Physics 7240: Advanced Statistical Mechanics

Lecture 3: Landau Theory of Phase Transitions and Its Breakdown

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I. INTRODUCTION

In condensed matter physics we aim to understand the nature of condensed phases and transitions between them, utilizing methods of statistical mechanics and field theory. As we have seen, all of the thermodynamic information is contained in the partition function, $Z = \mathrm{Tr}_{\{\mathbf{S}_i\}}[e^{-\beta H[\{\mathbf{S}_i\}]}]$, that is a Boltzmann-weighted trace over microscopic degrees of freedom. It is often useful to compute the latter for a particular set of macroscopic observables, that are functionals of microscopic ones.

However, for most, even idealized model systems the exact trace cannot be directly carried out and requires physically insightful approximation. Fortunately, properties of condensed matter phases and transitions exhibit considerable universality that is independent of details of microscopic models. This allows one to fruitfully trade a trace over microscopic degrees of freedom, e.g., $\mathbf{S}_i$ to one over fluctuating continuum coarse-grained fields, $\vec{\phi}(\mathbf{x})$, defined on mesoscopic scales, as we have done explicitly in Lecture set 2. The resulting effective field theory, $Z = \int \mathcal{D}\vec{\phi}(\mathbf{x}) e^{-\beta H_{\rm eff}[\vec{\phi}(\mathbf{x})]}$, defined by effective local free-energy functional requires a functional integral over fully fluctuating coarse-grained fields (that we will study in the next lecture 4, Field Theory Primer), and is thus still typically quite challenging to analyze exactly.

A simplest tractable approximation is the Landau's mean-field theory, that treats these fields as nonfluctuating and often uniform in space (although this latter approximation is unnecessary)[1, 4, 6, 7, 14]. In these lectures we will take this approach. We will construct Landau theories for a variety of systems and will explore their predictions. We will conclude with the analysis of the breakdown of Landau theory due to strong fluctuations, typically present near a continuous phase transition, and will thereby determine the range of validity of the mean-field theory.

Landau theory approach consists of (i) identifying the symmetry of the system, denoted by group G, (ii) identifying order parameter ϕ, invariant under group H, that qualitatively distinguishes ordered state from the disordered state, and (iii) constructing a Landau functional of ϕ that satisfies all the underlying symmetries. Going back to Landau and up until quite recently, it used to be thought that all phases of matter and associated transitions were completely classified by this Landau theory construction, characterized by various patterns of symmetry breaking, with H a subgroup of G, $H \in G$. However, it is now well understood,

that there are rich variety of phases of matter, such as the quantum Hall states, spin liquids, Fermi liquid, etc., that are not subject to Landau classification. These states usually exhibit "topological nature", that is used for their classification, and are thus referred to as topological states of matter. We leave the discussion of these more exotic states to other courses, and refer the student to many excellent expositions, most notably lectures by X. G. Wen, one of the main architects of the subject.[28]

II. LANDAU THEORY OF PHASE TRANSITIONS

A. General Landau approach

Instead of deriving the effective free energy from a microscopic lattice theory (which most of the time is not rigorously possible nor is microscopic theory always known or can be literally trusted), Lev Landau (1937) made a bold and physically insightful proposal of simply writing down the answer. Namely, in treating a phase transition, one must first identify the so-called "order parameter" local field, $\vec{\phi}(\mathbf{x})$ (a generic name at this point, representing e.g., the local magnetization at the PM-FM transition), that qualitatively distinguishes the two phases of matter, vanishing in the disordered (e.g., high $T > T_c$) state and nonzero in the ordered (e.g., low $T < T_c$) state. Then, guided by the underlying symmetries of the disordered phase, we then construct a Landau functional $H[\vec{\phi}(\mathbf{x})]$ (an effective Hamiltonian, a local free-energy functional) that satisfies these symmetries. Because we are focussed on the vicinity of a continuous or weakly first-order phase transitions, where the order parameter remains small, we take the Landau functional to be a local, analytic expansion in powers of the order parameter, limiting it to lowest qualitatively necessary order,

$$H[\vec{\phi}(\mathbf{x})] = \int_{\mathbf{x}} \left[\frac{1}{2} K (\boldsymbol{\nabla} \vec{\phi})^2 + \frac{1}{2} t |\vec{\phi}|^2 + \frac{1}{4} u |\vec{\phi}|^4 + \frac{1}{6} v |\vec{\phi}|^6 + \cdots - \vec{h} \cdot \vec{\phi} \right] . \tag{1}$$

Near T_c the corresponding coefficients (couplings) can all be taken to be constant smooth functions of temperature (ultimately determined experimentally), except for the quadratic coefficient, $t(T)$, that drives the transition changing sign at T_c and thus can be generically be taken to be a linear function, $t(T) \approx a(T - T_c)$. This is consistent with our lecture 2 findings, based on explicit calcualtions for the Heisenberg model for the PM-FM transition, where in particular we found $t(T) \propto 1 - J_0/k_B T$.

It is important to remember that even though for computational convenience, we write everything in the continuum, in reality, in any physical system there is a short scale "UV" cutoff a, set by lattice spacing and a large scale "IR" cutoff L, set by system size, with $a \ll x \ll L$. Because $\frac{1}{2}K(\boldsymbol{\nabla}\vec{\phi})^2$ term keeps $\vec{\phi}(\mathbf{x})$ smooth and nearly uniform in space, most of the time, a, L will not play an important role, but are there to make the model well-defined.

Within the Landau description, (rather than integrating over $\vec{\phi}(\mathbf{x})$) we simply minimize the Landau functional, $H[\vec{\phi}(\mathbf{x})]$ with respect to the order parameter, thereby determine its value $\vec{\phi}_0(\mathbf{x}, \vec{h}, T)$ as a function of temperature T and the external field, e.g., $\vec{h}$, thereby obtaining the equation of state.

