Physics 7240: Advanced Statistical Mechanics
Lecture 5: The Renormalization Group

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Abstract

In these lecture notes, I will present an introduction to renormalization group (RG). Starting with a general motivation, discussion, scaling and real-space coarse-graining in one-dimension, I will focus on field-theoretic approach of perturbative momentum-shell RG. Using this approach I will derive RG flow equations for the ϕ^4 (Ising) model, controlling it by $\epsilon = 4 - d$ expansion. I will then demonstrate how this can be combined with the "matching analysis" to compute physical observables inside the critical region, obtaining nontrivial universal exponents and scaling functions.

- Introduction, motivation and scaling theory
- General philosophy of renormalization group (RG)
- 1d Ising model via real space RG
- Momentum-shell RG diagrammatics
- RG flows and fixed points
- Physical observables by matching analysis

I. INTRODUCTION AND BACKGROUND

A. Motivation

As we have seen in earlier lectures, Landau theory provides a solid framework for treatment and understanding of continuous symmetry-breaking phase transitions. It is characterized by minimizing the Landau energy functional illustrated below. leading to power-law

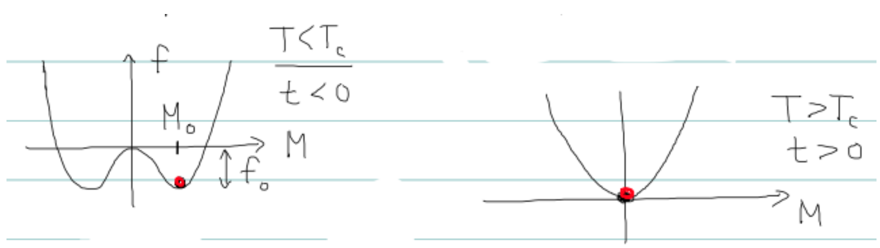


FIG. 1: Free energy of Ising model for $T < T_c$ and $T > T_c$, indicating spontaneous Z_2 symmetry breaking at the PM-FM transition below which a finite magnetization emerges.

dependence with the reduced temperature $t \sim T - T_c$ of the magnetization $m \sim |t|^{\beta}$, susceptibility $\chi \sim |t|^{-\gamma}$, heat capacity $C \sim |t|^{-\alpha}$, and correlation length $\xi \sim |t|^{-\nu}$, with mean-field exponents. However, as we discussed, this fluctuations-neglecting mean-field theory fails to describe a variety of phase transitions that display non-mean-field critical exponents, that for a 2d Ising model are observed to be

$$\beta = \frac{1}{8},\ \gamma = \frac{7}{4},\ \nu = 1,\ \alpha = 0,\ \eta = \frac{1}{4}, \tag{1}$$

where η is the exponent characterizing the critical power-law decay of two-point magnetization correlation function, $G(x) = \langle m(x)m(0)\rangle \sim 1/x^{d-2+\eta}$.

We understood this failure theoretically as stemming from the importance of fluctuations in dimensions $d < d_{uc}$ (where $d_{uc} = 4$ for the classical $O(n)$ model), sufficiently close to T_c, within the so-called Ginzburg region, defined by

$$|t| \equiv \frac{|T - T_c|}{T} < \left(\frac{uk_BT}{J}\right)^{\frac{2}{4-d}} \equiv t_G. \tag{2}$$

Inside this critical region (for $d < d_{uc} = 4$) it is even qualitatively incorrect to neglect

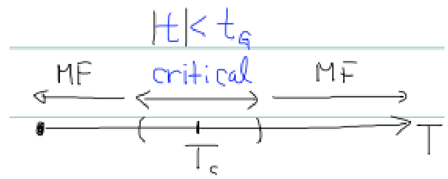


FIG. 2: Ginzburg region around T_c, where mean-field theory, valid outside t_G is guaranteed to fail, requiring a nonperturbative treatment of fluctuations

fluctuations, as for example indicated by free energy barrier in a correlation volume is small compared to a typical thermal energy,

$$\delta F \simeq \frac{|t|^2}{u}\xi^d \sim |t|^{2-d/2} \sim |t|^{(4-d)/2} \overset{t\to 0}{\ll} k_BT_c, \tag{3}$$

and leads to a failure of Landau mean-field theory. In technical terms, this shortcoming was reflected in the divergence a direct perturbation theory in nonlinearities of the field theory for $d < d_{uc}$ and sufficiently close to T_c.

B. Scaling theory

One phenomenological approach, pioneered by Widom, Kadanoff and Migdal is the so-called scaling theory. It is based on a key observation that near a critical point, the physics is controlled by a single diverging length scale, the correlation length, $\xi(t, h)$, assumed to be given by a homogeneous function of the reduced temperature t and external field h (more generally a set of fields $\{h_i\}$),

$$\xi(t, h) = t^{-\nu} g_\xi(h/t^\Delta), \tag{4}$$

$$= \begin{cases} \xi_t(t) \sim t^{-\nu}, & \text{for } h \ll t^\Delta, \text{ (equivalent to } \xi_h \gg \xi_t), \\ \xi_h(h) \sim h^{-\nu/\Delta}, & \text{for } h \gg t^\Delta, \text{ (equivalent to } \xi_h \ll \xi_t), \end{cases} \tag{5}$$

where ν is a universal (depends only on symmetry and dimensionality of the system) correlation length exponent, Δ is the so-called gap exponent and $g(x)$ is a homogeneous scaling function that is a constant for $x \to 0$ and vanishes as $x^{-\nu/\Delta}$ for $x \to \infty$. [21] The behavior of correlation functions and thermodynamics (free energy) then strongly depends on the ratio of other length scales (introduced by external, e.g., magnetic fields) to the correlation length, ξ. System's behavior is then qualitatively very different on scales shorter and longer than this the correlation length. Right at the critical point, the state is critical, $\xi(t \to 0) \to \infty$ and exhibits self-similar (as in a fractal) fluctuations on all scales. Away from the critical point, these critical fluctuations extend out to the correlation length ξ, beyond which they are strongly suppressed, and are characterized as (nearly) independent Gaussian degrees of freedom.

1. *Thermodynamics*

Given this picture, for a system of linear size L we expect a free energy of such nearly-critical state to be given by the number $N_\xi = (L/\xi)^d$ of independent critical domains, times the free-energy $f_0 = -k_B T \ln z_0$ of each ξ-sized domain. Thus, near a critical point of a continuous transition, we expect,

$$
\begin{aligned}
F(t,h) &= -k_B T \ln\left(z_0^{N_\xi}\right) = \left(\frac{L}{\xi(t,h)}\right)^d f_0, && (6)\\
&= \left(\frac{L}{\xi(t)}\right)^d g_1(\xi_h/\xi(t)) = L^d t^{d\nu} \tilde{g}_1(1/(h\xi(t)^{y_h}) && (7)\\
&\sim L^d t^{d\nu} g_f(h/t^\Delta), && (8)
\end{aligned}
$$

where $f_1(x), g_1(x), \tilde{g}_1(x), g_f(x)$ are scaling functions, all related to the scaling function $g_\xi(x)$ in (5), $\xi(t) \equiv \xi(t,0)$, y_h is the length "dimension" of field h (in terms of which the gap exponent is $\Delta = \nu y_h$), with equivalently, field-dependent length scale given by $\xi_h \sim h^{-1/y_h}$.

From the corresponding free-energy density $f(t,h) = t^{d\nu}\tilde{g}_1(h\xi(t)^{y_h})$, we can obtain all dominant singular contributions to thermodynamics by simply differentiating with respect to corresponding variable, t and h.

For example, the specific heat $c_v(t)$ at $h = 0$ is given by

$$
\begin{aligned}
c_v(t,h) &= \partial^2 f/\partial t^2, && (9)\\
&\sim t^{d\nu-2} \equiv t^{-\alpha}, && (10)
\end{aligned}
$$

giving the relation for the specific heat exponent,

$$\alpha = 2 - d\nu. \tag{11}$$

This exponent equation is the so-called hyperscaling relation, which clearly is violated by mft exponents since they do not depend on space dimension d. They do satisfy it however for $d = d_{uc} = 4$, not coincidentally as we will see from RG analysis later in these lectures.

The magnetization is also easily obtained

$$\begin{aligned} m(t,h) &= \partial f/\partial h, & (12)\\ &= t^{d\nu-\Delta} g_f'(h/t^\Delta) \sim \begin{cases} t^\beta, & \text{for } h \ll t^\Delta, \text{ (equivalent to } \xi_h \gg \xi),\\ h^{1/\delta}, & \text{for } h \gg t^\Delta, \text{ (equivalent to } \xi_h \ll \xi), \end{cases} & (13) \end{aligned}$$

which gives

$$\beta = d\nu - \Delta = 2 - \alpha - \Delta, \quad 1/\delta = \beta/\Delta, \tag{14}$$

with the last result obtained by requiring that the scaling function $g_2'(x) \sim x^{d\nu/\Delta - 1} \sim x^{\beta/\Delta}$, so that $m(t = 0, h)$ is t independent in the $t \to 0$ limit, as expected on general physical grounds.

From above we can then obtain linear magnetic susceptibility (in $h \to 0$ limit),

$$\begin{aligned} \chi(t) &= \partial^2 f/\partial h^2 = \partial m/\partial h, & (15)\\ &\sim t^{d\nu - 2\Delta} \sim t^{-\gamma}, & (16) \end{aligned}$$

with

$$\gamma = 2\Delta - d\nu = 2\Delta - 2 + \alpha. \tag{17}$$

Putting (18) together with (14) to eliminate Δ we obtain a relation,

$$\alpha + 2\beta + \gamma = 2. \tag{18}$$

2. Spatial correlations

So far, our discussion of scaling theory has been focused on thermodynamics, that does not explicitly give spatial correlations. Spatial correlations require a separate consideration. Given the above discussion on the nature of fluctuations below and above the correlation length $\xi(t, h)$, we expect that the field two-point connected correlation function to be a homogeneous function of x, t, and h, with t, h entering through the correlation length, $\xi(t, h)$,

$$\begin{aligned} C_c(x) &= \langle \phi(\mathbf{x})\phi(0)\rangle_c, && (19)\\ &= \frac{1}{x^{d-2+\eta}} g(x/\xi), && (20)\\ &= \begin{cases} \frac{1}{x^{d-2+\eta}}, & \text{for } x \ll \xi, \\ \frac{1}{\xi^{d-2+\eta}}, & \text{for } x \gg \xi. \end{cases} && (21) \end{aligned}$$

As is intuitively appealing on scale below ξ, the correlation function is critical (same as $t \to 0$ self-similar form), but is then cutoff by ξ at longer scales, thus exhibiting nearly independent, "gapped" Gaussian correlations. Above, we defined a final exponent, η, that describes nontrivial deviation of the power-law correlation from the mean-field exponent of $d-2$, as obtained from a Fourier transform of $1/q^2$ correlator (that, in the absence of critical anomalies, i.e., for $\eta = 0$ reduces to the familiar Coulomb's law).

Recalling that the uniform linear magnetic susceptibility is given by the $q = 0$ Fourier component of $C_c(x)$, we have

$$\begin{aligned} \chi(t) &\sim \int d^d x C_c(x) \sim \int d^d x \frac{g(x/\xi(t))}{x^{d-2+\eta}}, && (22)\\ &\sim \xi^{2-\eta} \int d^d(x/\xi) \frac{g(x/\xi)}{(x/\xi)^{d-2+\eta}} \sim \xi^{2-\eta}, && (23)\\ &\sim t^{-(2-\eta)\nu} \equiv t^{-\gamma}, && (24) \end{aligned}$$

which gives another important exponent relation,

$$\gamma = (2 - \eta)\nu. \tag{25}$$

Given all the exponent relations, in the end standard systems like the O(n) and related models, are characterized by only two *independent* universal exponents.

We close this section by noting that although this scaling theory is quite powerful, extracting lots of valuable information, it is after all phenomenological, with shortcomings of (a) assumption of scaling homogeneous form of thermodynamic and correlation functions, (b) not being able to make predictions for the values of the critical exponents and the nature of the scaling functions. To go beyond this phenomenology requires actual explicit calculations of the correlation and partition functions.

