{1-e^{ikx_\perp - ik_z z}}{Kk_\perp^4 + Bk_z^2} \\ &= \frac{1}{B}\left[\left(\frac{|z|}{\pi\lambda}\right)^{1/2} e^{-x_\perp^2/(4\lambda|z|)} + \frac{|x_\perp|}{2\lambda}\mathrm{erf}\left(\frac{|x_\perp|}{\sqrt{4\lambda|z|}}\right)\right] \qquad (98) \\ &\approx \frac{1}{B}\begin{cases} (|z|/\pi\lambda)^{1/2} \ , & x_\perp \ll \sqrt{\lambda|z|}\ , \\ |x_\perp|/2\lambda &, \ x_\perp \gg \sqrt{\lambda|z|}\ , \end{cases} \qquad (99) \end{aligned}$$

where $\mathrm{erf}(x)$ is the error function.

Although above analysis suggests that for $d \leq 3$ fluctuations destabilize the perfectly ordered mean-field-like smectic state, this does not necessarily mean that smectic order is fully destroyed. Indeed, as we will see below, although fluctuations lead to large phonon fluctuations, a form of smectic order survives, though qualitatively modified by thermal fluctuations and nonlinearities, to which we now turn.

3. Perturbative analysis of smectic nonlinearities

Thus, given above seemingly divergent critical ground state fluctuations, it is important to examine the effect of nonlinearities in the full Hamiltonian (89). To this end, we first perform an anisotropic "power-counting", a zeroth-order RG, i.e., a dimensional analysis of nonlinearities. Rescaling

$$z = b^{\omega} z', \qquad \mathbf{x}_\perp = b\mathbf{x}'_\perp, \qquad u(\mathbf{x}_\perp, z) = b^{\chi} u'(\mathbf{x}'_\perp, z'), \tag{100}$$

and choosing the anisotropic exponent ω and field exponent χ such that $K(b) = K$ and $B(b) = B$ are invariant under the rescaling (stay at the fixed point under this zeroth order RG), we find that a single dimensionless coupling

$$g = C_{d-1}\Lambda_\perp^{d-3} T \left(\frac{B}{K^3}\right)^{1/2}, \tag{101}$$

$$\approx \frac{T}{2\pi}\left(\frac{B}{K^3}\right)^{1/2}, \tag{102}$$

flows according to $g(b) = b^{3-d} g$, where in the second form we approximated g by its value in 3d.

The importance of smectic nonlinearities

$$\mathcal{H}_{\mathrm{nonlinear}} = -\frac{1}{2} B(\partial_z u)(\nabla u)^2 + \frac{1}{8} B(\nabla u)^4 \tag{103}$$

can also be assessed by a direct perturbation theory.

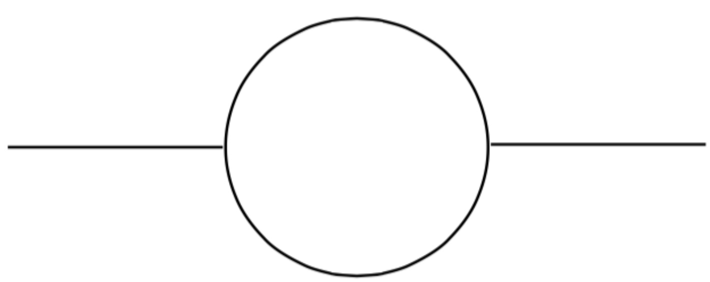

FIG. 8: Feynman graph that renormalizes the elastic moduli K, B of the LO superfluid.

To this end, we use a perturbative expansion in the nonlinear operators (103) to assess the size of their contribution to e.g., the free energy. Following a standard field-theoretic analysis these can be accounted for as corrections to the compressional B and bend K elastic moduli, with the leading contribution to δB, summarized graphically in Fig.8, and given by

$$\begin{aligned}
\delta B &= -\frac{1}{2}TB^2\int_{\mathbf{q}} q_\perp^4 G_u(\mathbf{q})^2 \;, &(104a)\\
&\approx -\frac{1}{2}TB^2\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\int_{L_\perp^{-1}}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\frac{q_\perp^4}{(Kq_\perp^4+Bq_z^2)^2}\;,\\
&\approx -\frac{1}{8}\frac{C_{d-1}T}{3-d}\left(\frac{B}{K^3}\right)^{1/2}L_\perp^{3-d}B = -\frac{1}{8}\frac{g}{3-d}\left(\Lambda L_\perp\right)^{3-d}B\;. &(104b)
\end{aligned}$$

In above, I cutoff the divergent contribution of the long wavelength modes via the infra-red cutoff $q_\perp > 1/L_\perp$ by considering a system of a finite extent $L_\perp$. Clearly the anharmonicity become important when the fluctuation corrections to the elastic constants (e.g., δB above) become comparable to the bare microscopic values. The divergence of this correction as $L_\perp \to \infty$ signals the breakdown of the conventional harmonic elastic theory on length scales longer than a crossover scale $\xi_\perp^{NL}$

$$\xi_\perp^{NL} \approx \begin{cases} \frac{1}{T}\left(\frac{K^3}{B}\right)^{1/2}, & d=2,\\ a e^{\frac{c}{T}\left(\frac{K^3}{B}\right)^{1/2}}, & d=3,\end{cases} \tag{105}$$

which I define here as the value of $L_\perp$ at which $|\delta B(\xi_\perp^{NL})| = B$. Within the approximation of the smectic screening length $\lambda = a$, these nonlinear crossover lengths reduce to the phonon disordering lengths that can be defined by the Lindemann-like criterion from u_{rms}. Clearly, on scales longer than $\xi_{\perp,z}^{NL}$ the perturbative contributions of nonlinearities diverge and therefore cannot be neglected. Their contribution are thus expected to qualitatively modify the predictions of the harmonic approximation above.

4. RG analysis of smectic nonlinearities

We thus conclude from above analysis that $d = 3$ is the upper critical dimension for these smectic nonlinearities. To describe the physics beyond the crossover scales, $\xi_{\perp,z}^{NL}$ – i.e., to make sense of the infra-red divergent perturbation theory found in Eq.104b – requires a renormalization group analysis. This was first performed in the context of conventional liquid crystals and Lifshitz points in a seminal work by Grinstein and Pelcovits (GP)[24], and extended to full momentum-shell RG in the $\epsilon = 3 - d$ expansion, in my own work on FFLO superconductors.[25]; see also a beautiful paper by Golubovic and Wang on a 2d smectic[26].

To this end I integrate (perturbatively in $\mathcal{H}_{\text{nonlinear}}$) short-scale Goldstone modes in an infinitesimal cylindrical shell of wavevectors, $\Lambda e^{-\delta\ell} < q_\perp < \Lambda$ and $-\infty < q_z < \infty$ ($\delta\ell \ll 1$ is infinitesimal). The leading perturbative momentum-shell coarse-graining contributions come from terms found in direct perturbation theory above, but with the system size divergences controlled by the infinitesimal momentum shell. The thermodynamic averages can then be equivalently carried out with an effective coarse-grained Hamiltonian of the same form (**??**), but with all the couplings infinitesimally corrected by the momentum shell. For smectic moduli B and K this gives

$$\delta B \approx -\frac{1}{8} g B \delta\ell, \tag{106a}$$

$$\delta K \approx \frac{1}{16} g K \delta\ell, \tag{106b}$$

where dimensionless coupling g is defined in (102). Consistent with physical intuition, Eqs.(106) show that B is softened and K is stiffened by the nonlinearities in the presence of thermal fluctuations, making the system effectively more isotropic.

For convenience we then rescale the lengths and the remaining long wavelength part of the fields $u^<(\mathbf{x})$ according to $\mathbf{x}_\perp = \mathbf{x}'_\perp e^{\delta\ell}$, $z = z' e^{\omega\delta\ell}$ and $u^<(\mathbf{x}) = e^{\chi\delta\ell} u'(\mathbf{x}')$, so as to restore the ultraviolet cutoff $\Lambda_\perp e^{-\delta\ell}$ back up to $\Lambda_\perp$. The underlying rotational invariance insures that the graphical corrections preserve the rotationally invariant strain operator $\left(\partial_z u + \frac{1}{2}(\nabla_\perp u)^2\right)$, renormalizing it as a whole. It is therefore convenient (but not necessary) to choose the dimensional rescaling that also preserves this form. It is easy to see that this choice leads to

$$\chi = 2 - \omega\ . \tag{107}$$

The leading (one-loop) changes to the effective coarse-grained and rescaled action can then be summarized by differential RG flows

$$\frac{dB(\ell)}{d\ell} = (d + 3 - 3\omega - \frac{1}{8} g(\ell)) B(\ell)\ , \tag{108a}$$

$$\frac{dK(\ell)}{d\ell} = (d - 1 - \omega + \frac{1}{16} g(\ell)) K(\ell)\ . \tag{108b}$$

From these we readily obtain the flow of the dimensionless coupling $g(\ell)$

$$\frac{dg(\ell)}{d\ell} = (3 - d) g - \frac{5}{32} g^2\ , \tag{109}$$

whose flow for $d < 3$ away from the $g = 0$ Gaussian fixed point encodes the long-scale divergences found in the direct perturbation theory above. As summarized in Fig.9 for

$d < 3$ the flow terminates at a nonzero fixed-point coupling $g_* = \frac{32}{5}\epsilon$ (with $\epsilon \equiv 3 - d$), that determines the nontrivial long-scale behavior of the system (see below). As with treatments of critical points[9], but here extending over the whole smectic phase, the RG procedure is quantitatively justified by the proximity to $d = 3$, i.e., smallness of ϵ.

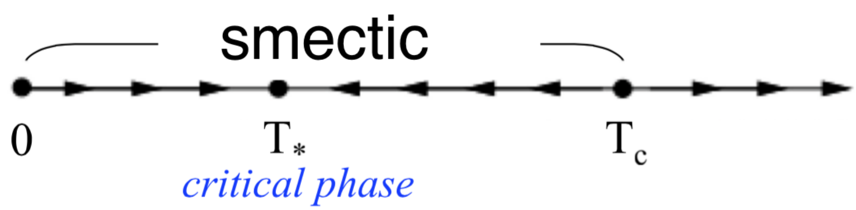

FIG. 9: Renormalization group flow for a smectic state in $d < 3$-dimensions, illustrating that at low T it is a "critical phase", displaying universal power-law phenomenology, controlled by a nontrivial infrared stable fixed point.

5. *Matching analysis: smectic as a critical phase*

I can now use a standard matching calculation to determine the long-scale asymptotic form of the correlation functions on scales beyond $\xi^{NL}_{\perp,z}$. Namely, applying above coarse-graining RG analysis to a computation of correlation functions allows us to relate a correlation function at long length scales of interest to us (that, because of infrared divergences is impossible to compute via a direct perturbation theory) to that at short scales, evaluated with coarse-grained couplings, $B(\ell)$, $K(\ell)$,.... In contrast to the former, the latter is readily computed via a perturbation theory, that, because of shortness of the length scale is convergent. The result of this matching calculation to lowest order gives correlation functions from an effective Gaussian theory

$$G_u(\mathbf{k}) \;\approx\; \frac{T}{B(\mathbf{k})k_z^2 + K(\mathbf{k})k_\perp^4}, \tag{110}$$

with moduli $B(\mathbf{k})$ and $K(\mathbf{k})$ that are singularly wavevector-dependent, latter determined by the solutions $B(\ell)$ and $K(\ell)$ of the RG flow equations (108a) and (108b) with initial conditions set by the "microscopic" values B and K, i.e., on scales much shorter than $\xi^{NL}_{\perp,z}$.

2d analysis: In $d = 2$, at long scales $g(\ell)$ flows to a nontrivial infrared stable fixed point $g_* = 32/5$, and the matching analysis predicts correlation functions characterized by anisotropic wavevector-dependent moduli

$$\begin{aligned} K(\mathbf{k}) &= K\left(k_\perp\xi_\perp^{NL}\right)^{-\eta_K} f_K(k_z\xi_z^{NL}/(k_\perp\xi_\perp^{NL})^\zeta) , && (111a)\\ &\sim k_\perp^{-\eta_K}, \\ B(\mathbf{k}) &= B\left(k_\perp\xi_\perp^{NL}\right)^{\eta_B} f_B(k_z\xi_z^{NL}/(k_\perp\xi_\perp^{NL})^\zeta) , && (111b)\\ &\sim k_\perp^{\eta_B}. \end{aligned}$$

Thus, on scales longer than $\xi_{\perp,z}^{NL}$ these qualitatively modify the real-space correlation function asymptotics of the harmonic analysis above. In Eqs.(111) the universal anomalous exponents are given by

$$\begin{aligned} \eta_B &= \frac{1}{8}g_* = \frac{4}{5}\,\epsilon , \\ &\approx \frac{4}{5} , \quad \text{for } d=2 , && (112a)\\ \eta_K &= \frac{1}{16}g_* = \frac{2}{5}\,\epsilon , \\ &\approx \frac{2}{5} , \quad \text{for } d=2 , && (112b) \end{aligned}$$

determining the $z-\mathbf{x}_\perp$ anisotropy exponent via (110) to be

$$\begin{aligned} \zeta &\equiv 2-(\eta_B+\eta_K)/2 , && (113a)\\ &= \frac{7}{5}, && (113b) \end{aligned}$$

as expected reduced by thermal fluctuations down from its harmonic value of 2. The $\mathbf{k}_\perp - k_z$ dependence of $B(\mathbf{k}), K(\mathbf{k})$ is determined by universal scaling functions, $f_B(x), f_K(x)$ that we will not compute here. The underlying rotational invariance (special to a LO state realized in an isotropic trap) gives an *exact* relation between the two anomalous $\eta_{B,K}$ exponents[**?**]

$$\begin{aligned} 3-d &= \frac{\eta_B}{2}+\frac{3}{2}\eta_K , && (114a)\\ 1 &= \frac{\eta_B}{2}+\frac{3}{2}\eta_K , \quad \text{for } d=2, && (114b) \end{aligned}$$

which is obviously satisfied by the anomalous exponents, Eqs.(112b),(112a), computed here to first order in $\epsilon = 3-d$[**?**].