B. A non-uniform solution to a generalized Landau theory

In the presence of nonuniform boundary conditions, as an example, illustrated in Fig.1, even a mean-field Landau treatment can be quite nontrivial, requiring a solution of a differ-

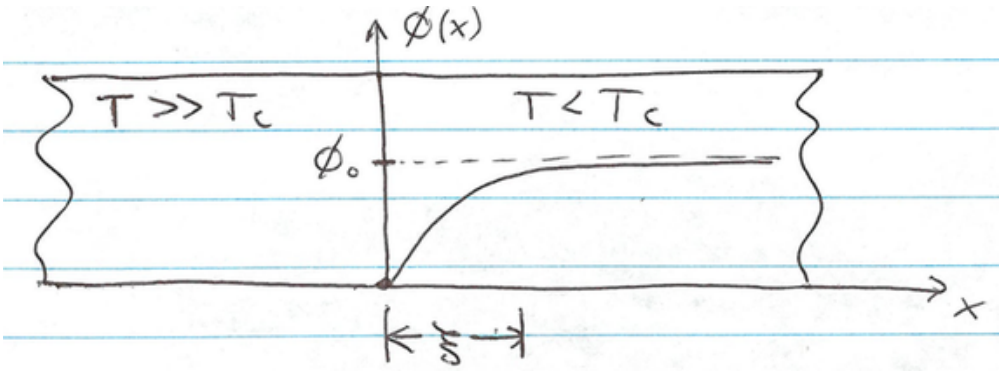


FIG. 1: Non-uniform boundary condition

ential Euler-Lagrange (EL) equation (specializing to the Ising, $N = 1$ case),

$$\frac{\delta H}{\delta \phi(\vec{r})} = 0 = -K\nabla^2\phi + t\phi(\vec{r}) + u\phi^3. \tag{2}$$

We take the boundary conditions to be $T \gg T_c$ for $x < 0$, with $\phi(x < 0) = 0$, and $T \ll T_c$ for $x > 0$, pinning $\phi(x \to +\infty) = \phi_0 = \sqrt{|t|/u}$ at positive infinity.

The full solution, pinned by these boundary conditions also satisfies EL differential equation,

$$K\partial_x^2\phi + |t|\phi - u\phi^3 = 0. \tag{3}$$

Because of 1d nature of the boundary conditions and the corresponding solution, this equation can be solve exactly by noting that it corresponds to a mechanics problem of a "fake" particle of "mass" K, at a one-dimensional position $\phi(x)$, moving in time x in an inverted quartic potential $V(R) = -V_{\rm Landau}(\phi) = \frac{1}{2}|t|\phi^2 - \frac{1}{4}\phi^4$, starting at the origin with "kinetic" energy E, sufficient to just climb to the top of the "hill" at a time $x \to \infty$,

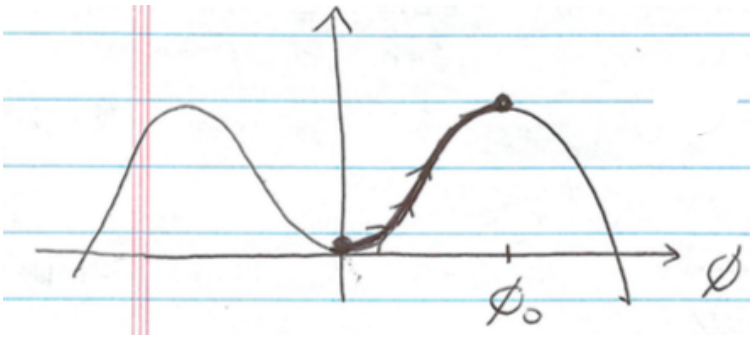


FIG. 2: A soliton solution satisfying above boundary conditions is captured by a position $\phi(x)$ of a "particle" moving in an effective potential $V(\phi) = -V_{\rm Landau}(\phi)$, starting at the origin with "kinetic" energy E, sufficient to just climb to the top of the "hill" at a time $x \to \infty$.

$$K\partial_x^2\phi = -\frac{\delta V}{\delta\phi}. \tag{4}$$

For a judiciously chosen E we know by mechanical analogy that such solution exists and can be obtained by utilizing an intergral of motion associated with conservation of energy,

$$K\partial_x\phi\partial_x^2\phi \;=\; -\frac{\delta V}{\delta\phi}\partial_x\phi = -\partial_x V(\phi(x), \tag{5}$$

$$\partial_x\left[\frac{1}{2}K(\partial_x\phi)^2 + V(\phi)\right] \;=\; 0, \tag{6}$$

which gives

$$\frac{1}{2}K(\partial_x\phi)^2 + \frac{1}{2}|t|\phi^2 - \frac{1}{4}\phi^4 = E. \tag{7}$$

Solving for $\partial_x\phi(x)$, the resulting first-order differential equation separable and can be integrated exactly, with E chosen so $\phi(x)$ satisfies above boundary conditions. The resulting solution is given by

$$\phi(x) = \phi_0 \tanh[x/(\sqrt{2}\xi)], \tag{8}$$

with the mean-field correlation length, $\xi(t) = \sqrt{K/|t|} \sim |t|^{-1/2}$, giving $\nu_{MF} = 1/2$. The latter scaling with K and t could have been read off from the Landau functional, (1) simply

by dimensional analysis. The correlation length is the scale over which the order parameter varies in space, and as expected, diverges as $T \to T_c$.

An alternative way to find a mean-field correlation length, ξ, is to consider a solution $\phi(\mathbf{x})$ in the disordered phase, $T > T_c$ ($t > 0$) with a delta-function source at the origin, i.e., a Green's function to the *harmonic* (linear) approximation to the Landau Euler-Lagrange equation above,

$$-K\nabla^2\phi_0 + t\phi_0 = h_0\delta^d(\mathbf{x}), \tag{9}$$

which gives the solution, illustrated in Fig.3,

$$\phi_0(\mathbf{x}) \simeq h_0 \frac{e^{-x/\xi_+}}{x}, \tag{10}$$

with $\xi_+ \approx \sqrt{K/t}$, illustrated in Fig.4.