3. Finite-size scaling

One important application of scaling is to assess a finite system size L on the critical behavior discussed above. This is crucial in numerical and experimental studies. For example, going back to the free energy, (69), extending it to include dependence on L and at first considering isotropic sample geometry $L \times L \times L \times \ldots$, gives,

$$\begin{aligned} f(h,t,u,L) &= b^{-d} f(b^{y_h} h, b^{y_t} t, u^*, L/b), && (26) \\ &\sim t^{d/y_t} f(h/t^{y_h/y_t}, t_0, u^*, L/t^{-1/y_t}) \equiv t^{d\nu} f(h/t^{\Delta}, L/t^{-\nu}) = t^{d\nu} f(\xi_h/\xi_t, L/\xi_t). && (27) \end{aligned}$$

Differentiating twice with respect to the external field h and setting $h = 0$, we find the linear susceptibility (being cavaliar about prefactors),

$$\begin{aligned} \chi(t,L) &\sim t^{-\gamma} g_\chi(L/t^{-\nu}) = t^{-\gamma} g_\chi(L/\xi_t), && (28) \\ &\sim L^{\gamma/\nu} \tilde{g}_\chi(tL^{1/\nu}), && (29) \\ &\sim \begin{cases} t^{-\gamma}, & \text{for } t \gg L^{-1/\nu}, \text{ (equivalent to } \xi_t \ll L), \\ L^{\gamma/\nu}, & \text{for } t \ll L^{-1/\nu}, \text{ (equivalent to } \xi_t \gg L). \end{cases} && (30) \end{aligned}$$

Clearly, in the infinite size limit $L \to \infty$ (with the criterion $\xi_t \ll L$), we recover original thermodynamic critical behavior. In contrast, sufficiently close to T_c (small t), when the correlation length ξ_t exceeds system size, the divergences are cutoff by L, with $\tilde{g}_\chi(x)$ an analytic function of its argument. The original divergent susceptibility peak is cutoff at $L^{\gamma/\nu}$ and is typically shifted to $t \sim L^{-1/\nu}$, from thermodynamic limit value of $t = 0$. This last observation is important for locating the thermodynamic critical point from finite size numerics. The sign of the shift depends on the nature of the boundary conditions that determine the details of the scaling function. On physical grounds one expects periodic boundary conditions to suppress fluctuations and thus favor the ordered state by raising T_c. In contrast, for less constraining free boundary conditions, fluctuations are enhanced, favoring the disordered phase and reducing T_c.

An important generalization of above analysis is to geometrically anisotropic systems, $L_x \times L_y \times L_z \times \ldots$. There is a rich set of possibilities that we will will only partially explore here.

One interesting case is that of a "film", with one dimension L_z much smaller than the two transverse dimensions, $L \times L \to \infty$. In this case we have,

$$\chi(t, L_z) \sim t^{-\gamma_3} g_\chi(L_z/t^{-\nu_3}) = t^{-\gamma_3} g_\chi(L_z/\xi_t), \tag{31}$$

$$\sim \begin{cases} t^{-\gamma_3}, \text{ for } t \gg L_z^{-1/\nu_3}, \text{ (equivalent to } \xi_t \ll L_z), \\ t^{-\gamma_2}, \text{ for } t \ll L_z^{-1/\nu_3}, \text{ (equivalent to } \xi_t \gg L_z), \end{cases} \tag{32}$$

which describes the crossover from the three-dimensional criticality, for the 3d correlation length below film thickness (that appears for $t_L \sim L_z^{-1/\nu_3} < t < t_G$) to the asymptotic two-dimensional criticality when the correlation length exceeds film thickness (that appears for $\xi_t \gg L_z$ for $t < t_L \sim L_z^{-1/\nu_3}$). Of course this crossover is preceded by the mean-field to bulk 3d criticality crossover, if $t_G > t_L \equiv L_z^{-1/\nu_3}$. Otherwise, for $t_G < L_z^{-1/\nu_3}$, the 3d criticality does not appear and the crossover is directly from mean-field to critical film criticality. Another qualitatively distinct scenario takes places when the effective asymptotic dimension falls below the lower-critical dimension of the model. In this case the crossover is a rounding of the 3d criticality by finite size effects.

4. Quantum statistical mechanics at finite temperature

One very important realization of dimensional "film" crossover takes place for quantum criticality at finite temperature. To see this, without getting into details at this stage, we note that in many cases the partition function for a d-dimensional quantum field theory, defined by noncommuting field operator $\hat{\phi}(\mathbf{x})$ and conjugate momentum field $\hat{\Pi}(\mathbf{x})$, can be recast as a $d+1$-dimensional classical statistical field theory over commuting fields $\phi(\mathbf{x}, \tau), \Pi(\mathbf{x}, \tau)$, with the imaginary compact time $0 \leq \tau < \beta\hbar$ as the extra dimension,

$$Z = \mathrm{Tr} e^{-\beta \hat{H}}, \tag{33}$$

$$= \int [d\Pi(\mathbf{x}, \tau)][d\phi(\mathbf{x}, \tau)] e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau L[\phi(\mathbf{x},\tau),\Pi(\mathbf{x},\tau)]}, \tag{34}$$

where

$$L[\phi(\mathbf{x}, \tau), \Pi(\mathbf{x}, \tau)] = H[\phi(\mathbf{x}, \tau), \Pi(\mathbf{x}, \tau)] - i\Pi(\mathbf{x}, \tau)\partial_\tau \phi(\mathbf{x}, \tau), \tag{35}$$

is the classical Lagrangian. We note that the effective $d+1$ system is confined to a slab of thickness $\beta\hbar$, with periodic (antiperiodic) boundary conditions imposed for bosons (fermions).

Even without any further detailed analysis it is clear that at high temperature the imaginary-time slab is thin, suppressing field variation with τ, as it "costs" large action. Taking the fields to be τ independent reduces imaginary action to the Hamiltonian, and the exponential factor simply to the Boltzmann weight,

$$e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau[H - i\Pi(\mathbf{x},\tau)\partial_\tau\phi(\mathbf{x},\tau)]} \approx e^{-\beta H[\phi(\mathbf{x}),\Pi(\mathbf{x})]}, \quad (36)$$

In more detail, at zero temperature, $\beta\hbar \to \infty$ and quantum criticality is that of a $d+1$-dimensional system. At finite temperature $\beta\hbar = L_\tau$ acts as the finite thickness "film" cutoff. The discussion of the previous subsection then immediately applies. For the correlation time $\tau_\xi \ll \beta\hbar$ (where τ_ξ is distinct from the correlation length ξ, at the least scaled by the effective "speed of light" velocity, but often when non-relativistic even scaling distinctly from ξ, with $\tau_\xi \sim \xi^z$ [z is the so-called dynamical critical exponent]), the quantum critical behavior is that of a zero-temperature $d+1$ dimensional system, with $\beta\hbar \to \infty$. However, at finite T, sufficiently close to the critical point, such that $\tau_\xi \geq \beta\hbar$, the system exhibits a crossover from the $d+1$-dimensional quantum to d-dimensional classical criticality. For a given reduced quantum coupling $g - g_c$, the corresponding crossover temperature scales as

$$T_\xi(g) \sim \hbar\omega_{k_\xi} \sim \hbar/\tau_\xi \sim \hbar\xi^{-z}, \quad (37)$$

$$\sim |g - g_c|^{\nu z}. \quad (38)$$

II. GENERAL STRUCTURE OF RENORMALIZATION GROUP (RG)

Having established the general phenomenology, we now would like to go further and actually derive above scaling phenomenology, that is support by numerics and experiments. However, as we have seen in previous lectures, direct attack on the problem, e.g., via a perturbative expansion in nonlinearities is impossible in the critical regime of interest, as perturbation theory diverges inside this Ginzburg region.

To handle this strongly interacting field theory challenge, a more sophisticate RG approach was developed in 1960-70s by Leo Kadanoff, Migdal, Ken G. Wilson [22], Michael E. Fisher, Sergey Pokrovsky, and many other founders of the field of critical phenomena, culminating in Wilson and Fisher's most practical calculational tool, the so-called momentum-shell RG and $\epsilon \equiv d_{uc} - d$-expansion, about the upper critical dimension d_{uc}, that we will study in the next section.

As an aside, I mention that complementary powerful field-theoretic methods were independently invented much earlier in the context of high energy physics (QED, QCD,. . . , working to develop and understand the "Standard Model") and in fact Ken Wilson was trained in this subject in his Ph.D. studies, working on the Gell Man-Low equation. However, these formal field-theoretic methods of Pauli-Villar, dimensional, and minimal subtraction regularization schemes, have missed the conceptual boat. In those early days (and long thereafter to at least early 90's) much of that particle physics community was focused on these formal RG methods to "hide" UV divergences, i.e., "renormalize" quantum field theory, rather than on the broader physical and conceptual implication of the RG discovery. Instead, here our philosophy will be complementary, namely to make physical sense of real physical IR divergences that represent strong fluctuations in a critical state, and lead to universal dependence of the system on long scales.

A. Real-space RG: coarse-graining and spin decimation

The crucial idea of RG, suggested by Kadanoff and built on by others, is based on the self-similar nature of the critical fluctuations. Namely, although the strong interacting critical state precludes full trace over all the degrees of freedom (leading to a divergent perturbation theory), instead we can trace over a small fraction of short-scale (high energy) degrees of freedom, in an iterative "coarse-graining" spin decimation procedure. It is guaranteed to be convergent since only a small fraction of short-scale degrees of freedom is traced out, and they are not divergent. As illustrated in Fig.3 for $b = 3$, we reexpress the original trace over the N microscopic degrees of freedom, $S_{\mathbf{x}}$ in terms of a trace over N/b^d coarse-grained degrees of freedom, $\tilde{S}'_{\mathbf{x}'}$, centered around every b^d-th lattice site

$$\mathbf{x}' = \mathbf{x}/b. \tag{39}$$

Definitions other than a straight diluted (decimated) spin degrees of freedom, e.g., averaged over b^d lattice sites around a site $\mathbf{x}$, $\tilde{s}_{\mathbf{x}} \equiv \frac{1}{b^d}\sum_{\mathbf{x}+\delta\mathbf{x}}^{b^d}$, or "majority" rule (decimated Ising spins replaced by the sign of their total) are possible alternatives for the coarse-graining field.

It is often convenient (but absolutely not necessary) to include a rescaling of the new coarse-grained degrees of freedom, $S'_{\mathbf{x}'}$ relative to the original ones, $S_{\mathbf{x}}$,

$$S'_{\mathbf{x}'} = b^{-\zeta} S_{b\mathbf{x}'}, \tag{40}$$

a freedom afforded by the fact that both are variables that are traced over. This will be important when we discuss the rescaling of correlation functions that explicitly depend on $S_{\mathbf{x}}$, but will not be done in this section.

Focussing on a partition function (though the procedure can be applied to any physical quantity, e.g., a correlator, as well) for a system with a Hamiltonian characterized by a set of operators $\{\phi_\alpha\}$ with coupling constants $\{h_\alpha\}$, we have

$$\begin{aligned} Z_N(\{h_\alpha\}) &= \mathrm{Tr}_{S_{\mathbf{x}}} e^{-\beta H[S_{\mathbf{x}}, h_\alpha]}, & (41)\\ &= \mathrm{Tr}_{S'_{\mathbf{x}'}} \left[\mathrm{Tr}_{S_{b\mathbf{x}'}} \prod_{\mathbf{x}'} \delta_{S'_{\mathbf{x}'}, S_{b\mathbf{x}'}} e^{-\beta H[S_{b\mathbf{x}'}, h_\alpha]} \right], & (42)\\ &\equiv \mathrm{Tr}_{S'_{\mathbf{x}'}} e^{-\beta H'[S'_{\mathbf{x}'}, h'_\alpha]} = Z_{N'=N/b^d}(\{h'_\alpha\}), & (43) \end{aligned}$$

where

$$e^{-\beta H'[S'_{\mathbf{x}'}, h'_\alpha]} \equiv \mathrm{Tr}_{S_{b\mathbf{x}'}} \prod_{\mathbf{x}'} \delta_{S'_{\mathbf{x}'}, S_{b\mathbf{x}'}} e^{-\beta H[S_{b\mathbf{x}'}, h_\alpha]} \tag{44}$$

is a trace over $N(1 - 1/b^d)$ spins $S_{\mathbf{x}}$ with a constraint that every b^d-th spin $S_{\mathbf{x}=b\mathbf{x}'}$ is fixed at values $S'_{\mathbf{x}'}$, to be traced over at the next coarse-graining iteration. The effective, coarse-grained Hamiltonian, $H'[S'_{\mathbf{x}'}, h'_\alpha(b)]$,

$$H[S_{\mathbf{x}}, \{h_\alpha\}] \stackrel{\mathcal{R}_b}{\to} H'[S'_{\mathbf{x}'}, \{h'_\alpha(b)\}] \tag{45}$$

is characterized by an infinite dimensional space of couplings

$$\{h'_\alpha(b)\} = \mathcal{R}[\{h_\alpha\}, b] \tag{46}$$

that "flow" with b under the coarse-graining procedure, as determined by the above RG transformation $\mathcal{R}_b$. Although for real-space Migdal-Kadanoff numerical implementation[7] of this RG procedure, b is a small integer, it is convenient to think of $b \to 1^+ = e^{\delta\ell}$, where $\delta\ell \to 0^+$, in which case the iteration procedure is continuous and can thus be characterized by a continuous evolution of the effective couplings $\{h'_\alpha(\ell)\}$. As we will see in the next section, the latter RG coupling constants "flow" can then be efficiently described by differential equations, with ℓ the RG "time".