Thus, we find that a finite temperature 2d smectic is highly nontrivial and qualitatively distinct from its mean-field perfectly periodic form. It is characterized by a universal nonlocal length-scale dependent moduli, Eq. (111). Consequently, on scales beyond $\xi_{\perp,z}^{NL}$ its effective

Goldstone mode fluctuations and the associated correlations are not describable by a local field theory, that is an analytic expansion in local field operators. Instead, in 2d, on length scales beyond $\xi_{\perp,z}^{NL}$ thermal fluctuations of a smectic are controlled by a nontrivial fixed point, characterized by universal anomalous exponents $\eta_{K,B}$ and scaling functions $f_{B,K}(x)$ defined above.

Above we obtained this nontrivial structure from an RG analysis and estimated these exponents within a controlled but approximate ϵ-expansion. Remarkably, in 2d an exact solution of this problem was discovered by Golubovic and Wang[26]. It predicts an anomalous phenomenology in a qualitatively agreement with the RG predictions above, and gives exact exponents

$$\eta_B^{2d} = 1/2, \tag{115a}$$

$$\eta_K^{2d} = 1/2, \tag{115b}$$

$$\zeta^{2d} = 3/2. \tag{115c}$$

3d analysis: In $d = 3$, the nonlinear coupling $g(\ell)$ is marginally irrelevant, flowing to 0 at long scales. Despite this, the marginal flow to the Gaussian fixed point is sufficiently slow (logarithmic in lengths) that (as usual at a marginal dimension[9]) its power-law in ℓ dependence leads to a universal, asymptotically *exact* logarithmic wavevector dependence[24]

$$K(\mathbf{k}_\perp, k_z = 0) \sim K|1 + \frac{5g}{64\pi}\ln(1/k_\perp a)|^{2/5}\ , \tag{116a}$$

$$B(\mathbf{k}_\perp = 0, k_z) \sim B|1 + \frac{5g}{128\pi}\ln(\lambda/k_z a^2)|^{-4/5}. \tag{116b}$$

This translates into smectic order-parameter correlations given by

$$n(z, \mathbf{x}_\perp = 0) = \langle n_{q_0}(\mathbf{x}) n_{q_0}(0)\rangle, \tag{117a}$$

$$\sim e^{-c_1(\ln z)^{6/5}}\cos(q_0 z), \tag{117b}$$

(c_1 a nonuniversal constant) as discovered in the context of conventional smectics by Grinstein and Pelcovits[24]. Although these 3d anomalous effects are less dramatic and likely to be difficult to observe in practice, theoretically they are quite significant as they represent a qualitative breakdown of the mean-field and harmonic descriptions, that respectively ignore interactions and thermal fluctuations.

I conclude by noting that all of the above analysis is predicated the validity of the purely elastic model, that neglects topological defects, such as vortices and dislocations. If these unbind (as they undoubtedly do in 2d at any nonzero temperature[27]), then our above prediction only hold on scales shorter than the separation ξ_v, ξ_d between these defects.

IV. O(N>2) NON-LINEAR σ-MODEL TRANSITION

We now return to the more general case of an $O(N)$ model, specifically focusing on $N > 2$, for which the Goldstone modes are interacting, in contrast to XY ($N = 2$) case studied above.

There are two complementary approaches to analyze the role of nonlinearities from the ordered (FM) state side. One is via a renormalization-group and $\epsilon = d - 2$-expansion[7, 28, 29], complementing our study from the disordered (PM) side in the previous lecture. Complementing this RG treatment is the so-called large-N expansion[10, 31**?**], namely an expansion in the powers of $1/N$, with N the number of spin $\vec{S}$ components, that we take to be large.

A. Large N expansion for the FM-PM transition

We first approach the problem of disordering of the FM state using large-N expansion. This approach rests on the observation that for large number of spin components N, the theory simplifies and in fact reduces to the so-called spherical model akin to, but distinct from a mean-field approximation. The reason for this simplification is that the challenging quartic term $|\vec{S}|^2|\vec{S}|^2$ in the $N \to \infty$ limit self-averages to an effective quadratic nonlinearity $\langle|\vec{S}|^2\rangle|\vec{S}|^2$ due to the central limit theorem; since $|\vec{S}|^2$ is a sum of large N number of random terms, it thus has relative fluctuations that scale as $1/\sqrt{N}$, allowing us to replace it by its average. This makes $N = \infty$ theory exactly solvable since the theory is quadratic with a coefficient $\langle|\vec{S}|^2\rangle$ self-consistently determined.

Alternatively, the simplifying feature of large-N limit can be seen purely diagrammatically by noticing that the renormalization of the quartic vertex is dominated by an infinite subset of bubbles (s-channel) that are all proportional to N and have the feature that they organize into a geometric series and can thus be resummed exactly. I encourage the reader to look back at the RG lectures for the $O(N)$ model and take the large N limit in that analysis, particularly in the β-function and the value of exponents.

As we discussed in the introductory sections, there are two models to describe the transition at hand, one the $O(N)$ Landau-Ginzburg model (ϕ^4-theory) and the nonlinear $O(N)$ σ-model. We will analyze these below, showing that they give consistent results.

1. nonlinear $O(N)$ σ-model: "hard"-spin description

As we discussed earlier, low-energy degrees of freedom of the ordered (FM) phase are fully captured by the nonlinear σ-model (often also referred to as the "hard"-spin description, emphasizing that $|\vec{n}|$ is fixed exactly at 1), with Hamiltonian,

$$H_{n\sigma m}[\hat{n}(\mathbf{x})] = \frac{1}{2}K \int d^d x (\boldsymbol{\nabla}\hat{n})^2, \tag{118}$$

and a nontrivial constraint $|\vec{n}(\mathbf{x})|^2 = 1$, that incorporates interactions in the ordered state, i.e., the fact that Goldstone modes are confined to a curved S_{N-1} (sphere) manifold.

The partition function is then given by (K is measured in units of $k_B T$)

$$Z = \int [d\vec{n}(\mathbf{x})]\delta[|\vec{n}(\mathbf{x})|^2 - 1]e^{-\frac{1}{2}K\int_{\mathbf{x}}(\boldsymbol{\nabla}\vec{n})^2}, \tag{119}$$

$$= \int [d\vec{n}(\mathbf{x})][d\lambda(\mathbf{x})]e^{-\frac{1}{2}K\int_{\mathbf{x}}(\boldsymbol{\nabla}\vec{n})^2 - i\int_{\mathbf{x}}\chi(\mathbf{x})(|\vec{n}(\mathbf{x})|^2-1)}, \tag{120}$$

where the constraint was implement by the functional δ-function, that in the second line was expressed in its functional Fourier transform form, i.e., through the functional integral over the auxiliary field $\chi(\mathbf{x})$. With this the Hamiltonian is quadratic in $\vec{n}(\mathbf{x})$, which can therefore be integrated out exactly. Before this, it is convenient to first separate out the uniform order parameter (magnetization in the case of a FM) $\vec{n}_0$, according to $\vec{n}(\mathbf{x}) = \vec{n}_0 + \delta\vec{n}(\mathbf{x})$, and integrate out the transverse fluctuations $\delta\vec{n}(\mathbf{x})$, obtaining (dropping a multiplicative constant),

$$Z = \int [d\lambda(\mathbf{x})]e^{-\frac{1}{2}\int_{\mathbf{x}}\lambda(\mathbf{x})(n_0^2-1)}\left[\det\left(-K\nabla^2 + \lambda(\mathbf{x})\right)\right]^{-\frac{1}{2}(N-1)}, \tag{121}$$

$$= \int [d\lambda(\mathbf{x})]e^{-\frac{1}{2}\int_{\mathbf{x}}\lambda(\mathbf{x})(\vec{n}_0^2-1) - \frac{1}{2}(N-1)\mathrm{Tr}\ln\left[-K\nabla^2+\lambda(\mathbf{x})\right]}, \tag{122}$$

$$\tag{123}$$

where I defined $\lambda \equiv i2\chi$ (think of taking the path integral over imaginary axis), and will look for solution where λ and K are of order N. With this the exponent scales with N and thus in the large $N \to \infty$ limit, the remaining functional integral over $\lambda(\mathbf{x})$ can be taken by method of steepest descent, i.e., by simply minimizing the effective Hamiltonian over λ. $1/N$ corrections correspond to a systematic loop expansion in $\lambda(\mathbf{x})$ fluctuations. To lowest zeroth order in $N \to \infty$ limit, we thus obtain,

$$Z = e^{-V\mathcal{H}_{\rm eff}[\lambda,\vec{n}_0]}, \tag{124}$$

where λ and n_0 are determined by minimizing the effective Hamiltonian according to $\partial\mathcal{H}_{\rm eff}/\partial\lambda = 0$, $\partial\mathcal{H}_{\rm eff}/\partial n_0 = 0$, which respectively give

$$\vec{n}_0^2 - 1 + N\int \frac{d^dk}{(2\pi)^d}\frac{1}{Kk^2+\lambda} = 0, \tag{125}$$

$$\lambda n_0 = 0. \tag{126}$$

From second equation we observe that there are two possible solutions: (i) $n_0 \neq 0$, requiring $\lambda = 0$ and corresponding to the ordered FM state, with a nonzero magnetization and a vanishing inverse correlation length, and (ii) $n_0 = 0$, allowing $\lambda \neq 0$ and corresponding to the disordered PM state, with a vanishing magnetization and a nonzero inverse correlation length. For these two cases, the first equation then determines the growth of the FM order parameter below T_c and the behavior of the correlation length $\xi = \sqrt{K/\lambda}$ in the disordered PM phase, respectively.

These solutions match at T_c, determined by $\lambda = n_0 = 0$, which gives (through $K_c \equiv (K/k_BT)_c$),

$$\frac{N}{K_c}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 1, \tag{127}$$

$$K_c^{-1} = \frac{d-2}{NC_d\Lambda^{d-2}}, \quad \text{for } d > 2, \tag{128}$$

noting that K_c^{-1} (i.e., T_c) is driven to zero for d below the lower-critical dimension $d_{lc} = 2$.

Using this expression for K_c^{-1} we can now return to the saddle point equations and determine the growth of the order parameter n_0 as K^{-1} drops below the critical point at K_c^{-1}. This gives

$$n_0^2 + \frac{N}{K}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 1, \tag{129}$$

$$n_0^2 + \frac{K_c}{K}\frac{N}{K_c}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 1, \tag{130}$$

$$n_0 = \left(1 - \frac{K_c}{K}\right)^{1/2}, \quad \text{for } K > K_c, \tag{131}$$

predicting $\beta = 1/2$ in this $N \to \infty$ limit, as in mean-field theory.

The divergence of the correlation length $\xi = \sqrt{K/\lambda}$ as the critical point K_c is approached from the PM phase is similarly determined by,

$$\frac{N}{K}\int_{\mathbf{k}} \frac{1}{k^2+\xi^{-2}} \;=\; 1, \tag{132}$$

$$\frac{N}{K}\int_{\mathbf{k}} \left[\frac{1}{k^2+\xi^{-2}} - \frac{1}{k^2}\right] \;=\; 1 - \frac{N}{K}\int_{\mathbf{k}} \frac{1}{k^2}, \tag{133}$$

$$N\xi^{-2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\xi^{-2})k^2} \;=\; K_c - K. \tag{134}$$

The final result for ξ depends strongly on the dimensionality d. For $d > 4$, the integral is IR convergent, dominated by the UV cutoff Λ and thus ony weakly depends on ξ in its denominator. It thus gives the mean-field result,

$$\xi_{d>4} \sim |K_c - K|^{-1/2}, \quad \text{for } K < K_c, \tag{135}$$

consistent with the earlier perturbative and RG analysis giving $d_{uc} = 4$.