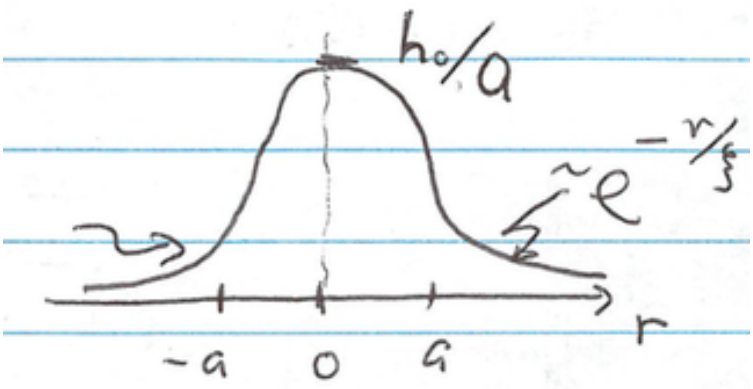


FIG. 3: Saddle point solution $\phi_0(x)$, Green's function in the disordered $t > 0$ phase.

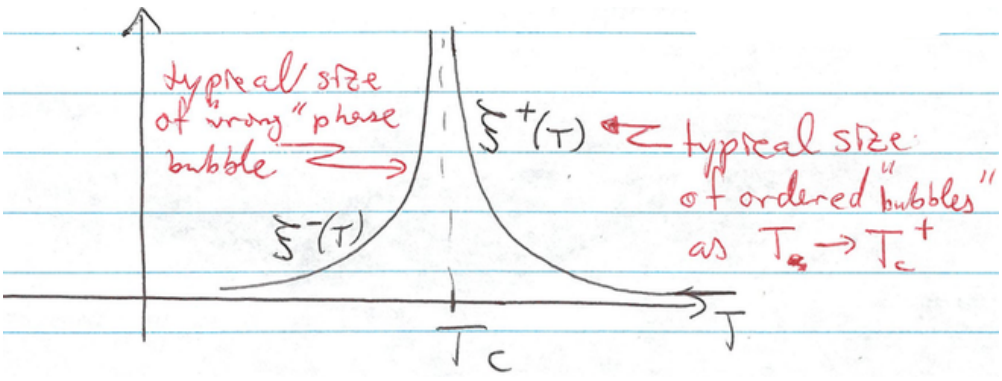


FIG. 4: Length scale on which $\phi(x)$ varies $\xi(T)$

C. Examples of Landau theories and their applications

We will now use above Landau description to analyze a number of phases and associated phase transitions. In a thermodynamic limit, with free boundary conditions, $\vec{\phi}_0(\vec{h})$ is spatially uniform. The free energy is then simply given by $F(\vec{h},T) = H[\vec{\phi}_0]$, and other thermodynamic quantities, such as the entropy, heat capacity, susceptibility, etc., are computed as derivatives with respect to the corresponding parameters, T, $\vec{h}$, etc. We emphasize that despite an analytic structure of Landau theory as a function of the order parameter, it will produce a *nonanalytic* behavior (singularities) of thermodynamic observables and of correlation functions.

As we will see below, while the detailed structure (nature of the order parameter, symmetries, etc.) will depend on the specific phase transition, the critical exponents will be super-universal, given by their MFT values of $\alpha = 0$ (heat capacity as function of t), $\beta = 1/2$ (order parameter as function of t), $\gamma = 1$ (susceptibility as function of t), $\delta = 3$ (order parameter as function of h), $\nu = 1/2$ (correlation length exponent as function of t), $\eta = 0$ (order parameter correlator as function of momentum q). Consistent with experiments, we will demonstrate in the following lectures that this MF super-universality does not survive fluctuations and in fact, while universal (independent of many microscopic model details), the exponents are determined by a universality class that depends on the dimensionality of space and symmetry of the system.

1. Ising PM-FM transition

As discussed in lecture 2, in real magnets the full spin rotational invariance of the Heisenberg model often reduces to a single scalar ($N = 1$) component order parameter, $\phi = m$, that represents the magnetization density m. In the absence of an external magnetic field the system exhibits Ising $\mathcal{Z}_2$ symmetry, $H[m] = H[-m]$, that requires the Landau free-energy density to be restricted to an even power-law series in m,

$$\mathcal{H}[m] = \frac{1}{2}tm^2 + \frac{1}{4}um^4 + \frac{1}{6}vm^6 + \dots . \tag{11}$$

For a weak external field h above is supplemented by a Zeeman term $-hm$. Minimizing this Landau free energy density over m, the saddle-point equation $\delta\mathcal{H}/\delta m = 0$ gives the equation

of state,

$$tm_0 + um_0^3 \approx h, \tag{12}$$

For a vanishing external field, $h = 0$, we find

$$m(t, h = 0) = \begin{cases} 0, & \text{for } t > 0 \ \ (T \ > T_c), \\ \pm\big(-\frac{t}{u}\big)^{1/2} \propto |t|^{\beta}, & \text{for } t < 0 \ \ (T \ < T_c), \end{cases} \tag{13}$$

where it can be verified that the $m_0 = 0$ solution for $T < T_c$ has a higher free energy and is in fact a local maximum. Thus, indeed the magnetization $m_0(T)$ exhibits a nonanalytic behavior as a function of temperature across T_c.

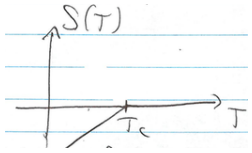
The more general solution $m_0(T, h)$ at finite h is schematically illustrated in Fig.5, showing that it is always finite at all T. At T_c this gives $m_0 \sim h^{1/\delta}$, with the mean-field value of $\delta_{MF} = 3$.

The free energy density is obtained by evaluating $f(T, h) = \text{Min}\,[\mathcal{H}[m_0(T, h), h]]$, which for $h = 0$ gives,

$$f(t, h = 0) = \begin{cases} 0, & \text{for } t > 0 \ \ (T \ > T_c), \\ -\frac{t^2}{4u} \propto |T - T_c|^2, & \text{for } t < 0 \ \ (T \ < T_c), \end{cases} \tag{14}$$

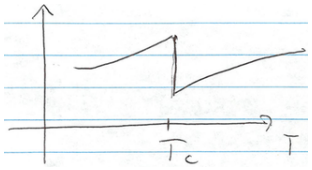
illustrated in Fig.6. The corresponding entropy and heat capacity are also straightforwardly computed, giving

$$S(T) = -\frac{\partial f}{\partial T} \sim -|t|, \tag{15}$$



and

$$C_v = -T\frac{\partial^2 f}{\partial T^2} = T\frac{\partial S}{\partial T} = \begin{cases} 0, & \text{for } T \ > T_c, \\ \frac{T_c}{2u}, & \text{for } T \ < T_c. \end{cases} \tag{16}$$



latter displaying jump-discontinuity. I emphasize, that, while there are many sources of T dependence in $f(T)$ (entering through β and all the coupling constants in the Landau Hamiltonian functional), the singular behavior is dominated by the dependence via the "reduced" temperature t. Thus, in computing T derivatives, for singular leading contributions only derivatives with respect to t need to be taken into account.