B. RG flows, fixed points, and critical exponents

While here we argued for the existence of the flows based on real-space RG, independent of the RG implementation, quite generally the flows contain all the information about the

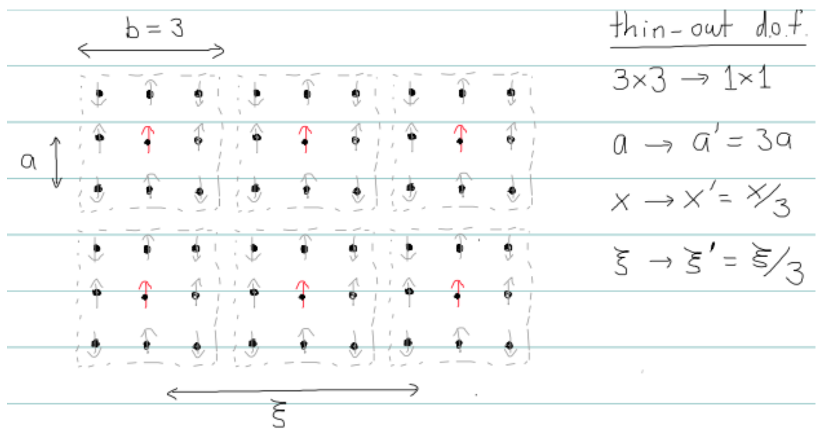


FIG. 3: Coarse-graining real-space RG procedure, that thins-out short-scale, high-energy degrees of freedom

universal critical behavior, as we now discuss.

The existence of a critical state (a critical point), corresponds to a *fixed point* $\{h_\alpha(b \to \infty)\} \to \{h^*_\alpha\}$ of the RG transformation

$$\{h^*_\alpha\} = \mathcal{R}[\{h^*_\alpha\}, b \to \infty], \tag{47}$$

as a consequence of self-similarity.

Linearizing the RG flow equations about a critical fixed point $\{h^*_\alpha\}$:

$$\begin{aligned} h'_\alpha(b) - h^*_\alpha &= R_{\alpha\beta}(h_\beta - h^*_\beta), & (48)\\ \delta h'_\alpha(b) &= R_{\alpha\beta}(b)\delta h_\beta, & (49)\\ e^i_\alpha \delta h'_\alpha(b) &= e^i_\alpha R_{\alpha\beta}(b) e^j_\beta e^j_\gamma \delta h_\gamma, & (50)\\ u_i(b) &= \lambda_i(b) u_i, & (51) \end{aligned}$$

where matrix transformation is given by

$$R_{\alpha\beta} = \frac{\partial h'_\alpha}{\partial h_\beta}|_*, \tag{52}$$

and is diagonalized by an orthonormal set of eigenvectors, e^i_α,

$$R_{\alpha\beta} e^i_\beta = \lambda_i e^i_\alpha, \tag{53}$$

with eigenvalues

$$\lambda_i(b) \equiv b^{y_i} = e^{y_i \delta\ell}, \tag{54}$$

and the eigen-couplings given by

$$u_i(b) = e^i_\alpha \delta h'_\alpha(b). \tag{55}$$

The above RG eigenvalue exponents

$$y_i = \frac{\ln \lambda_i}{\ln b} = \partial u'(\ell)/\partial \ell, \quad \text{for } \ell \to 0 \text{ limit} \tag{56}$$

are *universal* (since clearly they only depend on the long-scale properties of the system, such as dimensionality and symmetry of the Hamiltonian) and control the nature of the RG flow about the fixed point, as illustrated in Fig.4 for two couplings $h_1 = K_1, h_2 = K_2$.

We characterize the RG (exponent) eigenvalues y_i by their sign,

- $y_i > 0$ - *relevant*, with the corresponding $u_i(b)$ coupling growing under coarse-graining, i.e., getting further from the fixed point u_* under RG transformation

- $y_i < 0$ - *irrelevant*, with the corresponding $u_i(b)$ coupling vanishing under coarse-graining, i.e., approaching the fixed point u_* under RG transformation

- $y_i = 0$ - *marginal*, with the RG flow of the corresponding u_i coupling determined by the higher order terms in the couplings of the RG transformation, e.g., $C_{ijk}u_j u_k$.

As illustrated in Fig.4 for the PM-FM transition, there is only one relevant coupling, with all other couplings irrelevant, represented by coupling K_1 and a set of couplings $\{K_2\}$, respectively.

I now note a few key observations. One is that the behavior of all coupling constants in the large b limit fall into just three categories, those flowing to the left of, to the right of and along the separatrix into the critical point. These respectively correspond to two (e.g., ordered and disordered FM and PM) phases characterized by two attractive fixed points (not shown), separated by a continuous phase transition, and the third fine-tuned subspace capturing the critical state. As illustrated Fig.4, in a typical experiment the parameter space is traversed via a dotted red curve, tuned by external physical parameters such as e.g., temperature and/or field.

In most cases, irrelevant operators flow to zero (i.e., into the fixed point around which they are irrelevant) and can therefore can be ignored. However, there are physical observables, that depend singularly on an irrelevant coupling, e.g., diverging as the coupling flows to zero. We refer to such irrelevant operator (and associated coupling) as a *"dangerously irrelevant"*. In this case, the flow of such dangerously irrelevant coupling indeed enters the scaling behavior of the corresponding physical observable. One prominent example is that of the quartic coupling $u(b)$ in the ordered state for $d > 4$. Cavalierly neglecting $u(b)$ would imply that the free-energy density scales as $\sim \xi(t)^{-d} \sim t^{d/2}$. While correct for $d < 4$, this is clearly incorrect for $d > 4$, where mean-field theory should hold, predicting $f(t) \sim t^2/u$. The resolution is the need to include the flow of $u(b)$, which gives,

$$f(t) \sim \xi(t)^{-d}/u(b \sim t^{-\nu}) \sim \xi(t)^{-d}/t^{-(4-d)\nu}, \tag{57}$$

$$\sim t^{d/2+(4-d)/2} \sim t^2, \tag{58}$$

as required from mean-field theory.

C. "Matching" of physical observables: scaling theory

Remarkably, with this RG structure we can predict the behavior of physical observables implied by these RG flows through the so-called "matching" procedure, showing that it indeed leads to a universal scaling phenomenology discussed above. We do this by re-expressing each physical observable (that is impossible to calculate directly in a theory with a *long* correlation length ξ, due to strong fluctuation-driven infra-red divergences), to the observable in a coarse-grained theory, characterized by a *shortened* correlation length,

$$\xi'(\{h'_\alpha(b)\}) = b^{-1}\xi(\{h_\alpha\}), \tag{59}$$

due to rescaled lattice constant $a' = ba$. After coarse-graining to the level that the correlation length ξ' is reduced to a lattice constant a', the computation in the coarse-grained theory is easily done in a convergent perturbation theory (to lower order just the Gaussian theory), since, unlike the original model, the coarse-grained theory is short-range correlated.

One key observable is the free-energy density (from which all of thermodynamics can be deduced),

$$f(\{h_\alpha\}) = -\frac{k_B T}{N} \ln Z_N = -\frac{k_B T}{N' b^d} \ln Z'_{N'}(\{h'_\alpha(b)\}) = b^{-d} f(\{h'_\alpha(b)\}) + \delta f(b). \tag{60}$$

derived from the partition function $Z_N(h_\alpha) = Z'_{N'}(h'_\alpha(b))$, that, by an earlier definition is invariant under the RG transformation. Above, $\delta f(b)$ is nonsingular additive free-energy contribution that can therefore be neglected.

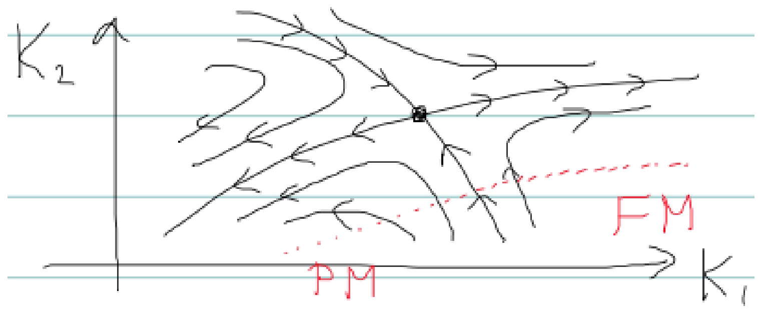


FIG. 4: RG flow and fixed point

For concreteness we apply this general discussion to the Ising model (ϕ^4 field theory),

$$H = \int d^d x \left[\frac{K}{2}(\nabla\phi_{\mathbf{x}})^2 + \frac{t}{2}\phi_{\mathbf{x}}^2 + u\phi_{\mathbf{x}}^4 + ... - h\phi_{\mathbf{x}} \right], \tag{61}$$

where $\{h_\alpha\} = (h, t, K, u, L, \ldots)$ and L is system size that can be thought of as just another coupling constant, $L'(b) = b^{-1}L$, that flows to infinity in thermodynamic limit $L \to \infty$. (To simplify notation we take $t = |t|$.) With this the correlation length rescales as,

$$\xi(h, t, K, u, \ldots) = b\xi(h(b), t(b), u(b), \ldots), \tag{62}$$

$$= b\xi(b^{y_h} h, b^{y_t} t, u^*), \tag{63}$$

where by assumption of a critical point $u(b)$ (along with all other couplings represented by $\ldots$) flows to the critical fixed point $u(b \to) = u^*$. We say that it is relevant around the Gaussian, $u = 0$ fixed point, but is irrelevant at the critical fixed point, u^*.

In contrast h and t are *relevant* perturbations around the critical point since they cut off critical fluctuations by ordering ϕ. At $h = 0$ and choosing b such that $b^{y_t} t = t_0 \approx \Lambda^2$, allows us to eliminate b in favor of t, (physically amounts to course-graining a critical theory out to the scale of the correlation length, $\xi' = a'$, beyond which the model is weakly coupled),

$$\xi(h = 0, t) = (t/t_0)^{-1/y_t} a \sim t^{-1/y_t} \equiv t^{-\nu}, \tag{64}$$

giving the universal correlation length critical exponent in terms of the RG eigenvalue

$$\nu = 1/y_t. \tag{65}$$

At the critical temperature, $t = 0$, but finite field h, choosing instead $b^{y_h} h = h_0$ we instead find the field-induced correlation length

$$\xi(h, t = 0) \sim h^{-1/y_h}. \tag{66}$$

We can apply similar matching analysis to the singular part of the free energy density, that, being a density scales as b^{-d},

$$\begin{aligned} f(h, t, u) &= b^{-d} f(h(b), t(b), u(b), \ldots), & (67)\\ &= b^{-d} f(b^{y_h} h, b^{y_t} t, u^*), & (68)\\ &\sim t^{d/y_t} f(h/t^{y_h/y_t}, t_0, u^*) \equiv t^{d\nu} f(h/t^{\Delta}) = t^{d\nu} f(\xi_h/\xi_t). & (69) \end{aligned}$$

In above we again eliminated the rescaling factor b by choosing $b^{y_t} t = t_0$, and thereby derived the scaling form proposed by Kadanoff, with

$$\Delta = y_h/y_t, \tag{70}$$

Armed with the free-energy density in a field h, we can now derive scaling expressions for any thermodynamic observable, by simply differentiating with respect to h and t. The uniform magnetization is then given by

$$\begin{aligned} m(h, t) &= \langle S \rangle = \frac{\partial f}{\partial h}, & (71)\\ &= b^{-d+y_h} f'(b^{y_h} h, b^{y_t} t), & (72)\\ &= t^{(d-y_h)/y_t} f'(h/t^{y_h/y_t}) \equiv t^{\beta} f'(h/t^{\Delta}), & (73)\\ &= \begin{cases} t^{\beta}, & \text{for } h \ll t^{\Delta}, \text{ (equivalent to } \xi_h \gg \xi), \\ h^{1/\delta}, & \text{for } h \gg t^{\Delta}, \text{ (equivalent to } \xi_h \ll \xi), \end{cases} & (74) \end{aligned}$$

where we expressed the critical exponents in terms of RG eigenvalues,

$$\beta = d\nu - \Delta, \quad \delta = \frac{\Delta}{\beta}. \tag{75}$$

Above we used the fact that magnetization is finite at $t = 0$ and nonzero h to deduce that the scaling function $g_1(x) \sim x^{\beta/\Delta}$ in order to have a finite $t \to 0$ limit (i.e., have t drop out).