In contrast, for $2 < d < 4$, the momentum integral is IR divergent and thus sensitively depends on the IR cutoff ξ, giving,

$$N\xi^{2-d} \;\sim\; K_c - K, \tag{136}$$

$$\xi \;\sim\; |K_c - K|^{-\nu}, \tag{137}$$

$$\tag{138}$$

with

$$\nu = \frac{1}{d-2}, \quad \text{for } 2 < d < 4. \tag{139}$$

I note that this prediction for ν is distinct from mean-field and in fact agrees with that from the ϵ-expansion in the $N \to \infty$ limit. Higher order $1/N$ corrections are obtained by including $\lambda(\mathbf{x})$ fluctuations about above saddle point in a systematic loop expansion.[30, 31]

2. $O(N)$ Landau-Ginzburg model: "soft"-spin description

It is enlightening to complement above analysis by the $O(N)$ Landau-Ginzburg model, also referred to as the "soft"-spin description (emphasizing that magnitude of $\vec{S}$ can fluctuate about S_0, "softly" biased to be at S_0 by the Mexican-hat potential, $(|\vec{S}|^2 - S_0^2)^2)$,

$$H[\vec{S}(\mathbf{x})] \; = \; \int_{\mathbf{x}} \left[\frac{1}{2} J(\boldsymbol{\nabla}\vec{S})^2 + \frac{1}{2} t|\vec{S}|^2 + \frac{1}{4} u|\vec{S}|^4\right], \tag{140}$$

$$= \; \int_{\mathbf{x}} \left[\frac{1}{2} J(\boldsymbol{\nabla}\vec{S})^2 + \frac{1}{4} u\left(|\vec{S}|^2 + t/u\right)^2\right] + \text{const.} \tag{141}$$

I introducing a Hubbard-Stratonovic transformation (i.e., a Gaussian integral over field $\lambda(\mathbf{x})$) in the partition function to decouple the quartic interaction in the Hamiltonian, and integrated out the resulting quadratic spin degrees of freedom,

$$Z \; = \; \int [d\vec{S}(\mathbf{x})][d\lambda(\mathbf{x})] e^{-\int_{\mathbf{x}}\left[\frac{1}{2}J(\boldsymbol{\nabla}\vec{S})^2 + \frac{1}{2}\lambda(\mathbf{x})(|\vec{S}|^2 + t/u) - \frac{1}{4u}\lambda^2(\mathbf{x})\right]}, \tag{142}$$

$$= \; \int [d\lambda(\mathbf{x})] e^{-\frac{1}{2}\int_{\mathbf{x}}\left[\lambda(\mathbf{x})(S_0^2 + t/u) - \frac{1}{2u}\lambda^2(\mathbf{x})\right] - \frac{1}{2}(N-1)\mathrm{Tr}\ln\left[-J\nabla^2 + \lambda(\mathbf{x})\right]}. \tag{143}$$

Noting that S_0^2 scales like N, and taking u to scale as $1/N$, gives a nontrivial $N \to \infty$ limit (since the effective Hamiltonian scales as N), obtained by the exact saddle-point method evaluation of the $\lambda(\mathbf{x})$ integral.

To this end, (as in the previous subsection for the nonlinear σ-model) I minimize over uniform S_0 and λ, obtaining saddle-point equations,

$$S_0^2/N + t/(Nu) - \lambda/(Nu) + \int \frac{d^d k}{(2\pi)^d} \frac{1}{Jk^2 + \lambda} \; = \; 0, \tag{144}$$

$$\lambda S_0 \; = \; 0. \tag{145}$$

Note that keeping uN fixed, makes all terms of order one, allowing for a well-defined $N \to \infty$ limit. I observe that these equations are nearly identical with the saddle-point equations for the nonlinear σ-model from the previous subsection, distinct by only the λ/u term, which, as we will see below is negligible.

In the FM phase, $\lambda = 0$, giving $S_0 \sim \sqrt{-t_R/u}$, for $t_R = t + Nu\int_{\mathbf{k}} \frac{1}{Jk^2} < 0$, i.e., $\beta = 1/2$. In the PM phase, $S_0 = 0$, and for the non-mean-field case of interest, $2 < d < 4$, the integral scales like $\lambda\lambda^{(4-d)/2}$, which for small λ allows us to neglect the λ/u term, and therefore gives $\xi \sim t_R^{1/(d-2)}$, consistent with the result from the nonlinear σ-model analysis above. Higher order $1/N$ corrections require analysis of $\lambda(\mathbf{x})$ fluctuations about this $N \to \infty$ saddle point, that can be performed in a systematic loop expansion.[30, 31]

B. $2+\epsilon$-expansion for the FM-PM transition

An analysis of the nonlinear σ-model complementary to the large-N treatment is through the renormalization group, controlled by the $\epsilon = d-2$ expansion about the $d = 2$ limit, where the transition is forbidden by the Hohenberg-Mermin-Wagner-Coleman theorem. To this end, we explicitly resolve the nonlinear fixed-spin constraint on $\hat{n}$ in terms of an $n-1$ transverse component field, $\vec{\pi}$,

$$\begin{aligned}\hat{n} &= (\sigma, \vec{\pi}), &(146)\\ &= \left(\sqrt{1-\pi^2}, \vec{\pi}\right), &(147)\\ &\approx \left(1-\frac{1}{2}\pi^2, \vec{\pi}\right). &(148)\end{aligned}$$

which leads to the effective Hamiltonian for the non-linear σ-model,

$$H_{n\sigma m}[\vec{\pi}(\mathbf{x})] = \frac{1}{2}K\int d^dx\left[(\boldsymbol{\nabla}\vec{\pi})^2 + (\boldsymbol{\nabla}\sqrt{1-\pi^2})^2\right] \quad (149)$$

It is possible to perform RG analysis directly on this model, as first done by A. Polyakov[28] and by Nelson and Pelcovits[29] and is somewhat involved[7].

Another, more streamlined approached, building on the simplicity of the $N = 2$, $d = 2$ XY model (which is harmonic in the absence of vortices) is the parameterization,[6]

$$\begin{aligned}n_1 &= \sqrt{1-\vec{t}^2}\cos\phi, &(150)\\ n_2 &= \sqrt{1-\vec{t}^2}\sin\phi, &(151)\\ n_i &= t_i, \quad \text{for } 2 < j \leq N, &(152)\end{aligned}$$

where $\vec{t}$ is a $N-2$ component vector field. In terms of this parameterization the non-linear σ-model Hamiltonian is given by,

$$H_{n\sigma m}[\vec{\pi}(\mathbf{x})] = \frac{1}{2}K\int d^dx\left[(1-t^2)(\boldsymbol{\nabla}\phi)^2 + (\boldsymbol{\nabla}\sqrt{1-t^2})^2 + (\boldsymbol{\nabla}\vec{t})^2\right], \quad (153)$$

which reduces to the linear XY model for $N = 2$.

The simplest and leading effect of coarse-graining comes from integrating out the transverse fields $\vec{t}$, which renormalize the XY part of the ϕ stiffness via $\langle t^2\rangle$ correction,

$$\begin{aligned}\langle\vec{t}(\mathbf{x})\cdot\vec{t}(\mathbf{x})\rangle &= (N-2)K^{-1}\int_{L^{-1}}^{a^{-1}}\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}, &(154)\\ &= (N-2)K^{-1}\frac{C_d}{d-2}\left(a^{2-d} - L^{2-d}\right), &(155)\\ &\approx \frac{N-2}{2\pi}K^{-1}\ln L/a, \quad \text{for } d = 2+\epsilon, &(156)\end{aligned}$$

giving

$$\delta K = -\frac{N-2}{2\pi} \ln L/a. \tag{157}$$

Combining this with the RG rescaling, $K(b) = b^{d-2}K$, taking θ to be dimensionless, we obtain the flow of the effective temperature, $K^{-1}(\ell) \equiv T(\ell)$ (since K is measured in units of $k_B T$ and it is more convenient to think of T as the coupling constant of the nonlinear σ-model, i.e., absorb K inside T),

$$\frac{dT}{d\ell} = -\epsilon T + \frac{N-2}{2\pi} T^2. \tag{158}$$

The vanishing of the perturbative correction for $N = 2$ is a reflection of the previously noted local linearity of the XY model. This low-temperature, ordered-state RG captures the FM-PM disordering transition and thereby complements our earlier high-temperature ϕ^4 treatment of the the PM-FM ordering transition. From above we find the critical point at

$$T_c \equiv T^* = \frac{2\pi}{N-2}\epsilon, \tag{159}$$

that controls this FM-PM criticality. The thermal eigenvalue at this critical point is controlled by the flow of $\delta T = T - T_*$,

$$\frac{d\delta T}{d\ell} = -\epsilon\delta T + 2\frac{N-2}{2\pi} T_* \delta T \equiv y_t \delta T, \tag{160}$$

with $y_t = \epsilon = d - 2$, which gives

$$\nu = 1/\epsilon = 1/(d-2). \tag{161}$$

This is consistent with our earlier, large-N prediction. Other critical exponents can be straightforwardly obtained.[6, 7]

We close by noting that above flow for T is indeed consistent with the earlier finding of $d_{lc} = 2$, a reflection of the Hohenberg-Mermin-Wagner-Coleman theorem, that requires that in two dimensions, for $N > 2$ the critical temperature $T_c = T^*$ is driven to zero; as we will see below for $N = 2$ and $d = 2$ the transition remains, but is the non-Landau topological Berezinskii-Kosterlitz-Thouless transition.[19, 20]

In 2d ($\epsilon = 0$) and $N > 2$ the flow for $T(\ell)$ is marginally relevant, with the solution,

$$1/T(\ell) = -\frac{N-2}{2\pi}\ell + 1/T, \tag{162}$$

such that $T(\ell)$ flows to infinite positive temperature, that is physically most simply interpretted as the paramagnetic phase; other logical possibilities remain but we will not explore them here.

Using the RG matching analysis, that for the correlation length gives $\xi(T) = e^{\ell}\xi(T(\ell))$ and choosing $\ell*$ such that $\xi(T(\ell^*)) = a$, I obtain the low temperature behavior of the 2d correlation length,

$$
\begin{aligned}
\xi(T) &= ae^{\ell^*}, &(163)\\
&\sim ae^{\frac{2\pi}{N-2}\frac{K}{k_BT}}, &(164)
\end{aligned}
$$

where in the last line I restored the units and stiffness K. The results demonstrates that indeed a 2d nonlinear σ-model is characterized by a finite correlation length $\xi(T)$ (i.e., gapped in the quantum context) that only diverges as $T \to 0$. In particle physics context it is often said to be “asymptotically free”, meaning at short scales (i.e., in UV limit) its coupling constant, T flows to zero and the theory is thus noninteracting. It is also said to exhibit “dimensional transmutation”, where a macroscopic length scale emerges out of an otherwise scale-invariant theory. It is believed that these features are shared by many generalizations of the nonlinear σ-model.

V. DISORDERING OF A 2D XY MODEL: KOSTERLITZ-THOULESS, ROUGHENING AND COMMENSURATE-INCOMMENSURATE PHASE TRANSITION

In our earlier analysis of the 2d XY model, we found that in the low-temperature ordered state, the phase correlations grow logarithmically,

$$
\begin{aligned}
C(\mathbf{r}) &= \frac{1}{2}\langle(\phi(\mathbf{r}) - \phi(0))^2\rangle = \langle\phi(\mathbf{r})\phi(\mathbf{r}) - \phi(\mathbf{r})\phi(0)\rangle, &(165)\\
&= \int \frac{d^2k}{(2\pi)^2}\frac{k_BT}{Kk^2}\left(1 - e^{i\mathbf{k}\cdot\mathbf{r}}\right), &(166)\\
&= \frac{k_BT}{2\pi K}\ln(r/a), \quad \text{for } r \gg a, &(167)
\end{aligned}
$$

and as a result the XY-order parameter correlations decay as a power law,

$$
\begin{aligned}
\langle\psi^*(\mathbf{r})\psi(0)\rangle &\approx n\langle e^{i(\phi(\mathbf{r})-\phi(0))}\rangle \sim e^{-C(\mathbf{r})}, &(168)\\
&\sim (a/r)^{\eta} \to |\langle\psi(\mathbf{r})\rangle|^2 \equiv |\Psi_0|^2 \to 0, \text{for } d = 2, \text{ QLRO}, &(169)
\end{aligned}
$$

which shows that even in the (supposedly) superfluid state in 2d (in contrast to $d > 2$), the Landau order parameter Ψ_0 vanishes, there is no long-range order and the system exhibits what we called a "quasi-long-range" order,[17, 18, 20] i.e., correlations fall off as a power-law, with exponent

$$\eta = \frac{k_B T}{2\pi K}. \tag{170}$$

These observations go back to Peierls and Landau in mid 1930s and are the expression of the Hohenberg-Mermin-Wagner-Coleman theorem of "no spontaneous breaking of continuous symmetry in 2d". One might emphasize this point by calling this power-law phase as quasi-long-range *disordered*.