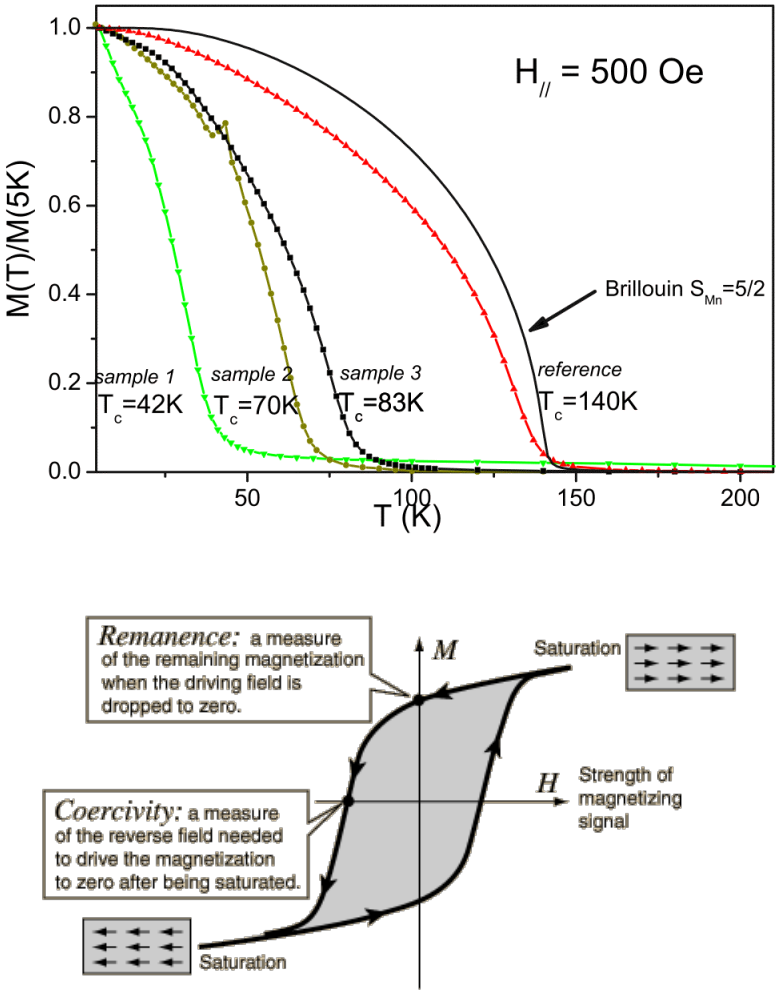


FIG. 5: (top) Magnetization as a function of temperature T for small nonzero external magnetic field H for various samples with varying T_c. Notice small nonzero magnetization (induced by H) even above T_c. (bottom) Magnetization as a function of H for $T < T_c$ showing finite spontaneous magnetization even as $H \to 0$, as well as the hysteresis for a finite rate of changing magnetic field, faster than the transition rate between the two magnetizaiton minima.

The magnetic linear susceptibility $\chi_\ell(T) \equiv \frac{\partial m}{\partial h} \mid_{h=0}$ can be computed by differentiating the equation of state, $tm + um^3 = h$,

$$(t + 3um^2)\frac{\partial m}{\partial h} = 1, \tag{17}$$

solving for χ_ℓ in the limit of $h \to 0$, and using the expression for the magnetization $m(t, h =$

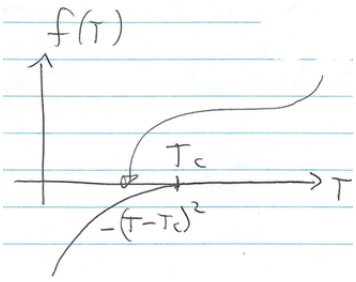


FIG. 6: Free energy density as a function of temperature, showing nonanalyticity around T_c.

0) obtained above. We thus find

$$\chi_\ell(t) = \begin{cases} \frac{1}{t} \ , & \text{for } T > T_c \ , \\ \frac{1}{2|t|} \ , & \text{for } T < T_c \ , \end{cases} \tag{18}$$

$$\tag{19}$$

that, more explicitly, below T_c gives

$$\chi_\ell \sim |T - T_c|^{-\gamma} \ , \text{with } \gamma_{MF} = 1. \tag{20}$$

I note that this generic form is qualitatively correct in all dimensions (with exponent $\gamma(d)$ a function of dimension of space, d) except in $d = 1$, where the FM phase and the associated Ising transition are unstable at finite T.

We will return to this question of stability of an ordered phase more generally and thoroughly in following lectures. However, we do note here in passing, that the 1d Ising FM phase is destabilized by domain-walls (kinks),

$$\text{kink}: \ \ldots \uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow \ldots .$$

Because these have a finite excitation energy, $\Delta E_{kink} \approx 2J$, the probability for them to be thermal activated at any one of N sites (since can occur in N places), is given by

$$P_{kink} \approx N e^{-\frac{2J}{k_B T}}, \tag{21}$$

which is always finite for $T > 0$. The kink density is thus given by $n_{kink} \approx e^{-\frac{2J}{k_B T}}$, corresponding to a finite distance between kinks, i.e., a finite FM correlation length, $\xi = e^{\frac{2J}{k_B T}}$ at any nonzero temperature. The FM state at finite T is thus unstable in 1d. In contrast, in d dimensions the domain-wall energy goes as $E_{dw} = 2J(L/a)^{d-1}$ and thus diverges in the

thermodynamic limit, ensuring a stability of the $d > 1$ Ising (and more generally discrete, Goldstone-mode-free) model to thermal fluctuations. Systems that break continuous symmetries and therefore exhibit Goldstone modes require a separate analysis, that will occupy us in subsequent lectures.