Continuing along these lines we can derive the scaling form for the uniform magnetic susceptibility,

$$\begin{aligned} \chi(h, t) &= \frac{\partial^2 f}{\partial h^2} = b^{-d+2y_h} f''(b^{y_h} h, b^{y_t} t), & (76)\\ &= t^{(d-2y_h)/y_t} f''(h/t^{y_h/y_t}) \equiv t^{-\gamma} f''(h/t^{\Delta}), & (77)\\ &\sim t^{-\gamma}, \text{for } h \to 0, & (78) \end{aligned}$$

and the corresponding universal γ exponent,

$$\begin{aligned}\gamma &= 2y_h/y_t - d/y_t = 2\Delta - d\nu, &(79)\\ &= (2-\eta)\nu. &(80)\end{aligned}$$

The two-point correlation function can also be obtained. In Fourier space it is given by

$$\begin{aligned}\chi(q,t) &= b^{2-\eta}\chi_0(bq, b^{1/\nu}t) = t^{-(2-\eta)\nu}\chi_0(qt^{-\nu}), &(81)\\ &\sim \chi(q,t=0) \sim q^{-(2-\eta)}, \text{for } q\xi(t) \gg 1, &(82)\end{aligned}$$

where η is the universal critical exponent related to the field rescaling ζ above (often called "wavefunction renormalization" b^ζ), that determines the correction to the power-law of the correlation function. From this the real-space correlator is obtained by inverse Fourier transform

$$\begin{aligned}G(x,t) &= \langle\phi(\mathbf{x})\phi(0)\rangle \stackrel{x\to\infty}{\to} \langle\phi_{\mathbf{x}}\rangle^2 = \int \frac{d^dq}{(2\pi)^d}\chi(q,t)e^{i\mathbf{q}\cdot\mathbf{x}}, &(83)\\ &= b^{-(d-2+\eta)}\int \frac{d^d(bq)}{(2\pi)^d}\chi(bq, b^{y_t}t)e^{i(b\mathbf{q})\cdot(\mathbf{x}/b)} = b^{-(d-2+\eta)}G(x/b, b^{1/\nu}t), &(84)\\ &\sim \begin{cases} 1/x^{d-2+\eta}, & \text{for } x \ll t^{-\nu}, \text{(equivalent to } x \ll \xi(t)),\\ 1/\xi^{d-2+\eta} \sim t^{(d-2+\eta)\nu} \equiv t^{2\beta}, & \text{for } x \gg t^{-\nu}, \text{(equivalent to } x \gg \xi(t)),\end{cases} &(85)\end{aligned}$$

where

$$\beta = \frac{1}{2}(d-2+\eta)\nu, \tag{86}$$

a hyperscaling relation satisfied by mean-field exponents only at the upper-critical dimension $d = 4$. More generally, a requirement that the partition function is invariant under RG transformation leads to an n-point correlator of an operator O_α, that couples to a field h_α with dimension y_α, scales as $1/x^{n(d-y_\alpha)}$, with d coming from a d-dimensional integral. Combining above hyperscaling exponent relation with (80) to eliminate η, I find another relation

$$\gamma + 2\beta = d\nu. \tag{87}$$

The heat capacity is also given by

$$\begin{aligned}c_v(t) &= -T\frac{\partial^2 f}{\partial T^2} \sim \frac{\partial^2 f}{\partial t^2} \sim \frac{\partial^2\left[\xi(t)^{-d}\right]}{\partial t^2}, &(88)\\ &\sim |t|^{d\nu-2} \equiv t^{-\alpha}, &(89)\end{aligned}$$

with

$$\alpha = 2 - d\nu \tag{90}$$

These scaling behavior and the associated critical exponents are summarized in the figure below,

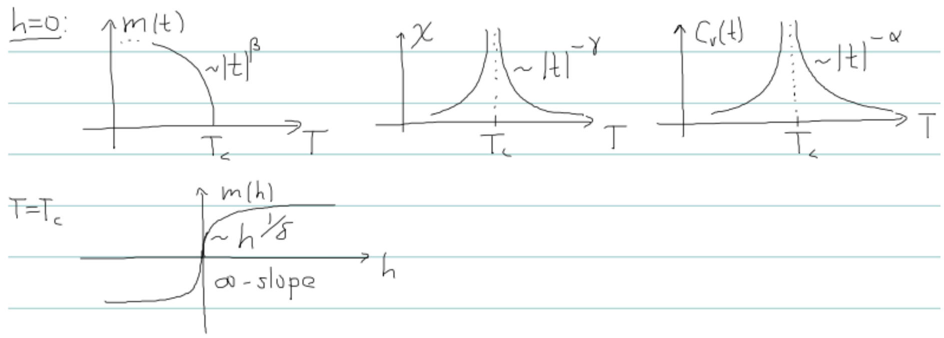


FIG. 5: Critical behavior of various physical observables and associated critical exponents.

III. REAL-SPACE RG FOR ISING MODEL

A. One-dimensional Ising model

Although a 1d this model can be solved exactly by a number of other ways (e.g., by introducing decoupled bond "spins" $\sigma_x = s_x s_{x+1}$, or by diagonalizing its transfer matrix $T_{s_x s_{x+1}} = e^{K s_x s_{x+1}}$), it is instructive to use a real-space spin-decimation RG method. Unlike higher dimensions or in very specially constructed models, in 1d Ising model this coarse-graining Migdal-Kadanoff RG can be performed exactly[7, 12].

As illustrated in the Fig.6, we decimate the lattice into blocks of triplets ($b = 3$), keeping every third spin $s'_{x'} = s_{3x'}$ fixed as the effective spin representing each block, we trace over two-thirds of all the spins s_x, thereby obtaining an effective coarse-grained Hamiltonian.

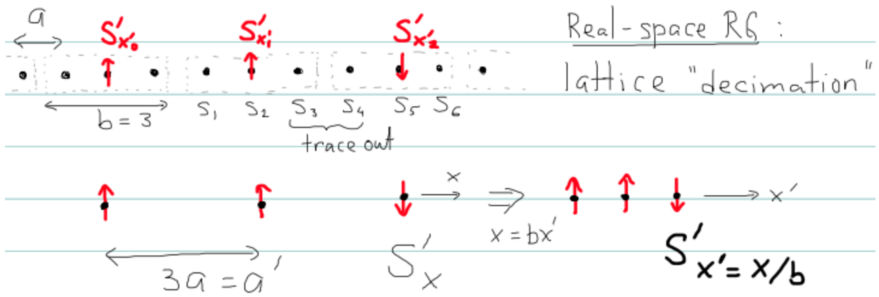


FIG. 6: RG procedure for 1D Ising model

To this end, we execute this block-decimation RG on the partition function,

$$
\begin{aligned}
Z &= \mathrm{Tr}_{\{s_x\}} e^{\sum_x^N K s_x s_{x+1}} = \mathrm{Tr}_{\{s_x\}} \prod_x^N e^{K s_x s_{x+1}} = \mathrm{Tr}_{\{s_x\}} \left[T_{s_1 s_2} T_{s_2 s_3} \dots T_{s_{N-1} s_N} \right], && (91)\\
&= \mathrm{Tr}_{\{s'_{x'}\}} \mathrm{Tr}'_{\{s_x\}} \left[\dots e^{K s'_1 s_3} e^{K s_3 s_4} e^{K s_4 s'_2} \dots \right], && (92)\\
&= \mathrm{Tr}_{\{s'_{x'}\}} \mathrm{Tr}'_{\{s_x\}} \left[\dots \{ \cosh^3 K (1 + s'_1 s_3 \tanh K)(1 + s_3 s_4 \tanh K)(1 + s_4 s'_2 \tanh K) \} \dots \right], \\
&= \mathrm{Tr}_{\{s'_{x'}\}} \left[\dots \{ 2^2 \cosh^3 K (1 + s'_{1'} s'_{2'} \tanh^3 K) \} \dots \right], && (93)\\
&\equiv \mathrm{Tr}_{\{s'_{x'}\}} \prod_{x'}^{N'=N/3} e^{-3\delta f + K' s'_{x'} s'_{x'+1}} && (94)
\end{aligned}
$$

where we defined $K \equiv J/k_B T$ and re-expressed the transfer matrix in a convenient form

$$
T_{s_x s_{x+1}} = e^{K s_x s_{x+1}} = \cosh K (1 + s_x s_{x+1} \tanh K). \tag{95}
$$

In above real-space RG computation we used the fact that in tracing over decimated spins the only terms that do not vanish are those where each traced-over spin appears quadratically, giving a factor of 2 for each spin sum. In the last line above we noted that after decimation the partition for the remaining spins $s'_{x'}$ is governed by a Hamiltonian with the identical form (hence the transformation is exact),

$$
H'(s'_{x'}, K') = N\delta f(K) - K' \sum_{x'} s'_{x'} s'_{x'+1}, \tag{96}
$$

where the first term is the spin-independent coarse-graining correction to the overall free energy, and

$$
\begin{aligned}
K' &= \tanh^{-1}[(\tanh K)^3], && (97)\\
\delta f &= -\frac{1}{3} \ln \left[\frac{(\cosh K)^3}{\cosh K'} \right] - \frac{2}{3} \ln 2 && (98)
\end{aligned}
$$

that can be more simply re-expressed in terms of the magnetization-like coupling, $m < 1$,

$$m \equiv \tanh K = \tanh\left(\frac{J}{k_B T}\right) \tag{99}$$

giving,

$$m' = m^3, \tag{100}$$

$$\tag{101}$$

that more generally is given by $m'(b) = m^3$, with $b = 3$.

The solution to this flow equation for $m(b)$ is straightforwardly analyzed. Because m gets raised to b-th power under decimation, it is clear that unless $m = 1$ (corresponding to infinite $J/k_B T$, i.e., zero temperature), $m(b)$ always flows to zero, corresponding to $J/k_B T \to 0$, i.e., infinite temperature. Thus, as illustrated in Fig.7, there 1d Ising model is characterized by two $T^* = 0$ and $T^* = \infty$ fixed points (or equivalently $J^* = \infty$ and $T^* = 0$, respectively), with $T^* = 0$ unstable to arbitrary small thermal fluctuations. This behavior is consistent to

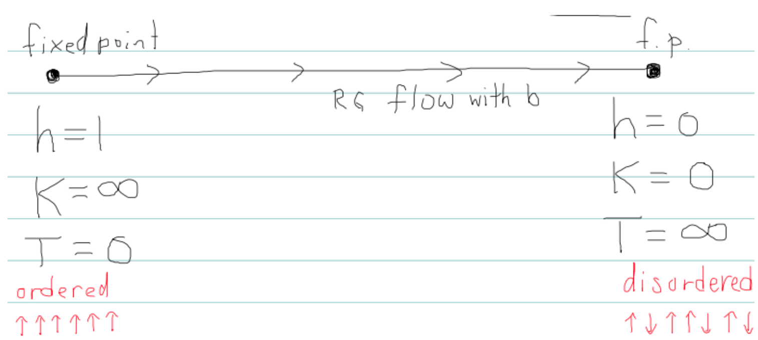


FIG. 7: RG flow and fixed points for 1d Ising model.

our discussion in earlier lectures on fluctuations-instability of the ferromagnetic state in 1d Ising model due to nonzero probability of spin-flip (domain-wall) excitations that destroy the ordered state. This is also a reflection of the more general phenomena that discrete classical systems have a lower-critical dimension $d_{lc} = 1$, at and below which order is unstable to thermal fluctuations.

We can now use above RG analysis to extract the correlation length. The physical *dimensionful* correlation length ξ is the same, whether computed using microscopic or coarse-grained degrees of freedom. Because the Hamiltonian form is unchanged under RG transformation, the *dimensionless* correlation lengths (i.e., measured in units of corresponding lattice constants a and $a' = ba$) are identical functions of m, related by a ratio of lattice constants,

$$\begin{aligned} \xi(m,a) &= \xi(m',a'), && (102)\\ a\hat{\xi}(m) &= a'\hat{\xi}(m(b)), && (103)\\ \xi(m) &= b\xi(m^b), && (104) \end{aligned}$$

which is solved exactly by

$$\begin{aligned} \xi(m) &= \frac{const.}{-\ln m} = \frac{const.}{-\ln\tanh(J/k_BT)}, && (105)\\ &\overset{T\to 0}{\approx} \frac{1}{-\ln(1-2e^{-J/k_BT})} \sim ae^{J/k_BT}, \quad \text{at low T, for } k_BT \ll J. && (106) \end{aligned}$$

Alternatively, we can employ a "matching" calculation by choosing b such that m^{b_*} is a small constant c (corresponding to infinite temperature), at which $\xi(m^{b_*}) = \xi(c) = a$, giving $b_* = -const./\ln m$ and leads to the same result as the exact expression, above.

The solution shows that the correlation length is always finite, though diverging exponentially as $T \to 0$ (a common feature at the lower critical dimension). Thus, at finite T the FM phase is unstable in a 1d Ising model and there is only single PM phase, characterized by a finite ξ.

B. Higher dimensions

The above Migdal-Kadanoff decimation can and has been applied to higher dimensions, though it can no longer be done exactly, even asymptotically[7].

However, in higher dimensions, quite generally at low T, we have

$$K' \sim b^{d-1}K, \tag{107}$$

showing that for $d > 1$ the effective coupling grows under coarse-graining. This is associated with the $d-1$ dimensional size of the domain wall (that vanishes for $d = 1$), whose energy grows as L^{d-1}. Thus, for $d > 1$ the $K^* = \infty$, $T^* = 0$ fixed point and the associated FM phase are *stable* to thermal fluctuations. Clearly at high temperature the $K^* = 0$, $T^* = \infty$ fixed point is stable and model exhibits a stable PM phase. This thus generically predicts flows illustrated in Fig.8 and a genuine PM-FM critical point and the associated phase transition.

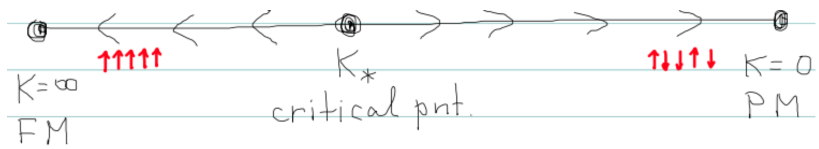


FIG. 8: Critical point and fixed points

To make further progress in a controlled analysis, we next turn to field-theoretic formulation of the RG.