However, as was first noted by Berezinskii and by Kosterlitz and Thouless[20], a vanishing of Ψ_0, amazingly, does *not* imply absence of e.g., a normal-superfluid phase transition in finite temperature helium films or in a 2d XY model in general. The reason is, that, as we showed in earlier lectures, that ψ correlations are short-ranged, falling off exponentially in a fully disordered high T state. Since we have just rigorously demonstrated that inside the superfluid film they fall off as a power-law, with an arbitrary small exponent $\eta(T \to 0) \to 0$, it is clear that there must be a genuine phase transition, with the two phases distinguished by a qualitatively distinct behavior of correlation functions, rather than the order parameter.

Because in both 2d XY phases, Landau parameter vanishes, this transition between two qualitatively distinct *disordered* states, is often referred to as the Berezinskii-Kosterlitz-Thouless (BKT)[20], is not of Landau type and, in fact, is a first example of what's now referred to as the *topological* phase transition.

A. Vortices and heuristic analysis of BKT transition

Above observation, demanding a sharp transition between power-law and exponentially correlated disordered phases presents a puzzle as low-temperature correlation function in (169), (170) is analytic as a function of T. Although it does become shorter-range with increasing T, it remains a power-law, unable to give exponential correlations at arbitrary high temperatures.

The resolution of this apparent paradox is the existence of vortex configurations, where the phase $\phi(\mathbf{r})$ winds by integer-multiples of 2π about a singular points $\mathbf{r}_i$, namely

$$\oint_{C\in\mathbf{r}_i} d\phi = \oint_{C\in\mathbf{r}_0} d\mathbf{r}\cdot\boldsymbol{\nabla}\phi(\mathbf{r}) = 2\pi n, \tag{171}$$

where $n \in Z$ is the integer charge of the vortex, required by single-valuedness of $\psi(\mathbf{r})$. In the strict sense, in the continuum, such vortex $\phi_v(\mathbf{r})$ is not well defined at the vortex center.

As a result vortices were tacitly neglected in the low-temperature harmonic XY model treatment, that at face value only includes nonsingular "spin-wave" modes, missing topological nontrivial spin fluctuations, (Fig.10). While spin-waves (nonsingular $\phi(\mathbf{r})$ configurations) disorder FM (and SF) order, reducing correlations, alone they are not sufficient and vortices are required to fully disorder the XY model, as we explore below.

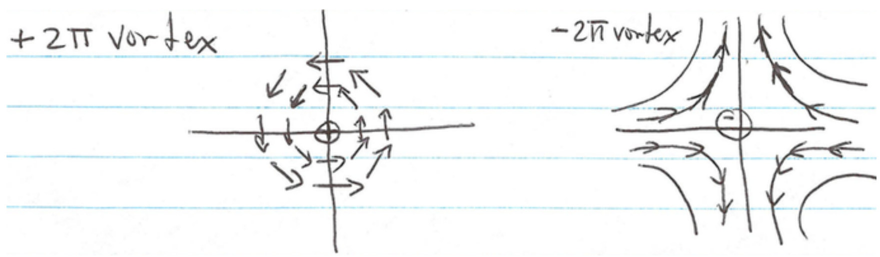

FIG. 10: Illustration of $+2\pi$ and -2π unit vortices, with arrows indicating the XY spins, not the supercurrents of a superfluid. The superfluid velocity is a gradient of the orientation angle, $v_s = \hbar\boldsymbol{\nabla}\phi/m$ and corresponds to counter-clockwise and clockwise flow, respectively.

Vortex configurations can be included through lattice regularization of the XY model,

$$H_{XY} = \frac{1}{2}K\int_{\mathbf{r}} |\boldsymbol{\nabla}\phi|^2, \tag{172}$$

$$\rightarrow -K\sum_{\mathbf{r},\mathbf{r}'}\cos(\phi_{\mathbf{r}} - \phi_{\mathbf{r}'}), \tag{173}$$

requiring no singular configurations of $\phi_{\mathbf{r}}$. Alternatively, vortices are automatically included in the 'soft-spin' Ginzburg-Landau description in terms of $\psi(\mathbf{r})$, where both the magnitude and phase can fluctuate, with the core of the vortex a ξ_0-size circular region around $\mathbf{r}_0$, where magnitude of ψ vanishes, as illustrated in Fig.11

A third approach, that we will utilize is to explicitly allow for a singular vortex part of $\phi(\mathbf{r})$, as encoded in (171). To this end we explicitly decompose $\phi(\mathbf{r}) = \phi_v(\mathbf{r}) + \phi_s(\mathbf{r})$ into its singular vortex part ϕ_v and spin-wave component ϕ_s. It is convenient to decouple the two

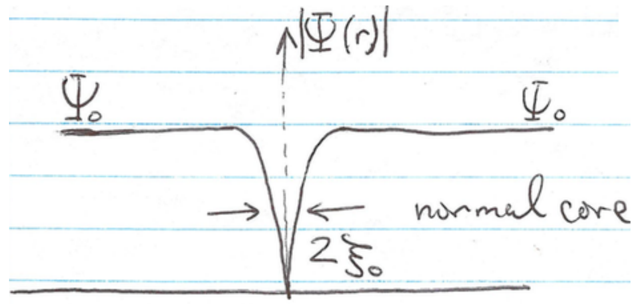

FIG. 11: A schematic illustration of the vortex core in terms of the vanishing magnitude of the Ginzburg-Landau order parameter.

contributions in the Hamiltonian, eliminating the cross-term, by choosing $\phi_v(\mathbf{r})$ to satisfy the Euler-Lagrange equation

$$\nabla^2\phi_v \;=\; 0. \tag{174}$$

To encode the vortex winding $2\pi n$, I supplement the saddle-point equation with the differential statement of (171), obtained through the Stokes theorem

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v \;=\; 2\pi n\delta(\mathbf{r}-\mathbf{r}_0)\hat{\mathbf{z}}. \tag{175}$$

familiar from the Gauss's law in electrostatics of charges. Indeed such $\phi(\mathbf{r})$ must be a singular function since curl of a $\boldsymbol{\nabla}$ of a nonsingular function vanishes identically.

It is easy to verify that the solution to these two simultaneous equations is given by

$$\phi_v(\mathbf{r}) \;=\; n\varphi = n\arctan(y/x) = n\mathrm{Im}\ln(x+iy), \tag{176}$$

with φ the azimuthal angle coordinate and for simplicity I have taken $\mathbf{r}_0 = 0$. The corresponding "current" $\boldsymbol{\nabla}\phi(\mathbf{r})$ is then given by

$$v_v(\mathbf{r}) \equiv \boldsymbol{\nabla}\phi_v(\mathbf{r}) \;=\; n\frac{\hat{\mathbf{z}}\times\mathbf{r}}{r^2} = \frac{n}{r^2}(-y,x) \equiv n\frac{\hat{\boldsymbol{\varphi}}}{r}, \tag{177}$$

from which we see that

$$\oint_{C\in 0} d\phi \;=\; n\int d\varphi = 2\pi n, \tag{178}$$

$$n\oint_{C\in 0} d\varphi r\hat{\boldsymbol{\varphi}}\cdot\frac{\hat{\boldsymbol{\varphi}}}{r} \;=\; 2\pi n, \tag{179}$$

Equivalently, the solution can be found by inverting the Fourier transform of (175),

$$\mathbf{v}_v(\mathbf{k}) \equiv \boldsymbol{\nabla}\phi_v(\mathbf{k}) \;=\; -2\pi n i\frac{\hat{\mathbf{z}}\times\mathbf{k}}{k^2}, \tag{180}$$

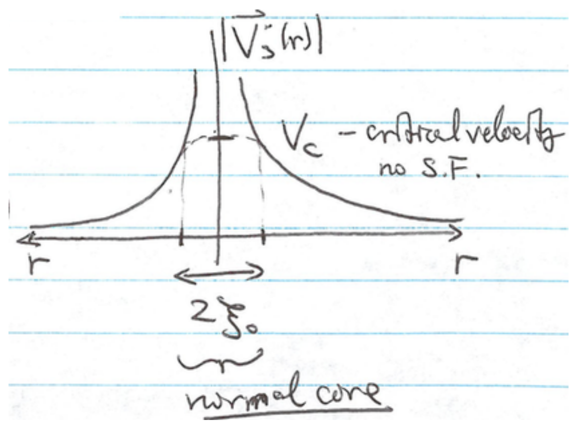

FIG. 12: "Velocity" $|\vec{v}_s(r)|$

that clearly also solves the Euler-Lagrange equation (174). In real space, the solution can also be found by utilizing rotational invariance, taking the ansatz to be $\boldsymbol{\nabla}\theta_v(\mathbf{r}) = v_v(r)\hat{\boldsymbol{\varphi}}$, inserting into (179)

$$\oint \boldsymbol{\nabla}\phi \cdot d\mathbf{r} = 2\pi n, \tag{181}$$

$$\int_0^{2\pi} v_v(r)\hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}} r d\varphi = 2\pi n, \tag{182}$$

$$v_v(r) 2\pi r = 2\pi n, \tag{183}$$

and solving for $v_v(r)$, which again leads to the solution $\boldsymbol{\nabla}\phi = n\hat{\boldsymbol{\varphi}}/r$.

As illustrated in Fig.12, superfluid velocity around a vortex diverges as $1/r$, reaching the critical velocity $v_c = \hbar/(m\xi_0)$ at $r = \xi_0$, at which the fluid becomes normal inside the vortex core.

To understand the role of vortices in the partition function I evaluate the energy of the above vortex solution $\phi_v(\mathbf{x})$ inside the 2d XY model Hamiltonian, obtaining

$$E_v = \frac{1}{2}K \int d\mathbf{r}(\boldsymbol{\nabla}\phi_v)^2, \tag{184}$$

$$= \frac{1}{2}K \int d\mathbf{r}\frac{n^2}{r^2}, \tag{185}$$

$$= \frac{1}{2}K \int \frac{d^2k}{(2\pi)^2}\frac{n^2}{k^2}, \tag{186}$$

$$= n^2\pi K \ln(L/a), \tag{187}$$

where $a \sim \xi_0$ is the UV cutoff set by the vortex core and L the system size. From E_v (187) we observe that vortices with the lowest winding number, $n = \pm 1$ have minimum energy and we thus focus on these fundamental ones.

Although vortices appear to be forbidden as their energy diverges logarithmically in thermodynamic, $L \to \infty$, at finite temperature they carry significant amount of translational entropy that for sufficiently high temperature can indeed out-compete the energy, lowering the overall free energy. To see this, we note that a single vortex entropy contribution is a logarithm of the number of states, in this case positions $\sim L^2/a^2$ available to it, giving total free-energy vortex contribution

$$\begin{aligned} F_v &= E_v - TS_v = \pi K \ln(L/a) - k_B T \ln(L^2/a^2), \qquad (188) \\ &= (\pi K - 2k_B T) \ln(L/a). \qquad (189) \end{aligned}$$

This thus indicates (ignoring the effects of screening by vortex dipoles) that vortex free-energy is positive for $k_B T < \frac{\pi}{2} K$ and negative for $k_B T > \frac{\pi}{2} K$. Thus, we expect a vortex unbinding KT phase transition at

$$k_B T_{KT} = \frac{\pi}{2} K, \qquad (190)$$

from a superfluid state with quasi-long-range order to a fully disordered normal state with short-range exponential correlations. Above transition temperature is equivalent to the condition

$$\begin{aligned} \eta(T_{KT}) &= \frac{T_{KT}}{2\pi K} = \frac{\pi}{2} \frac{1}{2\pi}, \qquad (191) \\ &= \frac{1}{4}, \qquad (192) \end{aligned}$$

on the range exponent of spatial correlations in the XY QLRO state.

As illustrated in Fig.13, for $T < T_{KT}$, free vortices are free-energetically costly and only appear paired into neutral dipoles. For $T > T_{KT}$, positional entropy dominates and free vortices proliferate, destroying QLRO and leading to fully disordered short-range correlated normal (PM) fluid.