Finally, I note that (as illustrated in Fig.7) for $T < T_c$ the transition as a function of the magnetic field h is first order. It corresponds to a jump of magnetization between the two minima at $\pm m_0(T)$, with symmetry between them broken by $h \neq 0$. The jump discontinuity vanishes as $T \to T_c$.

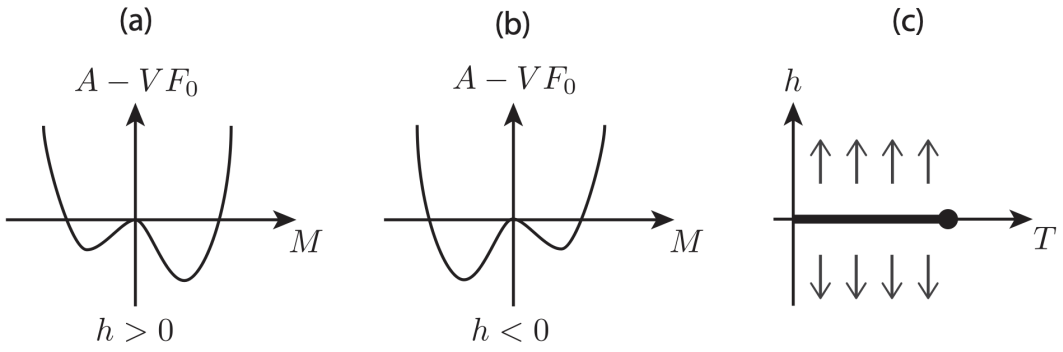


FIG. 7: Illustration of 1st order phase transition (with $A(h)$ the Helmholtz free energy) as a function magnetic field h (a) for $T < T_c$ and (b) smooth crossover for $T > T_c$ (from Nelson and Zhang)..

2. Liquid-gas transition near a critical point

Physically, liquid and gas phases *appear* to be quite different, strongly- and weakly-interacting, respectively. In fact these differences are only quantitative, with both phases disordered and ergodic, with former appearing at high and latter at low densities. We thus introduce a natural order parameter, particle density, for convenience measured relative the density of the gas, $\delta\rho(T, P) \equiv \rho - \rho_{gas}$. This insures that the order parameter $\delta\rho(T, P)$ vanishes in the "disordered" gas phase and is nonzero and equals to the liquid - gas density difference, $\rho_{liquid} - \rho_{gas}$.

At fixed pressure, P and high temperature, $T > T_c(P)$, the densities of the two phases are the same, $\rho_{liq} = \rho_{gas}$, corresponding to a vanishing order parameter, $\delta\rho = 0$. For $T <$

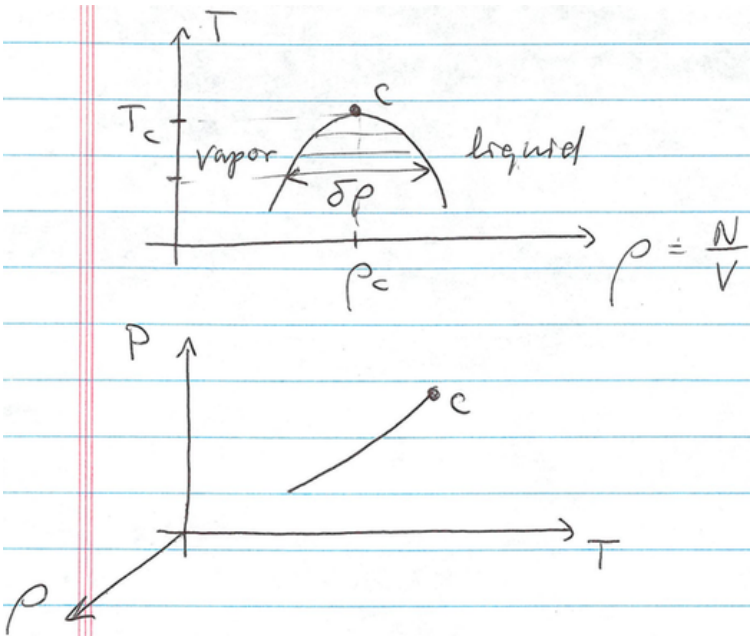


FIG. 8: Order parameter for a liquid-gas transition near the critical point, and the critical boundary $T_c(P)$, across which $\delta\rho$ jumps, and a continuous transition takes place at the critical point C.

$T_c(P)$, there is a condensed high-density liquid and low-density gas phases, with $\rho_{liq} > \rho_{gas}$, with $\delta\rho(T, P) > 0$. This condensation of a liquid below the critical point C, finely-tuning P, T to follow tangetially along the critical curve $T_c(P)$ (see Fig.8) is the liquid-gas (L-G) analog of the PM-FM phase transition, and $\delta\rho$ the counterpart of the magnetization density (difference between spin up and spin down densities), discussed in the previous subsection. The important qualitative difference is that unlike the PM-FM transition, where, for the coexistence curve $h = 0$, Ising $\pm m$ symmetry ensures that the critical transition is continuous, for L-G transition, there is no such natural simple curve and P, T must be finely tuned to cross the critical point C tangentially. If instead the $T_c(P)$ boundary is *not* crossed tangentially, then the transition is first-order akin to h-tuned transition in ferromagnet for $T < T_c$.

As for a FM, near the critical point C, we expand the free energy density functional $f[\delta\rho]$ in powers of the order parameter, the density difference, $\delta\rho$,

$$f[\delta\rho] \approx f_0 + h(T, P)\delta\rho + \frac{1}{2}t(T, P)\delta\rho^2 - \frac{1}{3}w(T, P)\delta\rho^3 + \frac{1}{4}u(T, P)\delta\rho^4, \tag{22}$$

which when approached tangentially along the coexistence curve, through C ($h(T_c(P)) = w(T_c(P)) = 0$) takes the form of the free energy density for an Ising PM-FM transition.