IV. FIELD-THEORETIC MOMENTUM-SHELL RG ANALYSIS OF ϕ^4 MODEL

Although real-space RG is conceptually very clear, outside of 1d it is not practical to implement analytically, nor is it typically "controlled" by a small parameter to ensure its even asymptotic accuracy. Instead we turn to the complementary implementation, the so-called momentum-shell field-theoretic RG and ϵ-expansion, a practical tool developed by Ken Wilson and Michael Fisher.[13]

A. Recap of general RG analysis

Let us begin by first summarizing qualitatively the key steps in the RG procedure.

1. Integrate out a $1/b^d$ fraction ($b \equiv e^{\delta\ell} > 1$, $\delta\ell > 0$) of the total number of degrees of freedom, $\phi_>$ corresponding to high-energy, short-scale, large k degree of freedom, $\Lambda/b < k < \Lambda \equiv 2\pi/a$, thereby obtaining corrections to the effective coarse-grained Hamiltonian $H_<[\phi_<, \Lambda/b]$, as a function of the remaining longer scale degrees of freedom $\phi_<$, with support in $k < \Lambda/b$. In contrast to real-space RG, in momentum-shell RG we organize the degrees of freedom by shells in momentum and energy space.

2. Rescale length scales $\mathbf{x} = b\mathbf{x}'$ and momenta $\mathbf{k} = \mathbf{k}'/b$ (sometimes anisotropically), such that the new UV cutoff $k < \Lambda/b$ (and in real space $x > ba$) is rescaled back to $k' < \Lambda$ (and in real space $x' > a$). Although this step is in principle unnecessary, it is convenient because it allows us not to have to keep track of the UV cutoff, as then the effective Hamiltonian $H'_b[\phi'(x')]$ is defined with the same cutoff as the original one. Along with this, it is sometimes convenient (but not necessary) to also rescale the

fields, $\phi_<(b\mathbf{x}') = b^\zeta \phi'(\mathbf{x}')$, or, equivalently in momentum space $\phi_<(\mathbf{k}'/b) = b^{d+\zeta}\phi'(\mathbf{k}')$, and select the rescaling dimension ζ so as to keep one of the couplings in H fixed under RG.

3. Identify new effective couplings, $\{h_i(b)\} = \{K(b), t(b), u(b)\}$ in terms of the original, short-scale (bare) couplings and the rescaling parameter b, such that the new Hamiltonian functional, $H'_b[\phi'(x')]$ has the same form as the original one, H. These associated RG flows with $b = e^{\delta\ell}$ are conveniently formulated in terms of differential equations for couplings $\{h_\alpha(\ell)\}$. For large b, some of couplings (e.g., $u(b)$) will flow to a fixed point, others will grow with b (e.g., $t(b)$, $h(b)$) and thus their physical (bare) values will need to be tuned to zero to access a critical fixed point. The eigenvalues y_i of the relevant couplings (e.g., $\delta t(b) = b^{y_t}\delta t$) around the fixed point of interest will give the universal critical exponents, with e.g., the correlation length exponent $\nu = 1/y_t$. The η exponent comes from the corrections to the exchange stiffness $K(b)$, with others exponents often obtainable from the scaling relations or by directly computing correlation functions, via matching analysis.

4. Establish a relation between correlation functions of interest, $C^{(n)}(\mathbf{k}, K, t, u, \ldots)$ at $b = 1$, that is difficult to calculate because the system is near-critical, with k and t small and correlation length large to $C_b^{(n)}(k(b), K(b), t(b), u(b), \ldots)$. Choosing $b > 1$, such that $k(b)$ and/or $t(b)$ are large (at UV cutoff) and other coupling constants go to a fixed point, $u(b \to \infty) \to u^*$, allows $C_{b_*}^{(n)}(\mathbf{k}b_*, K(b_*), t(b_*), u(b_*), \ldots)$ to be computed in a convergent perturbation theory, thereby obtaining the original, critical correlation function, $C^{(n)}(\mathbf{k}, K, t, u, \ldots)$.

A few conceptual comments are in order here. I emphasize that above, the first step is the most crucial and calculationally intensive one.

It is also important to note, that, because only a thin momentum shell of short-scale (non-critical) degrees of freedom are integrated out at each step, the procedure (in contrast to the direct perturbation theory where all degrees of freedom are traced over) is guaranteed to be IR divergences free. Nevertheless, the RG procedure is perturbative in the nonlinear couplings, u, and is thus controlled only if the coupling $u(b)$ remains small, requiring the fixed point u^* to be "close" to the Gaussian fixed point $u^* = 0$. Wilson and Fisher's insight[13] is the observation that this happens in dimension d below but close to the upper-critical dimension d_{uc} (= 4 for the O(n) model), thereby allowing the so-called ϵ-expansion in $\epsilon = d_{uc} - d$ to make RG well controlled, with $u^* = O(\epsilon)$. This requires computations in an arbitrary fractional dimension d close to d_{uc}, which is done as analytical continuation in d. At the end of the analysis, the physical predictions are obtained by evaluating exponents and other observables in the physical dimension of interest, sometimes extending ϵ to uncomfortably large values of 1 or 2. The expectation is that there is no qualitative change in behavior that takes place as a function of ϵ and thus results obtained for small ϵ (for d near d_{uc}) remain (at least) qualitatively valid in the physical dimension.

Finally, the rescaling $\mathbf{x} = b\mathbf{x}'$ may also generalize to anisotropic rescaling and in dynamical (e.g., quantum) problems can be supplemented by rescaling of time $t = b^z t'$, with dynamical rescaling exponent $z = 1$ only in effectively relativistic fixed point, but more general to be determined by the RG analysis.

Keeping this general protocol and caveats in mind, we now turn to the detailed implementation and analysis.

B. Scaling by dimensional analysis: zeroth-order RG

As I hope is clear from our general discussion above, the overarching goal of RG analysis is to assess the behavior of the system at long scales, as characterized by a coarse-grained effective Hamiltonian. More specifically, we need to quantify the relative amplitudes of various components (operators) of the Hamiltonian. In the RG procedure summarized above, this involves coarse-graining trace over a fraction of short-scale fields, done perturbatively in the nonlinearities of the Hamiltonian and rescaling length scales to reinstate the original cutoff to a.

To zeroth order in the nonlinear coupling one can simply neglect the first coarse-graining step, and only carry out the rescaling step

$$\mathbf{x} = b\mathbf{x}', \tag{108}$$

$$\phi_<(b\mathbf{x}') = b^\zeta \phi'(\mathbf{x}'), \tag{109}$$

or equivalently in momentum space

$$\mathbf{k} = \mathbf{k}'/b, \tag{110}$$

$$\phi_<(\mathbf{k}'/b) = b^{d+\zeta}\phi'(\mathbf{k}'), \tag{111}$$

in what amounts to dimensional analysis of various terms in the Hamiltonian. Applying this to the ϕ^4 theory in coordinate space, we obtain,

$$H[\phi(\mathbf{x})] = \int d^dx \left[\frac{1}{2}K(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}t\phi^2 + u\phi^4\right], \tag{112}$$

$$H[\phi'(\mathbf{x}')] = \int d^dx' \left[\frac{1}{2}(b^{d-2+2\zeta}K)(\boldsymbol{\nabla}'\phi')^2 + \frac{1}{2}(b^{d+2\zeta}t)\phi'^2 + (b^{d+4\zeta}u)\phi'^4\right], \tag{113}$$

$$= \int d^dx' \left[\frac{1}{2}K'(b)(\boldsymbol{\nabla}'\phi')^2 + \frac{1}{2}t'(b)\phi'^2 + u'(b)\phi'^4\right], \tag{114}$$

where the effective couplings on scale enlarged by b are given by

$$K(b) = b^{d-2+2\zeta}K, \tag{115}$$

$$t(b) = b^{d+2\zeta}t, \tag{116}$$

$$u(b) = b^{d+4\zeta}u. \tag{117}$$

To assess the physical implication of these flows it is crucial to look at the *dimensionless* measure of the nonlinear coupling u, and the relative measure of K and t. It is clear from the above flows that $t(b)$ grows by an extra factor of b^2 relative to $K(b)$, a reflection that in the Hamiltonian K multiplies a term with two extra powers of gradient. Physically this reflects the fact that at scales longer by b the gradient becomes smaller by a factor b^{-1}, which can be interpreted as an effectively weaker K by a factor of b^{-2}. In the RG parlance we say that t is a relevant perturbation at the critical point and must be tuned to zero to remain at the fixed point.

The dimensionless coupling $\hat{u} \sim \Lambda^{d-4}u/K^2$, measuring the strength of u relative to K, can be extracted from the perturbative expansion for $\Gamma^{(4)}$ in the Field Theory Primer Lecture 4, or simply by dimensional analysis, and is given by

$$\hat{u} = C_d\Lambda^{d-4}\frac{u}{K^2}, \tag{118}$$

$$\sim \left(\frac{a}{\xi_G}\right)^{4-d}, \tag{119}$$

with $\xi_G = (K/t_G)^{1/2}$ and constant factors of course arbitrary, included just for cosmetics.

With this we find that the flow for $\hat{t}(b) \equiv t(b)/K(b)$ and $\hat{u}(b)$ are given by

$$\hat{t}(b) = b^2 \hat{t}, \tag{120}$$

$$\hat{u}(b) = b^{4-d} \hat{u}, \tag{121}$$

importantly, we observe that the arbitrary rescaling exponent ζ dropped out from the flow of these physical couplings.

I also note that the same result can more conventionally obtained by choosing the field rescaling ζ, such that $K(b)$ does not flow, i.e., $K(b) = K$, which gives

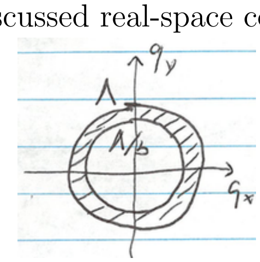
$$\zeta = (2-d)/2. \tag{122}$$

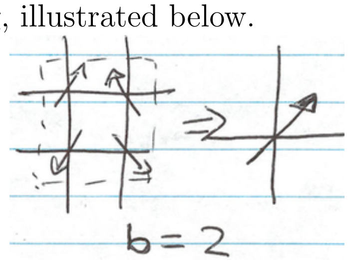
Then using this ζ inside the flow for $t(b)$ and $u(b)$ we obtain the same flows as those for the dimensionless couplings in (120), (**??**).

Either methods then demonstrates that the upper-critical dimension for the quartic nonlinear coupling u, is d $\epsilon = 4 - d$ is the RG eigenvalue for the quartic nonlinear coupling u, and $y_t = 2$ near the $u^* = 0$ (the so-called) Gaussian fixed point.

C. Momentum-shell field theory analysis: 1st order RG

We turn to the full momentum-shell RG analysis[**?**], that most importantly now involves the coarse-graining step, illustrated below, that is the momentum-space analog of previously discussed real-space course-graining, illustrated below.





To this end, perturbatively in nonlinearity u, we integrate out the high-momentum fields $\phi_>$, that take support in a infinitesimal momentum shell $\Lambda/b < k < \Lambda \equiv 1/a$ (vanishing outside of it), thereby obtaining the effective Hamiltonian as a function of the long-scale fields $\phi_<$, that take support at all lower momenta, $0 < k < \Lambda/b$ (vanishing outside of this disk of radius Λ/b). These field components are naturally related to the full field,

$$\phi = \phi_< + \phi_>, \tag{123}$$

where I did not specify whether the fields are in coordinate or momentum space, as the initial analysis below is independent of this choice, and can be done in either space.

After perturbatively integrating out $\phi_>$, purely for convenience, we will follow with rescaling of lengths and fields according to (160) and (111), so as to restore the UV cutoff $b^{-1}\Lambda \equiv e^{-\delta\ell}\Lambda$ back to $\Lambda = 1/a$.

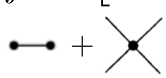
$$
\begin{aligned}
Z &= \int [d\phi] e^{-H[\phi]} = \int [d\phi_<][d\phi_>] e^{-H[\phi_< + \phi_>, \Lambda]}, && (124)\\
&\equiv \int [d\phi_<] e^{-H'[\phi_<, \Lambda/b]} = \int [d\phi'] e^{-H_b[\phi', \Lambda]} && (125)
\end{aligned}
$$

where $H'[\phi_<, \Lambda/b] = H_b[\phi', \Lambda]$ is the coarse-grained effective Hamiltonian of interest, that we will compute perturbatively below in the nonlinearity

$$
\begin{aligned}
u\phi^4 &= u(\phi_< + \phi_>)^4,\\
&= u\left(\phi_<^4 + 4\phi_<^3\phi_> + 6\phi_<^2\phi_>^2 + 4\phi_<\phi_>^3 + \phi_>^4\right). && (126)
\end{aligned}
$$

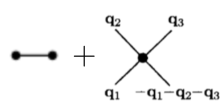
We recall that at long scales the Ising model is characterized by a field-theoretic Hamiltonian, that in coordinate and momentum spaces is given by,

$$
H[\phi] = \int d^d x \left[\frac{1}{2}K(\boldsymbol{\nabla}\phi_{\mathbf{x}})^2 + \frac{1}{2}t\phi_{\mathbf{x}}^2 + u\phi_{\mathbf{x}}^4 + \ldots\right], \qquad (127)
$$



$= \quad + \quad + \ldots, \qquad (128)$

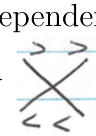
$$
= \frac{1}{2}\int_{\mathbf{q}} (Kq^2 + t)|\phi_{\mathbf{q}}|^2 + u\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} (2\pi)^d \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4)\phi_{\mathbf{q}_1}\phi_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\phi_{\mathbf{q}_4} + \ldots, \qquad (129)
$$



$= \quad + \quad + \ldots, \qquad (130)$

$$
= H_0[\phi] + H_i[\phi], \qquad (131)
$$

where H_i is the quartic interaction graphically represented by a 4-point vertex.