We conclude by noting that in a close analogy to the above KT transition, a melting of a 2d crystal can be described an unbinding of its topological defects, restoring the rotational and translational symmetries back to a liquid. In this case, there are two types of topological defects, the scalar disclination s (a deficit or surplus $s = \pm 2\pi/6$ of bond angle in e.g., a hexagonal 2d crystal) and vector dislocations (characterized by a Burgers vector $\mathbf{b}$), that are dipoles of $\pm 2\pi/6$ disclinations. In close analogy with the above KT transition description for scalar vortex defects, upon raising the temperature, first dislocations entropically unbind

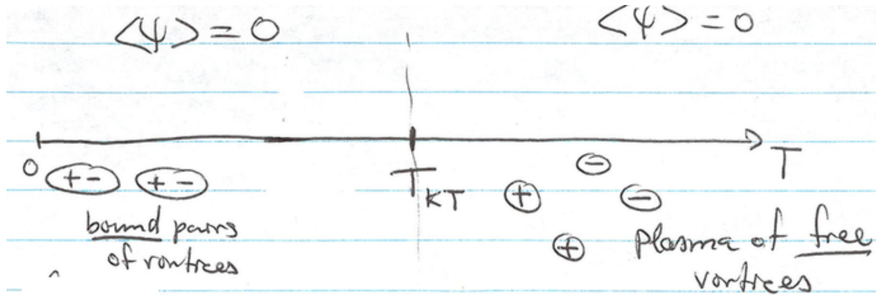

FIG. 13: Schematic phase diagram for a 2d superfluid film (or equivalently an XY FM-PM), illustrating Kosterlitz-Thouless vortex unbinding transition from a QLRO superfluid, with vortices logarithmically bound into neutral dipoles to a normal fluid of free vortices.

at the melting transition, T_m, thereby restoring the translational symmetry. However, as was first pointed out and worked out in great detail by Halperin and Nelson[21] and by Young[23], while unbinding of dislocations restores translational symmetry (thereby melting a crystal), it retains orientational order, that for triangular crystal is a hexatic liquid crystal order, where locally a fluid still retains a sense of bond orientation, θ_b, characterized by the hexatic orientational order parameter, $\psi_6 = e^{i6\theta_b}$. This continuous KT-like KTHNY melting tansition is then followed by the unbiding of disclination pairs s ("vortices" in the bond angle, $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\theta_b = s$), which destroy QLR orientational order of the hexatic liquid, converting it into an isotropic conventional liquid. This two-stage continuous topological melting transition is illustrated in Fig.14.

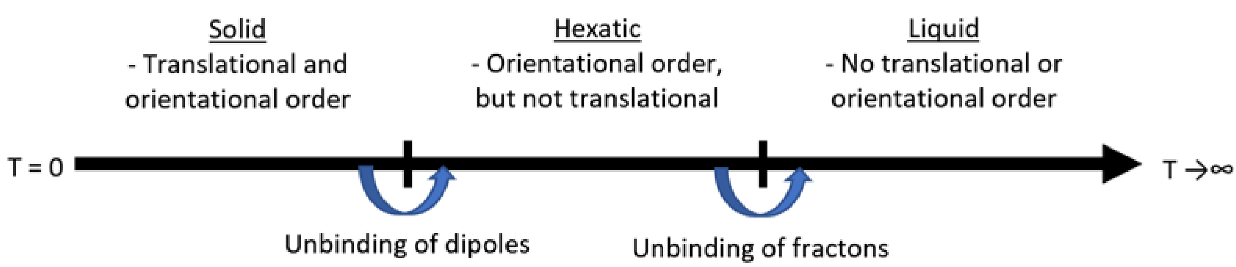

FIG. 14: Schematic phase diagram for a two-stage KTHNY melting transition of a hexagonal crystal, that upon increasing temperature proceeds by unbinding dislocations (that restore translational but not rotational symmetry, generating a hexatic liquid, that is a liquid crystal) followed by unbinding of disclinations that leads to the fully disordered isotropic liquid. (Figure created by Michael Pretko, from "Fracton-Elasticity Duality", M. Pretko and L. Radzihovsky, 2017.

B. Vortex Coulomb gas

The prediction of the KT transition in the previous subsection is somewhat heuristic as it based on a single vortex analysis, neglecting multi-vortex interaction and fluctuations. A complete analysis that we present here requires a full statistical treatment of vortex degrees of freedom.

To this end we study the partition function of the 2d XY model, but now including both the spin-waves and vortex degrees of freedom. We first generalize some of above relations to a state with a finite density of vortices,

$$\nabla \times \nabla \phi_v = \nabla \times \mathbf{v}_v \;=\; 2\pi n(\mathbf{r})\hat{\mathbf{z}} = \sum_{\mathbf{r}_i} 2\pi n_{\mathbf{r}_i} \delta(\mathbf{r} - \mathbf{r}_i)\hat{\mathbf{z}}. \tag{193}$$

The corresponding multi-vortex solution is given by

$$\mathbf{v}_v = \nabla \phi_v(\mathbf{r}) \;=\; \sum_{\mathbf{r}_i} \frac{\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^2} 2\pi n_{\mathbf{r}_i}, \tag{194}$$

$$= \int_{\mathbf{r}'} \frac{\hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} 2\pi n(\mathbf{r}'). \tag{195}$$

With this, I now transform XY model partition function into that of a Coulomb gas,

$$Z \;=\; \int [d\phi(\mathbf{r})] e^{-\frac{1}{2}K\int_{\mathbf{r}}(\nabla\phi)^2}, \tag{196}$$

$$= \int [d\phi(\mathbf{r})][d\mathbf{j}(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2 + i\mathbf{j}\cdot\nabla\phi\right]}, \tag{197}$$

$$= \int [d\phi_s(\mathbf{r})][d\mathbf{v}_v(\mathbf{r})][d\mathbf{j}(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2 - i(\nabla\cdot\mathbf{j})\phi_s + i\mathbf{j}\cdot\mathbf{v}_v\right]}, \tag{198}$$

$$= \int [d\mathbf{v}_v(\mathbf{r})][da(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\hat{\mathbf{z}}\times\nabla a)^2 + i\hat{\mathbf{z}}\times\nabla a\cdot\mathbf{v}_v\right]}, \tag{199}$$

$$= \int [d\mathbf{v}_v(\mathbf{r})][da(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\nabla a)^2 - ia\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}_v\right]}, \tag{200}$$

$$= \int [dn(\mathbf{r})][da(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\nabla a)^2 - i2\pi a n + E_c n^2\right]}, \tag{201}$$

where I (i) introduced a Hubbard-Stratonovich current field $\mathbf{j}(\mathbf{r})$ to decouple the elastic term, (ii) replaced $\nabla\phi = \nabla\phi_s + \mathbf{v}_v$, (iii) integrated out the spin-wave field $\phi_s(\mathbf{r})$, obtaining a constraint $\nabla \cdot \mathbf{j} = 0$, (iv) solved the constraint by introducing an effective gauge field $a(\mathbf{r}) = \hat{\mathbf{z}} \times \nabla a$, (v) integrated by parts, utilizing (193), $\nabla \times \mathbf{v}_v = 2\pi n(\mathbf{r})$, and (vi) introduced vortex core energy E_c to account for short-range part of the vortex energy.

Finally integrating over the field $a(\mathbf{r})$, we obtain a partition function for a Coulomb vortex gas,

$$
\begin{aligned}
Z &= \int [dn(\mathbf{r})] e^{\frac{1}{2}K\int_{\mathbf{r},\mathbf{r}'}(2\pi)^2 n(\mathbf{r})n(\mathbf{r}')\ln|\mathbf{r}-\mathbf{r}'| - E_c\int_{\mathbf{r}} n^2(\mathbf{r})}, &(202)\\
&= \sum_{\{n_{\mathbf{r}_i}\}} e^{\frac{1}{2}K\sum_{\mathbf{r}_i,\mathbf{r}_{i'}}(2\pi)^2 n_{\mathbf{r}_i}n_{\mathbf{r}_{i'}}\ln|\mathbf{r}_i-\mathbf{r}_{i'}| - E_c\sum_{\mathbf{r}_i} n^2_{\mathbf{r}_i}}, &(203)
\end{aligned}
$$

where lengths are measured in units of the lattice cutoff. In above, after intergrating out a, we obtained the 2d Coulomb Hamiltonian, $H_{\text{Coul}} = \frac{1}{2}K\int_q n(q)n(-q)/q^2 = \frac{1}{2}\int_{\mathbf{r},\mathbf{r}'} n(\mathbf{r})n(\mathbf{r}')V(|\mathbf{r}-\mathbf{r}'|)$, where $V(r) = -\frac{1}{2}K(2\pi)^2\ln|\mathbf{r}-\mathbf{r}'|) + \frac{1}{2}K(2\pi)^2\ln(L/a) + E_c\delta^{(2)}(r)$. We note a few important observations: (i) The minus sign in front of the logarithmic potential makes potential attractive between opposite charges. (ii) To obtain the logarithm (to do the Fourier transform of $1/q^2$ potential in 2d) we had to add and subtract a constant that is logarithmically divergent with L. This constant suppresses the partition function Z by a factor $(a/L)^{4\pi^2N^2}$, making it *vanish* in the thermodynamic limit due to the infinite energetic cost of overall charged Coulomb plasma $N = \int_{\mathbf{r}} n(\mathbf{r}) \neq 0$; this constant in the partition function is simply 1 for a neutral plasma $N = 0$; (iii) the last constant is the UV cutoff dependent core energy E_c not directly accounted for by a continuum limit analysis above.

As was first done by Kosterlitz and Thouless, it is possible to carry out RG in real space by coarse-graining the vortex degrees of freedom, integrating out dipoles from short to long scales, thereby obtaining a dielectric like screening (as in electrostatics) of vortex 2d Coulomb interaction. This coarse-graining and screening effect is summarized by the celebrated KT RG flow equations,

$$
\begin{aligned}
\frac{dy}{d\ell} &= (2-\pi K)y, &(204)\\
\frac{dK^{-1}}{d\ell} &= 4\pi^3 y^2, &(205)
\end{aligned}
$$

for the inverse dimensionless stiffness $K^{-1}(\ell) = k_BT/K$ and vortex fugacity $y(\ell) \equiv e^{-E_c/k_BT}$. This gives length-scale $b = e^{\ell}$-dependent reduction (screening) of $K(\ell)$, along with core energy $E_c(\ell)$,[7, 19, 20], where for sufficiently small K and E_c, a vortex proliferation takes place at,

$$
\eta(T_{KT}) = \frac{T_{KT}}{2\pi K(T_{KT})} = \frac{1}{4}, \tag{206}
$$

via the celebrated Berezinskii-Kosterlitz-Thouless transition. Above criterion of $\eta(T_{KT}) =$

1/4 is distinct from the heuristic one found earlier, (192), as the one above, (206) takes into account the renormalization through dipole screening of the stiffness $K(T_{KT})$.

Instead of following this Coulomb gas analysis, below I will utilize the duality that transforms the XY model and the Coulomb gas into the sine-Gordon model, and perform RG analysis on it.

C. 2d XY to sine-Gordon model (boson-vortex) duality

To derive the 2d XY model to sine-Gordon model duality (often also called boson-vortex duality), I return to Eq.(201) and sum over the vortex charges $n_{\mathbf{r}}$ at lattice sites $\mathbf{r}$, obtaining

$$
\begin{aligned}
Z &= \int [da(\mathbf{r})] e^{-\frac{1}{2}K^{-1}\int_{\mathbf{r}}(\boldsymbol{\nabla} a)^2} \prod_{\mathbf{r}} \left[\sum_{n_{\mathbf{r}}} e^{-i2\pi a_{\mathbf{r}} n_{\mathbf{r}} - E_c n_{\mathbf{r}}^2} \right], && (207)\\
&= \int [da(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla} a)^2 + V_{\text{Villain}}(2\pi a(\mathbf{r}))\right]}, && (208)\\
&\approx \int [da(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla} a)^2 - g\cos(2\pi a(\mathbf{r}))\right]}, && (209)\\
&\approx \int [d\theta(\mathbf{r})] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}(4\pi^2 K)^{-1}(\boldsymbol{\nabla}\theta)^2 - g\cos(\theta(\mathbf{r}))\right]}. && (210)
\end{aligned}
$$

In above $V_{\text{Villain}}(2\pi a))$ is the so-called Villain potential, defined by

$$
\begin{aligned}
e^{-V_{\text{Villain}}(\theta)} &\equiv \sum_n e^{-i\theta n - E_c n^2}, && (211)\\
&\approx \left[1 + e^{-E_c}\left(e^{-i\theta} + e^{i\theta}\right) + \ldots\right], && (212)\\
&\approx e^{g\cos(\theta)}, && (213)
\end{aligned}
$$

with periodic property $V_{\text{Villain}}(\theta + 2\pi) = V_{\text{Villain}}(\theta)$ under shift by 2π, obvious from its definition of summing over *integers* charges n. In the last equality in (210), I approximated the $V_{\text{Villain}}(\theta)$ by single harmonic, valid in the large E_c, small fugacity y limit, with $g = 2e^{-E_c}$.