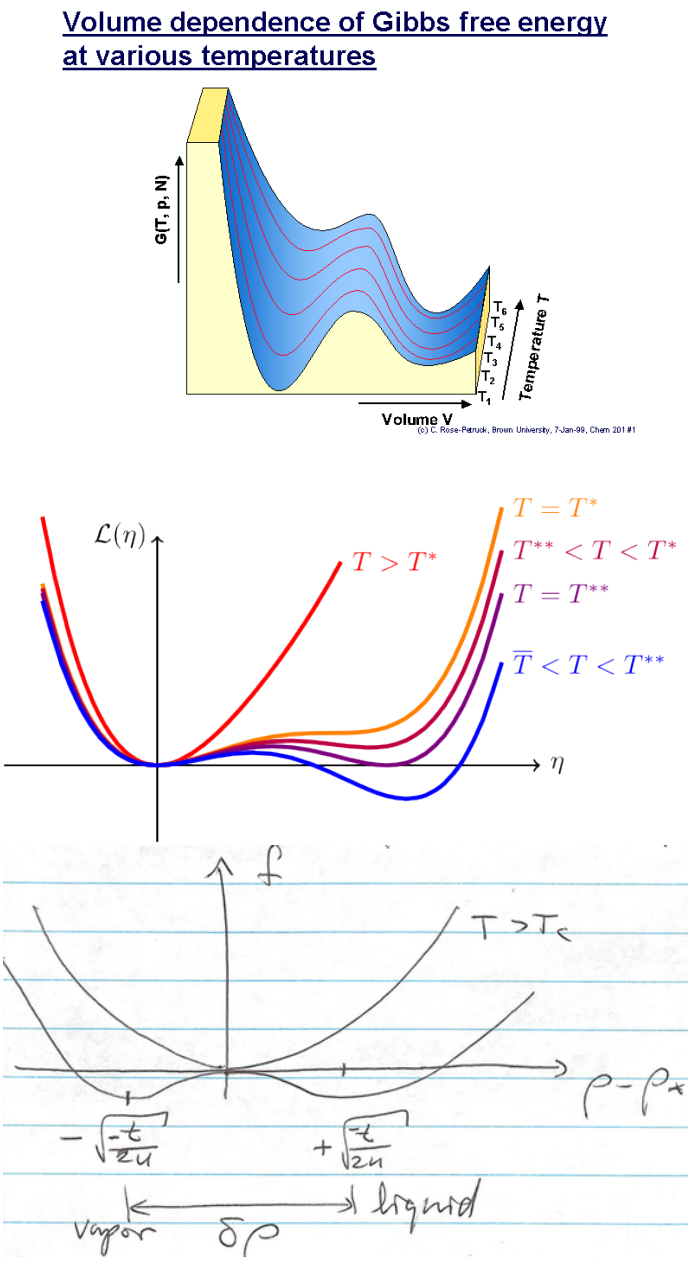


FIG. 9: (top) Free energy density near the L-G transition with two minima corresponding to a liquid and a gas and T, P controlling which of the two minima is the lowest. (middle) General form of odd Landau theory around a first-order transition. (bottom) Free energy density tuned through a critical point for ($T > T_c$), where there is no (even quantitative) distinction between a liquid and a gas, and $T < T_c$, where liquid and gas have different densities.

More generically, at low $T < T_c$ $f[\delta\rho]$ allows for odd terms in $\delta\rho$ and therefore exhibits two minima, whose relative value determines the thermodynamic state and corresponding density difference.

Thus, along the coexistence curve the phase transition is of Ising universality class, that in MFT exhibits the very same exponents as found in the previous subsection. Instead, crossing this coexistence foundary, gives a first-order transition between liquid and gas.

Near this transition, the compressibility (the analog of magnetic susceptiblity) defined by $\chi^{-1} \equiv \frac{\partial^2 f}{\partial \rho^2}$, diverges as

$$\chi \sim |t|^{-\gamma} \text{ , with } \gamma_{MF} = 1 \tag{23}$$

near the critical point C. This corresponds to the fact that the system is infinitely easy to compress at $T_c(P_c)$, since at C it does not "care" at what density to be, ρ_{gas} or ρ_{liq}. Namely, infinitesimally small compressional pressure, δh leads to a finite changes in the density ρ.

I note that compressibility χ is also related by fluctuation-dissipation theorem[7] to the variance of density fluctuations, $\chi \sim \langle \delta\rho^2 \rangle$. As a result, near a critical point density fluctuations are large, and manifest themselves in light scattering (through fluctuations of the dielectric constant, ϵ) and critical opalescence, These critical properties near C have

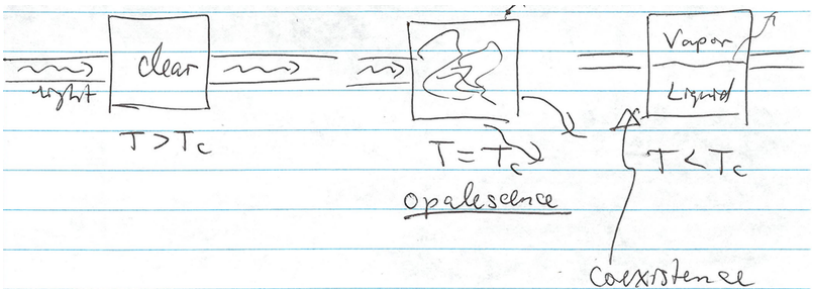


FIG. 10: The phenomenon of critical opalescence.

been studied extensively in large variety of experiments and are summarized in Fig.11.

3. Normal-to-superfluid transition: XY-model

Most fluids when cooled freeze into a crystal or a glass, long before the degeneracy temperature below which quantum statistical effects begin manifest. However, for lighter elements like the bosonic isotope of Helium, He-4, quantum fluctuations are strong prevent-