We carry out the momentum-shell coarse-graining analysis, defined by (125) by expanding e^{-H_i} to second order in u, tracing over $\phi_>$ (that will generate and infinite number of terms), and re-exponentiating the result (which eliminates disconnected graphs) to assess the correction $\delta H[\phi_<]$, that can be absorbed into the redefinition of scale-dependent couplings $K(b), t(b), u(b)$. A typical leading term $u\phi_<^2\phi_>^2$ is graphically given by . Diagrams with no external $\phi_<$ legs, arise from the last term in (126) involving only $\phi_>$, lead to field-independent constants, which correct the actual full free energy (the constant part of the coarse-grained Hamiltonian functional), usually not of interest.

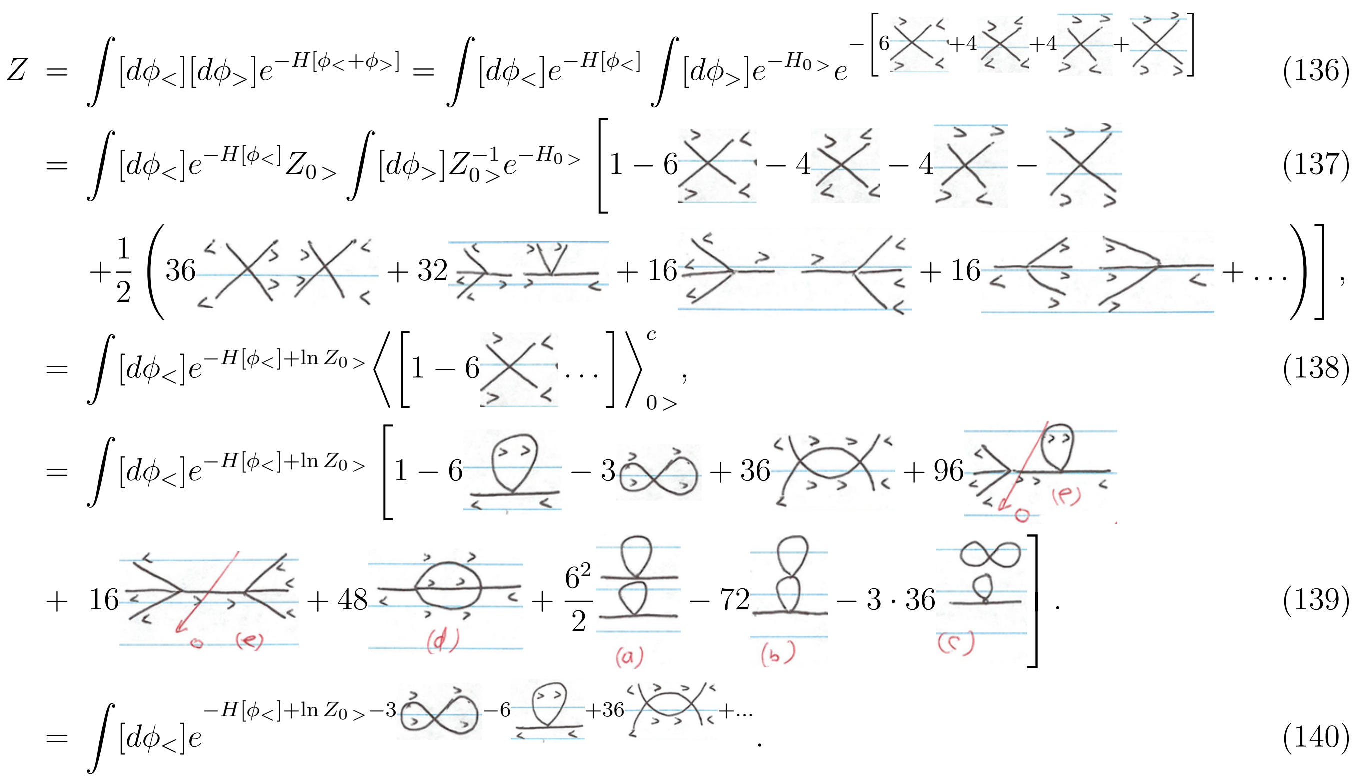
Formally, the perturbative RG expansion is given by,

$$
\begin{aligned}
Z &= \int [d\phi] e^{-H[\phi]} = \int [d\phi_<][d\phi_>] e^{-H[\phi_< + \phi_>, \Lambda]}, && (132)\\
&= \int [d\phi_<] e^{-H[\phi_<]} \int [d\phi_>] e^{-H_{0>} - H_{i>}} = \int [d\phi_<] e^{-H[\phi_<]} \int [d\phi_>] e^{-H_{0>}} \left[1 - H_{i>} + \frac{1}{2!} H_{i>}^2 - \ldots \right], \\
&= \int [d\phi_<] e^{-H[\phi_<]} Z_{0>} \left[1 - \langle H_{i>} \rangle_{0>} + \frac{1}{2} \langle H_{i>}^2 \rangle_{0>} - \ldots \right], && (133)\\
&\approx \int [d\phi_<] e^{-H[\phi_<] - \delta H_b[\phi_<]}, && (134)
\end{aligned}
$$

where

$$
\delta H_b[\phi_<, \Lambda/b] = -\ln Z_{0>} + \langle H_{i>} \rangle_{0>} - \frac{1}{2} \langle H_{i>}^2 \rangle_{0>}^c + \ldots, \qquad (135)
$$

Diagrammatically and more explicitly, we have,



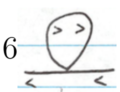
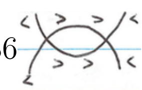
A number of comments are in order:

- Upon exponentiating the terms (a) and (c) have disappeared in the last line since they are disconnected graphs, that do not contribute to $\delta H_b[\phi_<]$, dropping out upon exponentiation. It is a reflection of observation in Eq.135 that it is only the connected graphs that contribute.

- (b) and (d) are higher order in u, contributing $O(u^2)$ to $\Gamma_{\mathbf{k}}(t)$. We note, however, that the diagram (d) is in fact the lowest order correction that is k-dependent and thus is an important contribution to the RG flow of $K(b)$, that otherwise does not flow under coarse-graining (as we already saw for perturbation theory in the previous lecture).

- two diagrams (e) vanish by momentum conservation, since they have a single line involving high momentum fields $\phi_>$ that is nonzero only in a shell $\Lambda/b < k < \Lambda = 1/a$, while external momenta are all at $k < \Lambda/b$. In complementary field-theoretic (HEP) language these graphs are not 1PI's and are thus do not contribute to the Γ generating functional.

From above, we can now read off the graphical corrections to the coarse-grained Hamiltonian,

$H'[\phi_<, \Lambda/b] = H[\phi_<, \Lambda] - \ln Z_{0>} + 3$  $+ 6$  $- 36$  $+ \ldots,$ (141)

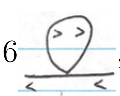
$$\approx \int_{ba} d^d x \left[\delta f(b) + \frac{1}{2}(K + \delta K(b))(\boldsymbol{\nabla}\phi_{\mathbf{x}})^2 + \frac{1}{2}(t + \delta t(b))\phi_{\mathbf{x}}^2 + (u + \delta u(b))\phi_{\mathbf{x}}^4 + \ldots \right],$$

where,

$\delta\Gamma^{(0)} \approx -\ln Z_{0>} + 3$  $= -\frac{1}{2}\ln\left(\prod_{\mathbf{q}}^{>} \frac{2\pi}{Kq^2 + t}\right) + 3$ , (142)

$$= \delta_g f(b), \tag{143}$$

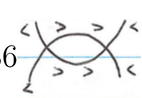
$$= \frac{1}{2}\int_{\mathbf{q}}^{>} \ln[(Kq^2 + t)/2\pi] + 3u\left[\int_{\mathbf{q}}^{>} \frac{1}{Kq^2 + t}\right]^2, \tag{144}$$

$\delta\Gamma^{(2)}(k) \approx -2 \cdot 48$  $+ 2 \cdot 6$ , (145)

(d)

$$= -96u^2 \int_{\mathbf{q}_1}^{>}\int_{\mathbf{q}_2}^{>} \frac{1}{(Kq_1^2 + t)(Kq_2^2 + t)(K(k - q_1 - q_2)^2 + t)} + 12u\int_{\mathbf{q}}^{>} \frac{1}{Kq^2 + t},$$

$$\overset{k\to 0}{=} \delta_g K(b)k^2 + \delta_g t(b), \tag{146}$$

$\delta\Gamma^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = -36$  $\overset{k\to 0}{=} \int_{\mathbf{q}}^{>} \frac{1}{(Kq^2 + t)^2},$ (147)

$$\overset{k\to 0}{=} \delta_g u(b). \tag{148}$$

where the momentum-shell integral is given by $\int_{\mathbf{q}}^{>} \ldots \equiv \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \ldots$, and $\delta_g h_i(b)$ stands for a "graphical" correction to a coupling h_i.

Thus, we find to first-order in interaction strength u and with $b \to 1^+$,

$$\begin{aligned}
\delta_g K(b) &= c\hat{u}^2 K\delta\ell \equiv \eta(\hat{u})K\delta\ell \approx 0, && (149)\\
\delta_g t(b) &= 12u\int_{\mathbf{q}}^{>}\frac{1}{Kq^2+t} = 12uC_d\int_{\Lambda/b}^{\Lambda}\frac{dqq^{d-1}}{Kq^2+t}, && (150)\\
&\approx 12\frac{uC_d\Lambda^{d-2}}{K}\frac{1}{1+t/(K\Lambda^2)}\left(1-b^{-1}\right) \approx \frac{12\hat{u}K\Lambda^2}{1+a^2/\xi_0^2}\ln b, && (151)\\
&\approx 12\hat{u}\left(K\Lambda^2-t\right)\delta\ell, && (152)\\
\delta_g u(b) &= -36u^2\int_{\mathbf{q}}^{>}\frac{1}{(Kq^2+t)^2} = -36u^2C_d\int_{\Lambda/b}^{\Lambda}\frac{dqq^{d-1}}{(Kq^2+t)^2}, && (153)\\
&\approx -36\frac{u^2C_d\Lambda^{d-4}}{K^2}\frac{1}{(1+t/(K\Lambda^2))^2}\left(1-b^{-1}\right) \approx -\frac{36\hat{u}u}{(1+a^2/\xi_0^2)^2}\ln b, && (154)\\
&\approx -36\hat{u}u\delta\ell, && (155)
\end{aligned}$$

where through perturbative RG we "rediscovered" the dimensionless measure of the nonlinearity u, given by $\hat{u} \equiv \frac{C_d\Lambda^{d-4}u}{K^2}$ and near a critical point I expanded to lowest order in t, or more accurately in the vanishing ratio $a/\xi_0(t)$ of the lattice constant a to the Gaussian correlation length $\xi_0 = (K/t)^{1/2}$. Note also, that, although to one-loop order the diagrammatic correction to K vanishes, to two-loop order (with the actual valued packaged inside a dimensionless constant c) it is nonzero, proportional to $\hat{u}^2$ (which will go to ta fixed point value) and, as we will see below, determines the correlation function exponent $\eta \equiv \eta(\hat{u}_*)$.