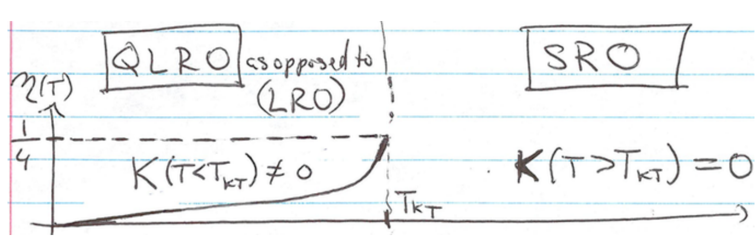

FIG. 15: Correlation length $\eta(r)$ in KT transition

D. 2d classical XY model to sine-Gordon model duality

1. *Duality via Coulomb gas*

I start out with the 2d classical XY model Hamiltonian

$$H = \frac{1}{2}K \int d^2r(\boldsymbol{\nabla}\phi)^2, \tag{214}$$

implicitly supplemented by non-single valued vortex part of the phase, ϕ_v, with $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\phi_v + \boldsymbol{\nabla}\phi_s = \mathbf{v} + \boldsymbol{\nabla}\phi_s$. This incorporates the vortex (in addition to spin-wave) degrees of freedom, under a condition

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 2\pi n(\mathbf{r})\hat{\mathbf{z}}, \tag{215}$$

where $n(\mathbf{r}) = \sum_i n_i \delta^2(\mathbf{r} - \mathbf{r}_i)$ the vortex density, and n_i are integer charges.

One approach is to eliminate $\phi(\mathbf{r})$ in favor of the vortex field $n(\mathbf{r})$ and the single-valued phase field $\phi_s(\mathbf{r})$ (giving purely longitudinal current).

$$\boldsymbol{\nabla}\phi(\mathbf{r}) = 2\pi\frac{\hat{\mathbf{z}} \times \boldsymbol{\nabla}}{\nabla^2}n(\mathbf{r}) + \boldsymbol{\nabla}\phi_s(\mathbf{r}), \tag{216}$$

$$(\boldsymbol{\nabla}\phi)_{\mathbf{k}} = -2\pi\frac{i\hat{\mathbf{z}} \times \mathbf{k}}{k^2}n(\mathbf{k}) + i\mathbf{k}\phi_s(\mathbf{k}), \tag{217}$$

With the above judicious choice of the vortex component, it decouples from the spin-wave component, leading to the Coulomb gas ($\ln r$ in real space) Hamiltonian, $H = H_{CG} + \frac{1}{2}K\int d^2r(\boldsymbol{\nabla}\phi_s)^2$,

$$H_{CG} = \frac{1}{2}K\int_{\mathbf{k}} \frac{(2\pi)^2}{k^2}|n_{\mathbf{k}}|^2. \tag{218}$$

Although H_{CG} can be analyzed directly (as originally done by KT) it is simpler to dualize it into sine-Gordon model. To this end I consider the corresponding partition function and perform standard analysis, starting with the Hubbard-Stratanovich transformation via "gauge" field $a(\mathbf{r})$ to decouple the Coulomb interaction, introduce the vortex core energy E_c, etc.

$$
\begin{aligned}
Z &= \int [da] e^{-\frac{1}{2}\int_{\mathbf{r}} K^{-1}(\nabla a)^2} \sum_{\{n_{\mathbf{r}}\}}{}^{\prime} \prod_{\mathbf{r}} e^{i2\pi a_{\mathbf{r}} n_{\mathbf{r}} - E_c n_{\mathbf{r}}^2} && (219)\\
&= \int [da] e^{-\frac{1}{2}\int_{\mathbf{r}} K^{-1}(\nabla a)^2} \left[1 + e^{-2E_c} \int_{\mathbf{r}_1,\mathbf{r}_2} e^{i2\pi a_{\mathbf{r}_1}} e^{-i2\pi a_{\mathbf{r}_2}} + \ldots\right]\\
&= \int [da] e^{-\frac{1}{2}\int_{\mathbf{r}} K^{-1}(\boldsymbol{\nabla} a)^2} \left[1 + e^{-E_c} \int_{r_1} \left(e^{i2\pi a_{\mathbf{r}_1}} + e^{-i2\pi a_{\mathbf{r}_1}}\right)\right.\\
&\quad \left. + \frac{1}{2!} e^{-2E_c} \int_{\mathbf{r}_1,\mathbf{r}_2} \left(e^{i2\pi a_{\mathbf{r}_1}} + e^{-i2\pi a_{\mathbf{r}_1}}\right)\left(e^{i2\pi a_{\mathbf{r}_2}} + e^{-i2\pi a_{\mathbf{r}_2}}\right) + \ldots\right]\\
&\equiv \int [da] e^{-H_{SG}}, && (220)
\end{aligned}
$$

where

$$
H_{SG} = \int_{\mathbf{r}} \left[\frac{1}{2} K^{-1} (\boldsymbol{\nabla} a)^2 - g\cos(2\pi a)\right], \tag{221}
$$

and the cosine coupling is given by $g = 2e^{-E_c}$. In above I've summed up over ± 1 charges and thus obtained cosine potential capturing the quantization/discreteness of vortex charges. Alternatively, I could have summed over all integer values of charges $n_{\mathbf{r}}$ in (219), obtaining the Villain potential, which is a periodic function

$$
\begin{aligned}
e^{-V[2\pi a]} &= \sum_n e^{i2\pi a n - E_c n^2}, && (222)\\
&= \int d\sigma e^{i2\pi a\sigma - E_c\sigma^2} \sum_n \delta(\sigma - n), && (223)\\
&= \int d\sigma e^{i2\pi a\sigma - E_c\sigma^2} \sum_p e^{i2\pi\sigma p}, && (224)\\
&= \sqrt{\pi/E_c} \sum_p e^{-\pi^2 E_c^{-1}(a-p)^2}, && (225)
\end{aligned}
$$

that contains a Fourier series of harmonics, but clearly in the same universality class as the above sine-Gordon model.

2. Lattice XY model duality

An alternative to the above approach of working with harmonic theory with singular fields ϕ, we can deform the Hamiltonian, putting it on the lattice,

$$
\begin{aligned}
H &= \frac{1}{2}K\int d^2r(\nabla\phi)^2, && (226)\\
&\to K\sum_{\mathbf{r},\boldsymbol{\delta}}\left(1-\cos\phi_{\mathbf{r},\mathbf{r}'}\right), && (227)\\
&\to \sum_{\mathbf{r},\boldsymbol{\delta}} V[\phi_{\mathbf{r},\boldsymbol{\delta}}], && (228)
\end{aligned}
$$

but keeping it in the same universality class, with $V(\phi_{\mathbf{r},\boldsymbol{\delta}})$ the Villain potential above, and compact field $\delta\phi_{\mathbf{r},\boldsymbol{\delta}} = \phi_{\mathbf{r}'}-\phi_{\mathbf{r}} = \Delta_{\boldsymbol{\delta}}\phi_{\mathbf{r}}$ living on the centers of the bond $\mathbf{r},\mathbf{r}'$, defined by $\mathbf{x}=\mathbf{r},\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is a two-element basis around site $\mathbf{r}$.

In terms of $V[\phi_{\mathbf{r},\boldsymbol{\delta}}]$ the partition function can be decoupled and dualized into short-range interacting, integer-valued bond fields,

$$
\begin{aligned}
Z &= \int_0^{2\pi}[d\phi_{\mathbf{r}}]e^{-\sum_{\mathbf{r},\boldsymbol{\delta}}V[\phi_{\mathbf{r},\boldsymbol{\delta}}]}, && (229)\\
&= \int_0^{2\pi}[d\phi_{\mathbf{r}}]\sum_{\{p_{\mathbf{r},\boldsymbol{\delta}}\}} e^{-\frac{1}{4E_c}\sum_{\mathbf{r},\boldsymbol{\delta}}(\Delta_{\boldsymbol{\delta}}\phi_{\mathbf{r}}-2\pi p_{\mathbf{r},\boldsymbol{\delta}})^2}, && (230)\\
&= \int[dj_{\mathbf{x}}]\int_0^{2\pi}[d\phi_{\mathbf{r}}]\sum_{\{p_{\mathbf{x}}\}} e^{-\sum_{\mathbf{x}}[E_c j_{\mathbf{x}}^2+ij_{\mathbf{r},\boldsymbol{\delta}}(\Delta_{\boldsymbol{\delta}}\phi_{\mathbf{r}}-2\pi p_{\mathbf{r},\boldsymbol{\delta}})]}, && (231)\\
&= \int[dj_{\mathbf{x}}]\sum_{\{n_{\mathbf{x}}\}}\delta(n_{\mathbf{x}}-j_{\mathbf{x}})e^{-\sum_{\mathbf{x}}E_c j_{\mathbf{x}}^2+i\left(\sum_{\boldsymbol{\delta}\in\mathbf{r}}j_{\mathbf{r},\boldsymbol{\delta}}\right)\phi_{\mathbf{r}}}, && (232)\\
&= \sum_{\{n_{\mathbf{x}}\}}{}' e^{-\sum_{\mathbf{r},\boldsymbol{\delta}}E_c n_{\mathbf{r},\boldsymbol{\delta}}^2}, && (233)\\
& && (234)
\end{aligned}
$$

where integer-valued bond currents $n_{\mathbf{r},\boldsymbol{\delta}}$ are constrained to have a vanishing lattice divergence, i.e., $\nabla_{\boldsymbol{\delta}}\cdot n_{\mathbf{r},\boldsymbol{\delta}} = \sum_{\boldsymbol{\delta}\in\mathbf{r}} n_{\mathbf{r},\boldsymbol{\delta}} = 0$.

Equivalently, one can solve the divergenless constraint, expessing the current as a curl of a dual field $a_{\tilde{\mathbf{r}}}$ living on a dual lattice $\tilde{\mathbf{r}}$, in 2d given by $n_{\mathbf{r},\boldsymbol{\delta}} = a_{\tilde{\mathbf{r}}+\hat{\mathbf{z}}\times\boldsymbol{\delta}}-a_{\tilde{\mathbf{r}}} = -\boldsymbol{\delta}\cdot\hat{\mathbf{z}}\times\nabla_{\boldsymbol{\delta}}a_{\tilde{\mathbf{r}}}$, giving the dual Hamiltonian

$$\tilde{H} = \sum_{\tilde{\mathbf{r}}}E_c(\nabla a_{\tilde{\mathbf{r}}})^2. \qquad (235)$$

To connect this to the result in the previous subsection, I "soften" up the integer constraint on the $a_{\tilde{\mathbf{r}}}$ degrees of freedom, implementing it instead via a $-g\cos(2\pi a)$, which reduces it to the sine-Gordon model above.

3. Continuum XY to sine-Gordon duality

I now review a complementary analysis, that will conveniently generalize to 3d. To this end, I decouple the elastic term, separate $\boldsymbol{\nabla}\phi$ into vortex and spin-wave part, and integrate over the spin-wave, getting a $\boldsymbol{\nabla}\cdot\mathbf{j}=0$ constraint, resolved by $\mathbf{j}=\hat{\mathbf{z}}\times\boldsymbol{\nabla}a$. Integrating by parts, this naturally introduces vortices via $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v=2\pi n(\mathbf{r})$ into the partition function. Summing over vortices then directly leads to the sine-Gordon model (with cosine for large E_c) or Villain model (for small E_c leading to many harmonics),

$$
\begin{aligned}
Z &= \int[\phi]e^{-\frac{1}{2}K\int_{\mathbf{r}}(\boldsymbol{\nabla}\phi)^2}, && (236)\\
&= \int[d\mathbf{j}][d\phi_s][d\phi_v]e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2+i\mathbf{j}\cdot(\boldsymbol{\nabla}\phi_v+\boldsymbol{\nabla}\phi_s)\right]}, && (237)\\
&= \int[d\mathbf{j}][d\phi_v]\delta(\boldsymbol{\nabla}\cdot\mathbf{j})e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2+i\mathbf{j}\cdot\boldsymbol{\nabla}\phi_v\right]}, && (238)\\
&= \int[da][d\phi_v]e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla}a)^2-ia\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v\right]}, && (239)\\
&= \int[da]\sum_{\{n_{\mathbf{x}}\}}{}^{\prime}\, e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla}a)^2-i2\pi a_{\mathbf{x}}n_{\mathbf{x}}+E_c n_{\mathbf{x}}^2\right]}, && (240)\\
&\equiv \int[da]e^{-H_{SG}}, && (241)
\end{aligned}
$$

consistent with the earlier result via Coulomb gas. Note that for $E_c=0$, the summation over $n_{\mathbf{x}}$, quantizes $a_{\mathbf{x}}$ at integers just like in Eq.(235).

I also note that for small vortex core energy E_c and small stiffness (high temperature) K, vortex density can be treated as a continuous field. Gaussian integral over n (or equivalently Taylor expanding and quadratically approximating the $-g\cos(2\pi a)\approx const.+2\pi^2ga^2$) then gives the high temperature paramagnetic gapped phase with the Hamiltonian

$$
H_{T>T_c} = \int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla}a)^2+\pi^2E_c^{-1}a^2\right]. \qquad (242)
$$

with correlation length $\xi\sim\sqrt{E_c/K}$ in the Villain case. (In the cosine approxmation case the correlation length is given by $\sqrt{1/(Kg)}=e^{E_c/2}/K^{1/2}$.