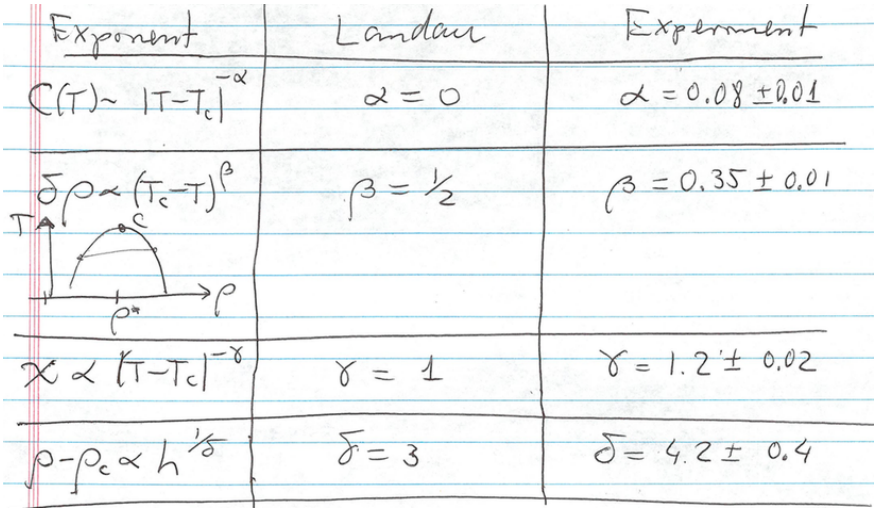


FIG. 11: Experiment vs. Landau theory for liquid-gas transition.

ing solidification. Instead, below the degeneracy temperature, T_c, He-4 undergoes a phase transition from a normal fluid to a superfluid.

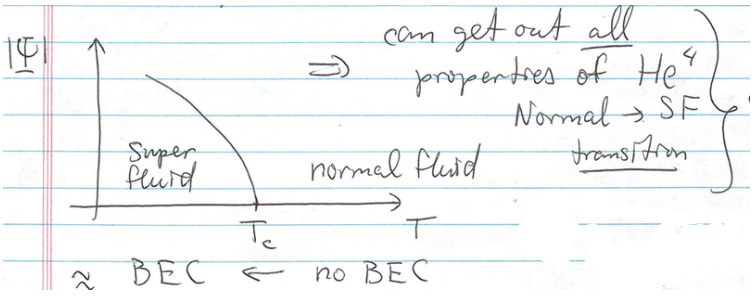


FIG. 12: Phase diagram for Normal-to-Superfluid transition.

Although generally the superfluid phenomenon is complicated by strong interactions in the fluid, the key ingredient of superfluidity (outside of two-dimensions) is captured by Bose-condensation. The latter corresponds to bosonic atoms all macroscopically occupying the same single-particle state $\Psi(\mathbf{x})$, that in a thermodynamic limit is a zero-momentum ground state. In the trapped degenerate atomic Bose gases realization of BEC, the single particle state is a ground state of the trap potential. The corresponding many-body wavefunction for N bosons is then given by $\Psi_N(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \prod_i^N \Psi(\mathbf{x}_i)$.

Thus, the transition into a simplest (there is a rich variety of more exotic superfluid states, requiring more complicated order parameters) superfluid is characterized by a complex scalar order parameter, $\Psi = |\Psi|e^{i\theta} = \Psi_{real} + i\Psi_{imag}$. The square of the amplitude $|\Psi|^2$

captures the density of Bose-condensed atoms and spatial gradients of the superfluid phase θ give the mass superflow current. Since an overall phase of a quantum wavefunction is not a physical observable, the Landau functional must be invariant under global $U(1)$ gauge transformations, $\theta \to \theta + \theta_0$, involving only powers of $|\Psi|^2$, and is thus given by,

$$\mathcal{H}[\Psi] = t|\Psi|^2 + \frac{1}{2}u|\Psi|^4 + \dots . \tag{24}$$

Since $|\Psi|^2 = \Psi_{real}^2 + \Psi_{imag}^2$, this $U(x)$ symmetric model is isomorphic to the $O(2)$ rotation symmetric XY-model with $\vec{\phi} = (\Psi_{real}, \Psi_{imag})$, as the order parameter with the phase shift invariance corresponding to the rotation of the two-component "spin" $\vec{\phi}$. The XY model in terms of $\vec{\phi}$, describing e.g., a planar ferromagnet requires that the Landau functional only involves dot-products of $\vec{\phi}$, ensuring $O(2)$ rotational invariance.

To make contact with your basic knowledge of BEC, recall that for BEC the condensate fraction N_0/N, which we identify with $|\Psi|^2$ below $T_c = T_{BEC}$ grows as

$$|\Psi|^2 = \frac{N_0}{N} \sim 1 - \left(\frac{T}{T_c}\right)^{3/2} \sim |T_c - T|, \tag{25}$$

consistent with our mean-field scaling of the order parameter with $\beta_{MF} = 1/2$.

In addition to these type of bosonic superfluids, where atoms are strongly bound "molecule" of fermions (protons, neutrons, electrons), there are weakly-paired superfluids as in e.g., degenerate gases of fermionic atoms that pair via the so-called Feshbach resonance into a weakly bound Cooper pair, with the bosonic pairs then Bose condensing into a single-particle state Ψ. Solid-state superconductors are charged version of this phenomena, electrons weakly pair into charged Cooper pairs and at mean-field level are also well described by above XY model for Ψ. The qualitatively important difference is that charged Cooper pairs respond to magnetic field and thus require an inclusion of minimal gradient coupling to the corresponding gauge vector potential $\mathbf{A}$, which will will explore in later lectures.

The key new ingredient exhibited by the XY (and more generally an $O(N)$ vector) model but not by the scalar Ising model is the appearance of Goldstone modes, that at zero momentum are zero-energy excitations, correspond to global reorientation of the order parameter, under which the Landau functional is invariant by construction of the model. In terms of the Fig.13 it corresponds to the change of the order parameter along the minimum of the "Mexican hat" potential,