Above coupling corrections are closely related in their structure to what we found in a direct perturbative expansion in the previous lecture. I note that for infinitesimal momentum-shell, the integral simply reduces to the value of the integrand evaluated $q \approx \Lambda$ times the infinitesimal shell width $\ln b \approx b-1 \approx \delta\ell$, since $b \to 1^+$,

$$\begin{aligned}
\int_{\mathbf{q}}^{>}\frac{1}{q^n} &\approx \frac{S_d}{(2\pi)^d}\left[\Lambda^{d-n}-(\Lambda/b)^{d-n}\right]\frac{1}{d-n} \simeq C_d\Lambda^{d-n}\ln b, && (156)\\
\int_{\mathbf{q}}^{>}\frac{1}{Kq^2+t} &\simeq \frac{C_d\Lambda^d}{K\Lambda^2+t}\delta\ell \overset{t\to 0}{\simeq} \frac{C_d\Lambda^{d-2}}{K}\delta\ell - \frac{C_d\Lambda^{d-4}}{K^2}t\delta\ell + \ldots, && (157)\\
\int_{\mathbf{q}}^{>}\frac{1}{(Kq^2+t)^2} &\simeq \frac{C_d\Lambda^d}{(K\Lambda^2+t)^2}\delta\ell \overset{t\to 0}{\simeq} \frac{C_d\Lambda^{d-4}}{K^2}\delta\ell, && (158)
\end{aligned}$$

Returning to Eq.(142) and for convenience rescaling in coordinate (that can equivalently be done in momentum) space according to,

$$\mathbf{x} = b\mathbf{x}', \qquad (159)$$

$$\phi_<(b\mathbf{x}') = b^{\zeta}\phi'(\mathbf{x}'), \qquad (160)$$

as in Eq.160, so as to restore the UV cutoff back to a, we find, to first-order in interaction strength u the RG flow of the coarse-grained coupling constants,

$$\begin{aligned}
K(b) &= b^{d-2+2\zeta}\left[1+\eta(\hat{u})\delta\ell\right]K, &(161)\\
t(b) &= b^{d+2\zeta}\left[t+\delta_g t\right], &(162)\\
&\approx b^{d+2\zeta}\left[1+12\hat{u}\left(\Lambda^2 K/t-1\right)\delta\ell\right]t, &(163)\\
u(b) &= b^{d+4\zeta}\left[u-\delta_g u\right]. &(164)\\
&\approx b^{d+4\zeta}\left[1-36\hat{u}\delta\ell\right]u, &(165)\\
& &(166)
\end{aligned}$$

In the limit of infinitesimal $\delta\ell$, these RG flows can be more conveniently expressed in the different equations form,

$$\begin{aligned}
\frac{dK}{d\ell} &\approx (d-2+2\zeta+\eta(\hat{u}))K, &(167)\\
\frac{dt}{d\ell} &\approx (d+2\zeta)t+12\hat{u}K\Lambda^2-12\hat{u}t, &(168)\\
\frac{du}{d\ell} &\approx (d+4\zeta)u-36\hat{u}u. &(169)
\end{aligned}$$

As for the zeroth order RG in previous subsection, we can use these equations to form the RG flow equation for the single dimensionless nonlinear coupling $\hat{u}(\ell)$ and a dimensionless measure of reduced temperature, $\hat{t}$,

$$\begin{aligned}
\frac{d\hat{u}}{d\ell} &\approx (4-d-2\eta(\hat{u}))\hat{u}-36\hat{u}^2,\\
&\approx (4-d)\hat{u}-36\hat{u}^2, &(170)\\
\frac{d\hat{t}}{d\ell} &\approx (2-12\hat{u}-\eta(\hat{u}))\hat{t}+12\hat{u}K\Lambda^2,\\
&\approx (2-12\hat{u})\hat{t}+12\hat{u}K\Lambda^2, &(171)
\end{aligned}$$

where we define a crucial expansion parameter $\epsilon \equiv 4-d$, that makes this analysis to be perturbatively "controlled" in the small ϵ limit.[13] Equivalently, the same equation can be conveniently obtained by choosing ζ so as to keep $K(b)=K$ fixed under RG, which to this $O(g^1)$ corresponds to $\zeta=(2-d)/2$. Keeping $\eta(\hat{u})$ correction will quite clearly reduce the eigenvalue exponent y_t by an additional correction η (at the fixed point). It will also require $\zeta=(2-d-\eta)/2$ to keep $K(b)$ fixed. Alternatively, we can choose $\zeta=(2-d)/2$, which will lead to a growth of $K(b)\sim b^{\eta}$. This will then in turn lead to a modification of the

"Coulomb's" law according to

$$G(q,K(b_*)) \sim \frac{1}{K(b_*)q^2} \sim \frac{1}{b_*^\eta q^2}, \tag{172}$$

$$\sim \frac{1}{q^{2-\eta}} \sim \frac{1}{x^{d-2+\eta}}, \tag{173}$$

where I chose $b_* \sim 1/q$. So the correlations are shorter range than in the Gaussian theory, as expected since interactions suppress fluctuations.

As discussed earlier in the lecture, from this by scaling we also get the order parameter exponent,

$$\beta = \frac{1}{2}(d-2+\eta)\nu. \tag{174}$$

For the record, for the O(N) model

$$\eta = 8(N+2)\hat{u}_*^2 = \frac{N+2}{2(N+8)^2}\epsilon^2. \tag{175}$$

D. RG flows and fixed points analysis

We now analyze the RG flows, by finding their fixed points and the corresponding eigenvalues that determine associated critical exponents.

1. $d > d_{uc} = 4$ *(*$\epsilon < 0$*)*

Consistent with our perturbative analysis and general fluctuations arguments (finite for $d > d_{uc} = 4$), we observe that $\hat{u}$ is irrelevant and flows into the Gaussian critical point

$$\text{Gaussian critical fixed point}: \hat{t}^* = 0, \hat{u}^* = 0. \tag{176}$$

It separates the high- and low-temperature disordered and ordered phases, for $t > 0$ (PM) and $t < 0$ (FM), respectively, described by the $t \to +\infty$ and $t \to -\infty$ attractive fixed points. We note that the $\hat{t}$-independent correction to $\hat{t}$ (first term in the $\hat{t}$ equation) results in the tilt of the RG flow, and (comparing to e.g., perturbative analysis of previous lectures) is associated with the simple downward T_c shift by fluctuations. As described in general discussion of RG flows, Sec.II B, to obtain the details of the critical behavior we need to Linearize the flows around critical fixed point and diagonalize them, to obtain the form

$$\frac{dX_i}{d\ell} = y_i X_i, \tag{177}$$

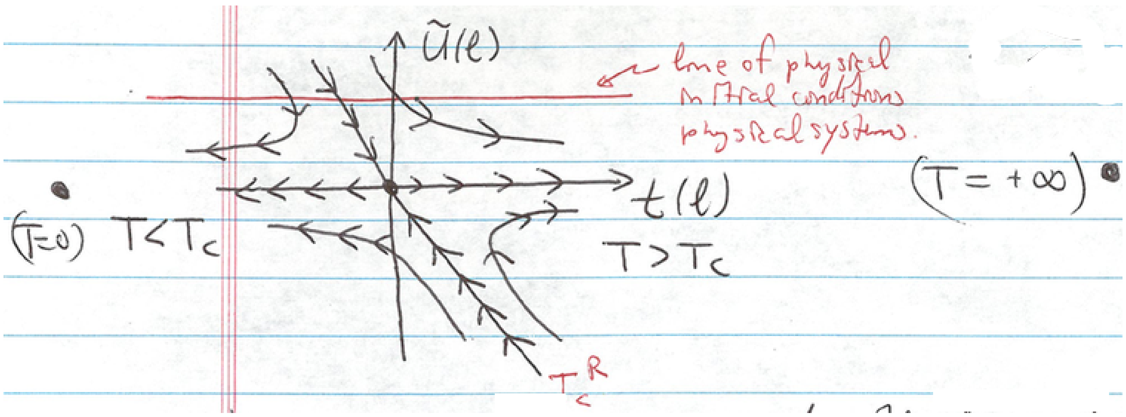


FIG. 9: Renormalization group flow for $d > d_{uc} = 4$ of two main couplings $u(\ell)$ and $t(\ell)$, illustrating stability of the Gaussian critical fixed point, the associated PM-FM phase transition controlled by mean-field exponents, and the high- and low-temperature fixed points for the PM and FM phases, respectively.

where the eigenvectors X_i are the scaling fields related to linear combination of the original fields (here t and u) and the corresponding eigenvalues y_i are related to critical exponents through matching analysis.

Indeed, linearizing the RG flows around the Gaussian critical point, we find

$$\frac{d}{d\ell}\begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix} = \begin{pmatrix} 2 & t_c \\ 0 & -|\epsilon| \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix}, \tag{178}$$

$$\tag{179}$$

where we defined $t_c = 12K\Lambda^2$ proportional to (negative) of the reduced critical temperature (see below), driven negative by fluctuations from its bare zero value (see below). Firstly, without any formal analysis, since $\hat{u}(\ell)$ is irrelevant, flowing to 0 at a rate $|\epsilon|$, we can neglect its correction to $\hat{t}$. This amounts to neglecting the off-diagonal term $12K\Lambda^2$ in the above RG flow matrix, $R_{\alpha\beta}$. It then becomes diagonal, with eigenvalues $y_t = 2$ and $y_u = -|\epsilon|$, and corresponding eigenvectors $e_1 = \hat{t}(1,0)^T$, $e_2 = \hat{u}(0,1)^T$. More simply, it shows that to this lowest order the original couplings (and the associated operators ϕ^2 and ϕ^4), $t(\ell) = te^{2\ell}$ and $u(\ell) = ue^{-|\epsilon|\ell}$ are the eigen-couplings of the RG flows.

A bit more formally, we do not drop the off-diagonal t_c term in $R_{\alpha\beta}$, but explicitly diagonalize it, finding two eigenvectors $e_1 = (1,0)^T$, $e_2 = (-t_c, 2+|\epsilon|)^T$ and the corresponding eigenvalues, $y_1 = 2$, $y_2 = -|\epsilon|$, giving

$$\begin{pmatrix} \hat{t}(\ell) \\ \hat{u}(\ell) \end{pmatrix}_1 = X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 e^{2\ell}, \quad \begin{pmatrix} \hat{t}(\ell) \\ \hat{u}(\ell) \end{pmatrix}_2 = X_2 \begin{pmatrix} -t_c \\ 2+|\epsilon| \end{pmatrix}_2 e^{-|\epsilon|\ell} \tag{180}$$

where X_1 and X_2 are the "initial" amplitudes of these eigenvectors, given by projecting the physical couplings $(\hat{t}, \hat{u})$ onto these eigenvectors,

$$\begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix} = X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X_2 \begin{pmatrix} -t_c \\ 2+|\epsilon| \end{pmatrix} \tag{181}$$

Equivalently, in terms of the two eigen-coupling combinations of $\hat{t}$ and $\hat{u}$, we have

$$X_1(\ell) = X_1 e^{2\ell}, \tag{182}$$

$$(2+|\epsilon|)\hat{t}(\ell) + t_c\hat{u}(\ell) = \left[(2+|\epsilon|)\hat{t} + t_c\hat{u}\right] e^{2\ell}, \tag{183}$$

$$X_2(\ell) = X_2 e^{-|\epsilon|\ell}, \tag{184}$$

$$\hat{u}(\ell) = \hat{u}e^{-|\epsilon|\ell}. \tag{185}$$

The fluctuation-renormalized critical point is identified by the vanishing of the relevant coupling, $X_1 = 0$, that gives

$$a(T - T_{c0}) \equiv \hat{t} = -t_c\hat{u}/(2+|\epsilon|), \tag{186}$$

and leads to a suppression of T_c by fluctuations,

$$T_c = T_{c0} - \frac{12K\Lambda^2\hat{u}}{a(2+|\epsilon|)}, \tag{187}$$

identical to what we found via perturbation theory in previous lectures.

With these results in hand, we can now readily identify critical exponents and other physical observables, utilizing the matching analysis from Sec.II C.

For example, to obtain the divergence of the correlation length $\xi(t,u)$ we use the relation between this physical correlation length and the rescaled on characterizing the coarse-grained theory

$$\xi(t,u) = e^{\ell}\xi(\hat{t}(\ell), \hat{u}(\ell)) = e^{\ell}\xi(e^{2\ell}X_1, e^{-|\epsilon|\ell}X_2), \tag{188}$$

$$\overset{\ell\to\ell_*}{=} \frac{\Lambda}{X_1^{1/2}}\xi(\Lambda^2, X_2^* = 0), \tag{189}$$

$$\sim \frac{1}{|\hat{t} + t_c\hat{u}/(2+|\epsilon|)|^{1/2}} \sim \frac{1}{|T - T_c|^{1/2}}, \tag{190}$$

where I chose ℓ_* such that $X_1(\ell_*)$ has rescaled to its UV cutoff value, which allowed me to eliminate ℓ_* in favor of X_1. Also, because X_1 is small (system is near critical $t \to 0$), $\ell_* \gg 1$, driving $X_2(\ell_*) \sim \hat{u}(\ell_*) \to 0$. Thus, this explicitly demonstrates the downward shift of T_c and that for $d > d_{uc} = 4$, the criticality is indeed controlled by the mean-field exponent $\nu = 1/2$.

2. $d < d_{uc} = 4$ $(\epsilon > 0)$

Consistent with our perturbative analysis and general fluctuations arguments, we observe already from Eq.(170), that, for $d < d_{uc} = 4$ the $\hat{u}$ coupling is now relevant around the Gaussian critical point, and flows into a nontrivial critical fixed point,

$$\text{Wilson-Fisher critical fixed point} : \hat{t}_* = -\frac{\epsilon K \Lambda^2}{3(2-\epsilon/3)}, \quad \hat{u}_* = \frac{1}{36}\epsilon. \tag{191}$$

It separates the high- and low-temperature disordered and ordered phases, for $\hat{t} > -t_c$ (PM) and $\hat{t} < -t_c$ (FM), respectively, described by the $t \to +\infty$ and $t \to -\infty$ attractive fixed points. We note that the $\hat{t}$-independent correction to $\hat{t}$ (first term in the $\hat{t}$ equation) results in the tilt of the RG flow, and (comparing to e.g., perturbative analysis of previous lectures) is associated with the simple downward T_c shift by fluctuations. The crossover from the Gaussian to Wilson-Fisher critical point for $d < 4$ is controlled by the Ginzburg length,

$$\xi_G \sim \left(K^2/u\right)^{1/(4-d)}, \tag{192}$$

that we already defined in (119), as the length that controls the divergence of the perturbation theory and beyond which the effects of fluctuations and interactions begin to be manifest.