E. 3d classical and 2+1d quantum XY models, superconductor, and boson-vortex duality

1. *Duality via Coulomb gas*

The above 2d classal analysis easily generalizes to 3d

$$H = \frac{1}{2}K \int d^3r(\boldsymbol{\nabla}\phi)^2, \tag{243}$$

implicitly supplemented by non-single valued vortex part of the phase, ϕ_v, with $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\phi_v + \boldsymbol{\nabla}\phi_s = \mathbf{v} + \boldsymbol{\nabla}\phi_s$. This incorporates the vortex (in addition to spin-wave) degrees of freedom, under a condition

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 2\pi\vec{n}(\mathbf{r}), \tag{244}$$

where $\vec{n}(\mathbf{r}) = \int ds \sum_i n_i \hat{\mathbf{n}}(s)\delta^2(\mathbf{r} - \mathbf{r}_i(s))$ the vortex line density, $\hat{n}(s)$ is the line tangent and n_i are integer charges conserved along i-th vortex line at $\mathbf{r}_i(s)$, parameterized by s.

As in 2d one approach is to eliminate $\phi(\mathbf{r})$ in favor of the vortex field $\vec{n}(\mathbf{r})$ and the single-valued phase field $\phi_s(\mathbf{r})$ (giving purely longitudinal current).

$$\boldsymbol{\nabla}\phi(\mathbf{r}) = 2\pi\frac{\boldsymbol{\nabla}}{\nabla^2} \times \vec{n}(\mathbf{r}) + \boldsymbol{\nabla}\phi_s(\mathbf{r}), \tag{245}$$

$$(\boldsymbol{\nabla}\phi)_{\mathbf{k}} = -2\pi\frac{i\mathbf{k}}{k^2} \times \vec{n}(\mathbf{k}) + i\mathbf{k}\phi_s(\mathbf{k}), \tag{246}$$

With the above judicious choice of the vortex component, it decouples from the spin-wave component, leading to the 3d Coulomb gas Hamiltonian $H = H_{CG} + \frac{1}{2}K\int d^2r(\boldsymbol{\nabla}\phi_s)^2$ of long-range ($1/r$ in real space) interacting vortex loops,

$$H_{CG} = \frac{1}{2}K \int_{\mathbf{k}} (2\pi\vec{n}_{-\mathbf{k}}) \cdot \frac{P^T_k}{k^2} \cdot (2\pi\vec{n}_{\mathbf{k}}) \tag{247}$$

where P^T is the transverse projection operator.

We can now dualize this Coulomb loop gas (as we will see) into short-range interacting loop gas or a superconductor with a fluctuating gauge field. To this end I consider the corresponding partition function and perform standard analysis, starting with the Hubbard-Stratanovich transformation via "gauge" field $\mathbf{a}(\mathbf{r})$ to decouple the Coulomb interaction, introduce the vortex core energy E_c, etc.

$$Z = \int [d\mathbf{a}][d\theta] e^{-\frac{1}{2}\int_{\mathbf{r}} K^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2} \sum_{\{n_{\mathbf{r}}\}}{}' \prod_{\mathbf{r}} e^{-i(\boldsymbol{\nabla}\theta_{\mathbf{r}} - 2\pi\mathbf{a}_{\mathbf{r}})\cdot\vec{n}_{\mathbf{r}} - E_c\vec{n}^2_{\mathbf{r}}} \tag{248}$$

$$\equiv \int [d\mathbf{a}][d\theta] e^{-H_{SC}[\mathbf{a},\theta]}, \tag{249}$$

where

$$H_{SC} = \int_{\mathbf{r}} \left[\frac{1}{2} K^{-1} (\boldsymbol{\nabla} \times \mathbf{a})^2 - g \sum_{i=0}^{3} \cos(\partial_i \theta - 2\pi a_{\mathbf{r},i}) \right], \tag{250}$$

$$\approx \int_{\mathbf{r}} \left[\frac{1}{2} K^{-1} (\boldsymbol{\nabla} \times \mathbf{a})^2 + \frac{1}{2} g (\boldsymbol{\nabla} \theta - 2\pi \mathbf{a})^2 \right], \tag{251}$$

with the cosine coupling given by $g = 2e^{-E_c}$, i indicates axes (x_1, x_2, x_3) and I utilized the gauge freedom of $\mathbf{a}$ to add the dual phase $\theta_{\mathbf{r}}$ degree of freedom, which imposes the vanishing divergence of the loop current $\vec{n}_{\mathbf{r}}$. In above I've summed up over ± 1 charges and thus obtained cosine potential capturing the quantization/discreteness of vortex charges. In the last equality I've focussed on the dual superconducting (Higgs) phase where cosine is relevant and so can be well approximated by a quadratic approximation of its argument, confined to a single well.

Alternatively, I could have summed over all integer values of charges $n_{\mathbf{r}}$ in (248), obtaining the Villain potential, which reduces to a sum of δ-functions in $E_c \to 0$ limit. This imposes the dual vector potentials to be integer valued. I can also define a dual magnetic field density to be $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$ constrained to zero divergence, which reduces to the short-range interacing loop model,

$$H_{loop} = \int_{\mathbf{r}} \frac{1}{2} K^{-1} \mathbf{b}^2 \tag{252}$$

with integer-valued and divergenless fields $\mathbf{b}_{\mathbf{r}}$, as found by Dasgupta and Halperin (1981 PRL). I emphasize that Coulomb-interacting vortex loop (XY) model is dual to the above short-range interacting dual loop model. The latter is suggestive of the duality of an XY model to a superconductor whose (dual) vortices are screened by the gauge field and thus are short-range interacting, as we will see below.

To recap, the divergenlessness of $\mathbf{b}$ allows it to be expressed as $\boldsymbol{\nabla} \times \mathbf{a}$, with its integer valuedness imposed by summation over integer valued divergenless loop degrees of freedom $\vec{n}_{\mathbf{r}}$ (which physically are the vortex loops introduced in the XY model above) via $\sum'_{\{n_{\mathbf{r}}\}} e^{i2\pi \mathbf{a}_{\mathbf{r}} \cdot \vec{n}_{\mathbf{r}}}$. Integrating continuous field $\mathbf{a}_{\mathbf{r}}$ out of the resulting partition function, then gives a Coulomb interacting partition function,

$$
\begin{aligned}
Z &= \int [d\mathbf{a}] e^{-\frac{1}{2}\int_{\mathbf{r}} K^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2} \sum_{\{n_{\mathbf{r}}\}}' \prod_{\mathbf{r}} e^{i2\pi \mathbf{a}_{\mathbf{r}}\cdot\vec{n}_{\mathbf{r}}} && (253)\\
&= \int [dn_{\mathbf{r}}] e^{-\frac{1}{2}K\int_{\mathbf{r},\mathbf{r}'}(2\pi\vec{n}_{\mathbf{r}})\cdot V_{Coulomb}(\mathbf{r}-\mathbf{r}')\cdot(2\pi\vec{n}_{\mathbf{r}'})}, && (254)\\
&= \int [d\mathbf{b}][d\phi] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{b}^2 + i(\boldsymbol{\nabla}\cdot\mathbf{b})\phi\right]}, && (255)\\
&= \int [d\mathbf{b}][d\phi] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{b}^2 + i\mathbf{b}\cdot\boldsymbol{\nabla}\phi\right]}, && (256)\\
&= \int [d\phi] e^{-\int_{\mathbf{r}}\frac{1}{2}K(\boldsymbol{\nabla}\phi)^2}. && (257)
\end{aligned}
$$

where ϕ was introduced to impose the continuity (i.e., divergenlessness) of the field line $\mathbf{b}$. Actually, to be more precise in the last three equalities, I need to implement the discreteness of $\mathbf{b}_{\mathbf{r}}$, which amounts to including vortices in resulting XY model. To this end I suppliment the integral over $\mathbf{b}_{\mathbf{r}}$ by a sum over integer charges $n_{\mathbf{r}}$ coupled to $\mathbf{b}_{\mathbf{r}}$ that imposes discreteness a la Poisson summation formula,

$$
\begin{aligned}
Z &= \int [d\mathbf{b}][d\phi] \sum_{\{n_{\mathbf{r}}\}}' e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{b}^2 + i(\boldsymbol{\nabla}\cdot\mathbf{b})\phi - i2\pi\mathbf{b}\cdot\vec{n}\right]}, && (258)\\
&= \int \sum_{\{n_{\mathbf{r}}\}}' [d\phi] e^{-\int_{\mathbf{r}}\frac{1}{2}K(\boldsymbol{\nabla}\phi - 2\pi\vec{n})^2}, && (259)\\
&\approx \int \sum_{\{n_{\mathbf{r}}\}}' [d\phi] e^{\int_{\mathbf{r}}\frac{1}{2}K\cos(\boldsymbol{\nabla}\phi - 2\pi\vec{n})}. && (260)
\end{aligned}
$$

2. Continuum 3d XY to superconductor duality

As a complementary approach I decouple the elastic term, separate $\boldsymbol{\nabla}\phi$ into vortex and spin-wave part, and integrate over the spin-wave, getting a $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ constraint, resolved by $\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{a}$. Integrating by parts, this naturally introduces vortices via $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v = 2\pi\vec{n}(\mathbf{r})$ into the partition function. Introducing dual phase θ to enforce the continuity of the vortex loops, and summing over vortices then directly leads to the superconductor model.

$$
\begin{aligned}
Z &= \int [\phi] e^{-\frac{1}{2}K\int_{\mathbf{r}}(\boldsymbol{\nabla}\phi)^2}, && (261)\\
&= \int [d\mathbf{j}][d\phi_s][d\phi_v] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2 + i\mathbf{j}\cdot(\boldsymbol{\nabla}\phi_v + \boldsymbol{\nabla}\phi_s)\right]}, && (262)\\
&= \int [d\mathbf{j}][d\phi_v]\delta(\boldsymbol{\nabla}\cdot\mathbf{j}) e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{j}^2 + i\mathbf{j}\cdot\boldsymbol{\nabla}\phi_v\right]}, && (263)\\
&= \int [d\mathbf{a}][d\phi_v] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2 - i\mathbf{a}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v\right]}, && (264)\\
&= \int [d\mathbf{a}][d\vec{n}] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2 + i(\boldsymbol{\nabla}\theta - 2\pi\mathbf{a})\cdot\vec{n} + E_c\vec{n}^2\right]}, && (265)\\
&\equiv \int [d\mathbf{a}][d\theta] e^{-H_{SC}[\mathbf{a},\theta]}, && (266)
\end{aligned}
$$

consistent with the result above via the Coulomb gas.

F. 3d classical superconductor to XY model duality

1. *Duality via equation of motion*

The generic classical superconductor Hamiltonian with fluctuating gauge fields is given by

$$H = \int d^3r \left[\frac{1}{2}n_s(\boldsymbol{\nabla}\theta - 2\pi\mathbf{A})^2 + \frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\times\mathbf{A})^2\right]. \qquad (267)$$

It is implicitly supplemented by non-single valued vortex part of the phase, θ_v, with $\boldsymbol{\nabla}\theta = \boldsymbol{\nabla}\theta_v + \boldsymbol{\nabla}\theta_s$. This incorporates the vortex (in addition to spin-wave) degrees of freedom, under a condition

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\theta = 2\pi\vec{n}(\mathbf{r}), \qquad (268)$$

where $\vec{n}(\mathbf{r}) = \int ds \sum_i n_i \mathbf{t}(s)\delta^3(\mathbf{r} - \mathbf{r}_i(s))$ the vortex line density, $\mathbf{t}(s)$ is the line tangent and n_i are integer charges conserved along i-th vortex line at $\mathbf{r}_i(s)$, parameterized by s.

As in the analyses above, one approach is to eliminate $\theta(\mathbf{r})$ in favor of the vortex field $\vec{n}(\mathbf{r})$ and the single-valued phase field $\theta_s(\mathbf{r})$ (giving purely longitudinal current). To this end

I examine the equations of motion,

$$\begin{aligned}
\frac{\delta H}{\delta \mathbf{A}} &= 0, && (269)\\
\tilde{K}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} &= 2\pi n_s(\boldsymbol{\nabla}\theta - 2\pi\mathbf{A}), && (270)\\
\tilde{K}\boldsymbol{\nabla}\times\mathbf{B} &= 2\pi\mathbf{J}, && (271)\\
\tilde{K}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B} &= (2\pi)^2 n_s(\vec{n} - \mathbf{B}), && (272)
\end{aligned}$$

where, $\mathbf{J} = n_s(\boldsymbol{\nabla}\theta - 2\pi\mathbf{A})$ is the current, whose curl gives the right hand side of the last equation above.

Solving for $\mathbf{B_k}$ and $\mathbf{J_k}$ in terms of $\vec{n}_\mathbf{k}$, using $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and the above Euler-Lagrange equation $-\tilde{K}\nabla^2\mathbf{B_k} + (2\pi)^2 n_s\mathbf{B_k} = (2\pi)^2 n_s\vec{n}$, I find

$$\begin{aligned}
\mathbf{B_k} &= \frac{\vec{n}_\mathbf{k}}{\lambda^2k^2+1}, && (273)\\
\mathbf{J_k} &= \frac{\tilde{K}}{2\pi}\frac{i\mathbf{k}\times\vec{n}_\mathbf{k}}{\lambda^2k^2+1}, && (274)
\end{aligned}$$

where the London penetration length $\lambda = \sqrt{\frac{\tilde{K}}{4\pi^2 n_s}}$. For divergent λ, the current reduces to a 3d XY model result.