Linearizing the RG flows around the Wilson-Fisher critical fixed point,

$$\delta\hat{t} = \hat{t} - \hat{t}_*, \quad \delta\hat{u} = \hat{u} - \hat{u}_*, \tag{193}$$

we find

$$\frac{d}{d\ell}\begin{pmatrix} \delta\hat{t} \\ \delta\hat{u} \end{pmatrix} = \begin{pmatrix} 2-12\hat{u}_* & t_c - 12t_* \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} \delta\hat{t} \\ \delta\hat{u} \end{pmatrix}, \tag{194}$$

$$\approx \begin{pmatrix} 2-\epsilon/3 & t_c \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} \delta\hat{t} \\ \delta\hat{u} \end{pmatrix}. \tag{195}$$

Diagonalizing this linearized RG matrix, $R_{\alpha\beta}$ we find (to lowest order in ϵ) two eigenvectors $e_1 = (1,0)^T$, $e_2 = (-t_c, 2+2\epsilon/3)^T$ and the corresponding eigenvalues, $y_1 = 2-\epsilon/3$, $y_2 = -\epsilon$, giving

$$\begin{pmatrix} \delta\hat{t}(\ell) \\ \delta\hat{u}(\ell) \end{pmatrix}_1 = X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 e^{y_1 \ell}, \quad \begin{pmatrix} \delta\hat{t}(\ell) \\ \delta\hat{u}(\ell) \end{pmatrix}_2 = X_2 \begin{pmatrix} -t_c \\ 2+2\epsilon/3 \end{pmatrix}_2 e^{-\epsilon\ell} \tag{196}$$

where X_1 and X_2 are the "initial" amplitudes of these eigenvectors, given by projecting the physical couplings $(\delta\hat{t}, \delta\hat{u})$ onto these eigenvectors,

$$\begin{pmatrix} \delta\hat{t} \\ \delta\hat{u} \end{pmatrix} = X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X_2 \begin{pmatrix} -t_c \\ 2+2\epsilon/3 \end{pmatrix} \tag{197}$$

I note that these eigenvectors describe tilted flows, that, as noted earlier, lead to the transition at reduced temperature $-t_c\hat{u}/2$.

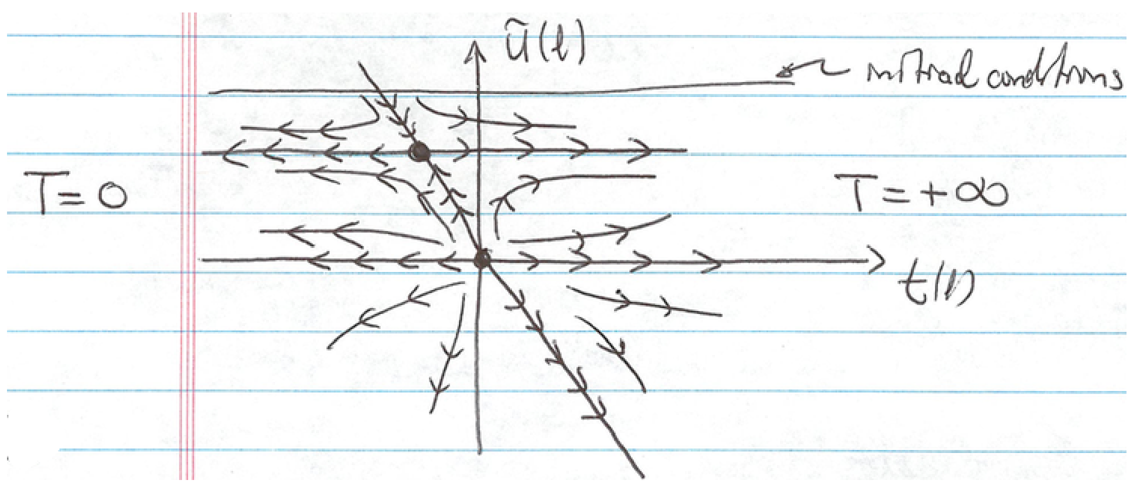


FIG. 10: Renormalization group flow for $d < d_{uc} = 4$ of two main couplings $u(\ell)$ and $t(\ell)$, illustrating the instability of the Gaussian critical point, the stability of the Wilson-Fisher critical fixed point, the associated PM-FM phase transition controlled by nontrivial critical exponents, and the high- and low-temperature fixed points for the PM and FM phases, respectively.

3. Matching analysis

Having characterized the flows near the W-F critical point we identify the critical exponents of physical observables, utilizing the matching analysis from Sec.II C.

Correlation length ξ:

As for the Gaussian critical point, above, the divergence of the correlation length $\xi(t, u)$ is determined by the relation between this physical correlation length and the rescaled on characterizing the coarse-grained theory

$$\xi(t, u) = e^{\ell}\xi(\hat{t}(\ell), \hat{u}(\ell)) = e^{\ell}\xi(e^{y_1\ell}X_1, e^{-\epsilon\ell}X_2), \tag{198}$$

$$\stackrel{\ell\to\ell_*}{=} \frac{\Lambda}{X_1^{1/y_1}}\xi(\Lambda^2, X_2^* = 0), \tag{199}$$

$$\sim \frac{1}{|\hat{t} + t_c\hat{u}/2|^{1/(2-\epsilon/3)}} \sim \frac{1}{|T - T_c|^{1/(2-\epsilon/3)}}, \tag{200}$$

where according to matching procedure, I chose ℓ_* such that $X_1(\ell_*)$ has rescaled to its UV cutoff value, which allowed me to eliminate ℓ_* in favor of X_1. Also, because X_1 is small (system is near critical $t \to 0$), $\ell_* \gg 1$, driving $X_2(\ell_*) \sim \hat{u}(\ell_*) \to \hat{u}_*$. Thus, this explicitly demonstrates the downward shift of T_c and that for $d < d_{uc} = 4$, the correlation-length criticality is indeed controlled by a universal nontrivial exponent,

$$\nu = 1/y_1 \approx \frac{1}{2 - \epsilon/3} \approx 1/2 + \epsilon/12, \tag{201}$$

$$\approx 3/5, \quad \text{for } d = 3,\ \epsilon = 1. \tag{202}$$

$\nu \backslash d$	4	3	2
ν_{exp}	$\sim 1/2$	0.6-0.7	1
ν_{theory}	$1/2$	0.6	$3/4$

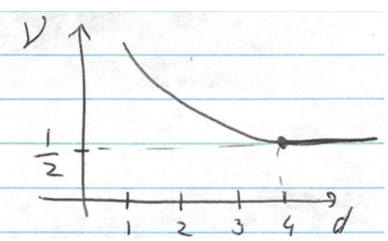


FIG. 11: Correlation length exponent, $\nu(d)$ for different dimensions d, comparing theory and experiments. On the right, the graph illustrates schematic variation of $\nu(d)$ with increased dimension, showing its pinning at the mean-field value of $\nu_{mf} = 1/2$ for $d \geq 4$.

Linear susceptibility χ:

The generalized linear susceptibility to an external field at momentum $\mathbf{k}$ is determined by the 2-point correlation function,

$$\chi(t, \hat{u}, K, \mathbf{k}_1) \equiv \frac{\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle}{(2\pi)^d\delta^d(\mathbf{k}_1 + \mathbf{k}_2)} \tag{203}$$

In contrast to the partition function, $Z = Z'$, that, by definition is invariant under coarse-graining, the correlation function,

$$\int [d\phi_{\mathbf{q}}]\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2} e^{-H[\phi_{\mathbf{q}}]} = \int [d\phi'_{\mathbf{q}'}]\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2} e^{-H_b[\phi_{\mathbf{q}'}]}, \tag{204}$$

is *covariant* under RG. Recalling that $\mathbf{k} = \mathbf{k}'/b, \phi(\mathbf{k}) = b^{d+\zeta}\phi'(\mathbf{k}b)$, the correlation function transforms nontrivially according to,

$$\frac{\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\rangle}{(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2)} = b^{2d+2\zeta-d}\frac{\langle\phi'(\mathbf{k}_1 b)\phi(\mathbf{k}_2 b)\rangle}{(2\pi)^d\delta^d(\mathbf{k}_1 b+\mathbf{k}_2 b)}, \tag{205}$$

$$\chi(t,\hat{u},K,\mathbf{k}) = b^{d+2\zeta}\chi(t(b),\hat{u}(b),K(b),\mathbf{k}b), \tag{206}$$

$$= b^{d+2\zeta}\chi(b^{y_1}\delta t,\hat{u}(b),b^{d-2+2\zeta}K,\mathbf{k}b). \tag{207}$$

Choosing for convenience $\zeta = (2-d)/2$ (since in ϕ^4 theory, to one-loop order there is no "wavefunction" renormalization, i.e., $\eta = 0$) keeps $K(b) = K$, taking rescaling to be b_* such that $b_*^{y_1}\delta t = \Lambda^2$, and noting that for small δt this $b_* \gg 1$ with $u(b_*) \to u_*$, for uniform ($k = 0$) susceptibility we obtain,

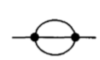
$$\chi(t,u,K,0) = |\delta t|^{-2/y_1}\chi(\Lambda^2,\hat{u}_*,K,0), \tag{208}$$

$$\simeq \frac{1}{|T-T_c|^\gamma}, \tag{209}$$

where for $\eta = 0$,

$$\gamma = 2/y_1 = 2\nu, \tag{210}$$

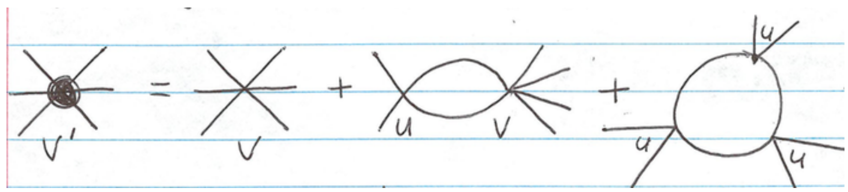
$$= \frac{2}{2-\epsilon/3} = \frac{6}{5}, \quad \text{for } d = 3,\ \epsilon = 1 \tag{211}$$

and where the scaling function $\chi(\Lambda^2, u_*, \mathbf{k})$ can be calculated perturbatively in small $\hat{u}_*$ near $d \leq 4$. More generally, as discussed earlier, $\gamma = (2-\eta)\nu$, with η arising from the diagrammatic corrections to K, to $O(\epsilon^2)$ given by .

I note that a (significantly) more detailed analysis, that goes to order $O(\epsilon^2)$ shows improvement of agreement between the ϵ-expansion and experiments and numerics. However, going further to include corrections of order $O(\epsilon^3)$ leads to worsening of this agreement, which is a reflection of the fact that ϵ-expansion is an *asymptotic* series. For such expansion, at any fixed order n accuracy improves with decreasing ϵ. However, for fixed ϵ, although at first the accuracy improves with increasing order n, it then worsens beyond a certain critical order n_c, with latter diverging with vanishing ϵ, i.e., $n_c(\epsilon \to 0) \to \infty$.

4. Irrelevant operators

The advocated power of RG, controlled by ϵ-expansion is the ability to neglect irrelevant operators. To see this explicitly in the current analysis we examine one of the least irrelevant operators $v\int d^dx\phi_{\mathbf{x}}^6$, that is clearly generated by the above momentum-shell RG coarse-graining, diagrammatically illustrated by



By now standard momentum-shell RG analysis gives

$$v(b) = b^{d+6\zeta}[v + A(t,b)uv + B(t,b)u^3], \tag{212}$$

that in infinitesimal form $b = e^{\delta\ell}$, choosing $\zeta = (2-d)/2$ is equivalent to a differential RG flow,

$$\frac{dv}{d\ell} = (-2+2\epsilon)v + \frac{c_1}{(1+t)^2}uv + \frac{c_2}{(1+t)^3}u^3. \tag{213}$$

Because of the power-counting contribution, $v(b)$ is strongly irrelevant, though because of the last $u_*^3 \sim \epsilon^3$ term, $v(b)$ flows to a fixed-point value of $O(\epsilon^3)$. One may worry that this will then feedback nontrivially into the the flow of $u(b)$. Although indeed it does contribute through a diagram 

, the correction to u_* is negligible and to this order can be neglected.

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[21] A homogeneous function $g(x_1, x_2, \ldots)$ of degree y_g is characterized by its scaling property, $g(b^{y_1}x_1, b^{y_2}x_2, \ldots) = b^{y_g}(x_1, x_2, \ldots)$, where y_i are "dimensions" of variables x_i.

[22] Ken Wilson worked on formulation of momentum-shell RG throughout 6 years of his tenure track at Cornell, not publishing any papers, and it took Hans Bethe to speak up on his behalf for Wilson to get tenure – a testament of value of deep long term research