This gives the Hamiltonian expressed in terms of short-range interacting vortex loops,

$$\begin{aligned}
H &= \int_\mathbf{k}\frac{1}{2}\frac{n_s^{-1}}{(2\pi)^2}\frac{\tilde{K}^2k^2}{(\lambda^2k^2+1)^2}\vec{n}_{-\mathbf{k}}\cdot P_k^T\cdot\vec{n}_\mathbf{k} + \frac{1}{2}\frac{\tilde{K}}{\lambda^2k^2+1}\vec{n}_{-\mathbf{k}}\cdot P_k^T\cdot\vec{n}_\mathbf{k}, && (275)\\
&= \frac{1}{2}\tilde{K}\int_\mathbf{k}\frac{\vec{n}_{-\mathbf{k}}\cdot P_k^T\cdot\vec{n}_\mathbf{k}}{(\lambda^2k^2+1)}, && (276)\\
&\approx \frac{1}{2}\tilde{K}\int_\mathbf{k}\vec{n}_{-\mathbf{k}}\cdot P_k^T\cdot\vec{n}_\mathbf{k}, && (277)
\end{aligned}$$

where in the last line in the long wavelength limit I neglected the momentum dependence and indeed found short-range interacting vortex loops, that, as we have seen from Eqs.(253) maps onto Coulomb gas of loops and then onto the XY model. Repeating this here, I find

$$\begin{aligned}
Z &= \int[d\vec{n}][d\phi]e^{-\int_\mathbf{r}\left[\frac{1}{2}\tilde{K}^{-1}\vec{n}^2 + i(\boldsymbol{\nabla}\cdot\vec{n})\phi\right]}, && (278)\\
&= \int[d\vec{n}][d\phi]e^{-\int_\mathbf{r}\left[\frac{1}{2}\tilde{K}^{-1}\vec{n}^2 + i\vec{n}\cdot\boldsymbol{\nabla}\phi\right]}, && (279)\\
&= \int[d\phi]e^{-\int_\mathbf{r}\frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\phi)^2}, && (280)
\end{aligned}$$

where ϕ was introduced to impose the continuity (i.e., divergenlessness) of the field line $\vec{n}$. Actually, to be more precise in the last three equalities, I need to implement the discreteness of $\vec{n}_{\mathbf{r}}$, which amounts to including vortices in resulting XY model. To this end I suppliment the integral over $\vec{n}_{\mathbf{r}}$ by a sum over integer charges $n_{\mathbf{r}}$ coupled to field lines $\mathbf{b}_{\mathbf{r}}$ that imposes discreteness a la Poisson summation formula,

$$
\begin{aligned}
Z &= \int [d\mathbf{b}][d\phi] \sum_{\{n_{\mathbf{r}}\}}' e^{-\int_{\mathbf{r}}\left[\frac{1}{2}K^{-1}\mathbf{b}^2 + i(\boldsymbol{\nabla}\cdot\mathbf{b})\phi - i2\pi\mathbf{b}\cdot\vec{n}\right]}, &(281)\\
&= \int [d\phi] \sum_{\{n_{\mathbf{r}}\}}' e^{-\int_{\mathbf{r}}\frac{1}{2}K(\boldsymbol{\nabla}\phi - 2\pi\vec{n})^2}, &(282)\\
&\approx \int [d\phi] \sum_{\{n_{\mathbf{r}}\}}' e^{\int_{\mathbf{r}}\frac{1}{2}K\cos(\boldsymbol{\nabla}\phi - 2\pi\vec{n})}. &(283)
\end{aligned}
$$

2. *Duality via functional integral*

As a complementary approach I decouple the kinetic energy term, separate $\boldsymbol{\nabla}\theta$ into vortex and spin-wave part, and integrate over the spin-wave, getting a $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ constraint, resolved by $\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{a}$. Integrating by parts, this naturally introduces vortices via $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi_v = 2\pi\vec{n}(\mathbf{r})$ into the partition function that can also be expressed in terms of the dual vortex current $\mathbf{j}_v \equiv \vec{n} = \boldsymbol{\nabla}\times\mathbf{a}_v \equiv \boldsymbol{\nabla}\phi$, and dual XY model phase ϕ. Carrying out these steps on the superconductor gives a dual 3d XY model.

$$
\begin{aligned}
Z &= \int [d\theta][d\mathbf{A}] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}n_s(\boldsymbol{\nabla}\theta - 2\pi\mathbf{A})^2 + \frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\times\mathbf{A})^2\right]}, &(284)\\
&= \int [d\mathbf{j}][d\theta_s][d\theta_v][d\mathbf{A}] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}n_s^{-1}\mathbf{j}^2 + i\mathbf{j}\cdot(\boldsymbol{\nabla}\theta_v + \boldsymbol{\nabla}\theta_s - 2\pi\mathbf{A}) + \frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\times\mathbf{A})^2\right]}, &(285)\\
&= \int [d\mathbf{a}][d\mathbf{a}_v][d\mathbf{A}] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}n_s^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2 - i\mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{a}_v - 2\pi\boldsymbol{\nabla}\times\mathbf{A}) + \frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\times\mathbf{A})^2\right]}, &(286)\\
&= \int [d\mathbf{a}][d\mathbf{a}_v] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}n_s^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2 + \frac{1}{2}\tilde{K}^{-1}(2\pi)^2\mathbf{a}\cdot P^T\cdot\mathbf{a} - i\mathbf{a}\cdot\boldsymbol{\nabla}\times\mathbf{a}_v\right]}, &(287)\\
&= \int [d\mathbf{a}][d\mathbf{j}_v] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}n_s^{-1}(\boldsymbol{\nabla}\times\mathbf{a})^2 + \frac{1}{2}\tilde{K}^{-1}(2\pi)^2\mathbf{a}\cdot P^T\cdot\mathbf{a} - i\mathbf{a}\cdot\mathbf{j}_v\right]}, &(288)\\
&= \int [d\mathbf{j}_v] e^{-\frac{1}{2}\frac{\tilde{K}}{(2\pi)^2}\int_{\mathbf{r}}\mathbf{j}_v\cdot\frac{P^T}{1-\lambda^2\nabla^2}\cdot\mathbf{j}_v}, &(289)\\
&\approx \int [d\mathbf{j}_v] e^{-\frac{1}{2}\frac{\tilde{K}}{(2\pi)^2}\int_{\mathbf{r}}|\mathbf{j}_v^T|^2} = \int [d\mathbf{a}_v] e^{-\frac{1}{2}\frac{\tilde{K}}{(2\pi)^2}\int_{\mathbf{r}}|\boldsymbol{\nabla}\times\mathbf{a}_v|^2}, &(290)\\
&\approx \int [d\mathbf{j}_v][d\phi] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}\frac{\tilde{K}}{(2\pi)^2}|\mathbf{j}_v|^2 + i(\boldsymbol{\nabla}\cdot\mathbf{j}_v)\phi\right]} = \int [d\mathbf{j}_v][d\phi] e^{-\int_{\mathbf{r}}\left[\frac{1}{2}\frac{\tilde{K}}{(2\pi)^2}|\mathbf{j}_v|^2 - i\mathbf{j}_v\cdot\boldsymbol{\nabla}\phi\right]}, &(291)\\
&\approx \int [d\phi] e^{-\frac{1}{2}(2\pi)^2\tilde{K}^{-1}\int_{\mathbf{r}}|\boldsymbol{\nabla}\phi|^2}, &(292)
\end{aligned}
$$

recoving the XY model.

From above I obtain an important equivalence between three different Hamiltonians,

$$H_{SC} = \int_{\mathbf{r}} \left[\frac{1}{2} n_s (\boldsymbol{\nabla}\theta - 2\pi \mathbf{A})^2 + \frac{1}{2}\tilde{K}(\boldsymbol{\nabla}\times\mathbf{A})^2 \right], \tag{293}$$

$$\longleftrightarrow H_j = \frac{1}{2}K\int_{\mathbf{r}} |\mathbf{j}^T|^2 \longleftrightarrow H_a = \frac{1}{2}K\int_{\mathbf{r}} |\boldsymbol{\nabla}\times\mathbf{a}|^2 \longleftrightarrow H_{XY} = \frac{1}{2}K^{-1}\int_{\mathbf{r}} |\boldsymbol{\nabla}\phi|^2. \tag{294}$$

G. RG of the sine-Gordon model: KT and Roughening transition

H. Commensurate-Incommensurate (Pokrovsky-Talapov) transition

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[1] Pathria: *Statistical Mechanics*, Butterworth-Heinemann (1996).

[2] L. D. Landau and E. M. Lifshitz: *Statistical Physics*, Third Edition, Part 1: Volume 5 (Course of Theoretical Physics, Volume 5).

[3] Mehran Kardar: *Statistical Physics of Particles*, Cambridge University Press (2007).

[4] Mehran Kardar: *Statistical Physics of Fields*, Cambridge University Press (2007).

[5] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman : *The Theory of Critical Phenomena*, Oxford (1995).

[6] John Cardy: *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics.

[7] P. M. Chaikin and T. C. Lubensky: *Principles of Condensed Matter Physics*, Cambridge (1995).

[8] M. E. Fisher, *Rev. Mod. Phys.* **42**, 597 (1974).

[9] K. G. Wilson and J. Kogut, *Phys. Rep.* **12** C, 77 (1974).

[10] J. Zinn-Justin: *Quantum Field Theory and Critical Phenomena*, Oxford (1989).

[11] see appendix A in Pierre Le Doussal, Leo Radzihovsky, "Anomalous elasticity, fluctuations and disorder in elastic membranes", arxiv.org/pdf/1708.05723.pdf.

[12] P. G. de Gennes and J. Prost: *The Physics of Liquid Crystals*, Oxford (1993).

[13] *Quantum Field Theory of Many-body Systems*, Xiao-Gang Wen.

[14] *Path Integrals*, R. P. Feynman and Hibbs. plore properties of this

[15] *Principles of Condensed Matter Physics*, by P. M. Chaikin and T. C. Lubensky.

[16] L.D. Landau, Phys. Z. Sowjetunion **II**, 26 (1937); see also S. Alexander and J. McTague, Phys. Rev. Lett. **41**, 702 (1984).

[17] R.E Peierls, Ann. Inst. Henri Poincaré **5**, 177 (1935); L.D. Landau, Phys. Z. Sowjetunion II, 26 (1937)

[18] N.D. Mermin and H. Wagner, "Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models", *Phys. Rev. Lett.* **17**, 1133-1136 (1966); P.C. Hohenberg, "Existence of Long-Range Order in One and Two Dimensions", *Phys. Rev.* **158**, 383, (1967); N.D. Mermin, *Phys. Rev.* **176**, 250 (1968).

[19] Berezinskii was the first to explore the consequences of such power-law order and correctly concluded about the importance of topological defects to drive a phase transition to the fully disordered phase. The theory of such transition was worked out by Kosterlitz and Thouless,[20], who were recognized with the 2016 Nobel Prize (shared with D. Haldane) for this achievement.

[20] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973); see also, V.L. Berezinskii, Zh. Eksp. Teor. Fiz. **59**, 907 (1970) [Sov. Phys. JETP **32**, 493 (1971)]; Zh. Eksp. Teor. Fiz. **61**, 1144 (1971) [Sov. Phys. JETP **34**, 610 (1972)];

[21] B.I. Halperin and D.R. Nelson, *Theory of two-dimensional melting*, Phys. Rev. Lett. **41**, 121 (1978); D.R. Nelson and B.I. Halperin, *Dislocation-mediated melting in two dimensions*, Phys. Rev. B **19**, 2457 (1979).

[22] S. Sachdev, "Quantum phase transitions" (Cambridge University Press, London, 1999).

[23] A. P. Young, *Melting and the vector Coulomb gas in two dimensions.* Phys. Rev. B 19, 1855 (1979)

[24] G. Grinstein and R. A. Pelcovits, Phys. Rev. Lett. **47**, 856 (1981).

[25] L. Radzihovsky, Phys. Rev. A **84**, 023611 (2011).

[26] L. Golubovic, Z. Wang, Phys. Rev. Lett. **69** 2535 (1992).

[27] J. Toner and D. R. Nelson, Phys. Rev. B **23**, 316 (1981).

[28] A. Polyakov, "Gauge, Fields and Strings" book.

[29] D. R. Nelson and R. Pelcovits.

[30] S. Coleman, "Aspects of Symmetry", Erice lectures. S. Coleman and Weinberg paper.

[31] Root, Large N-expansion to order $1/N^2$.

[32] Pokrovsky, Talapov transition.