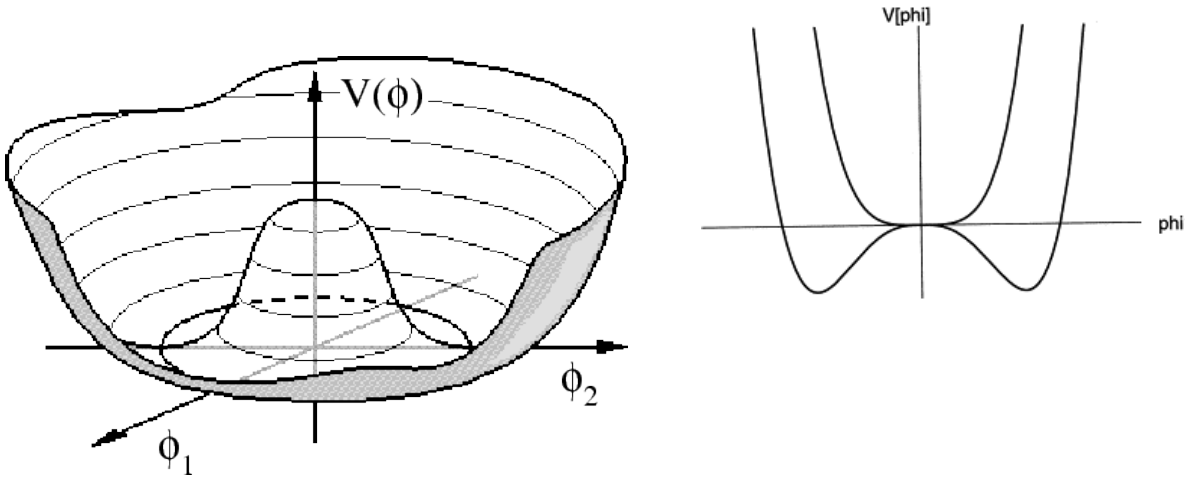


FIG. 13: A Mexican-hat potential and its cross-section controlling a continuous phase transition, illustrated for a two component order parameter $\mathbf{\Phi} = (\phi_1, \phi_2)$ (e.g., the normal-to-superfluid or XY PM-FM). Massive (gapped) amplitude (Higg's) and gapless Goldstone mode excitations, respectively correspond to radial and azimuthal fluctuations about $\mathbf{\Phi}_0$.

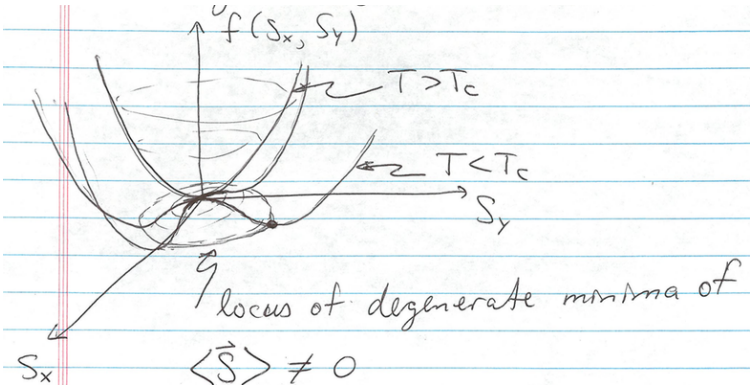


FIG. 14: Free energy of XY model for $T > T_c$ and $T < T_c$

Since such global excitations cost zero energy, such excitations at a finite wavevector will have very low energy, that will vanish in the limit of wavevector going to zero. Thus, Goldstone modes dominate low temperature thermodynamics and are therefore important to understand. We argued above that XY model exhibits a single Goldstone mode corresponding to the phase θ or equivalently fluctuations of $\vec{\phi}$ transverse to its spontaneous value

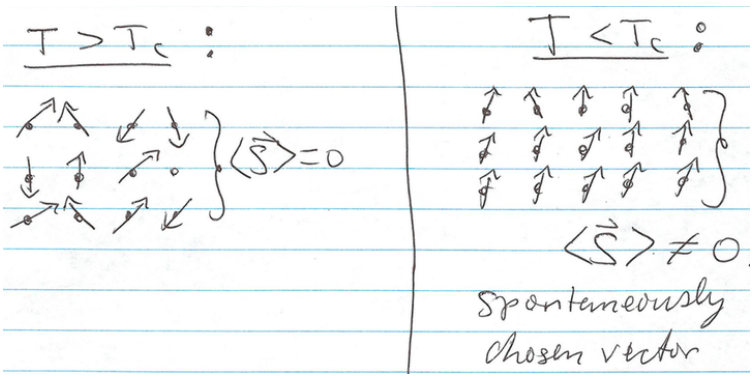


FIG. 15: Spontaneous symmetry breaking at $T < T_c$

$\vec{\phi}_0$.

More generally it is quite important to count and identify Goldstone modes for a broken symmetry phase. Systems with discrete symmetries (e.g., Ising, or q-states Potts model) have no Goldstone modes. For systems that break continuous symmetry, usually (but not always) Goldstone modes can be counted using the so-called G/H counting, where G is the symmetry group of the Hamiltonian, H is the symmetry group of the spontaneous order parameter $\vec{\phi}_0$, and G/H is a coset space constructed by identifying all the elements of G related by H. The reason for this protocol is that Goldstone modes are those excitations under which the free energy is invariant, but also the order parameter transforms nontrivially; otherwise it is not an excitation.

As an example, for $O(3)$ model (arising from e.g., Heisenberg FM), $G = O(3)$, $H = O(2)$ (corresponding to two-dimensional rotations about $\vec{\phi}_0$), and $G/H = O(3)/O(2) = S^2$, a two-dimensional surface of a three-dimensional ball, whose dimension is clearly 2. Thus, there are two Goldstone modes for $O(3)$ symmetry breaking transition, corresponding to two Euler angles of excitations along the surface of a 3d sphere. More generally for an $O(N)$ FM, there are $\dim[O(N)/O(N-1)] = S^{N-1} = N-1$. For $O(2)$ symmetric XY model this counting gives a single Goldstone mode.

We refer to this spontaneous choice of the multi-component order parameter as a "*spontaneous* symmetry breaking", to be contrasted with the "*explicit* symmetry breaking", by e.g., an external field. Of course in real materials, some symmetry breaking field, e.g. boundary of a sample, lattice effects, stray fields, etc will weakly choose the ordering of the order parameter, $\vec{\phi}$.

4. Tricritical point

As we have seen, ϕ^4 type Landau functionals generically exhibit a second-order transition, as the reduced temeperature (the quadratic coefficient), t changes sign and two degenerate, finite ϕ_0 minima continuous develop. Note, however, that this continuous development only happens if the quartic coupling, u is positive.

In contrast, for a negative quartic coupling, $u < 0$, the pure ϕ^4 theory is unstable and requires us to keep higher order terms in the order parameter, that, generically, is the next lowest order ϕ^6 term. It is clear from the general form of the resulting ϕ^6 Landau model for $u < 0$ and $v > 0$, that it exhibits a first-order transition at $t_c \sim u^2$ (when the finite ϕ_0

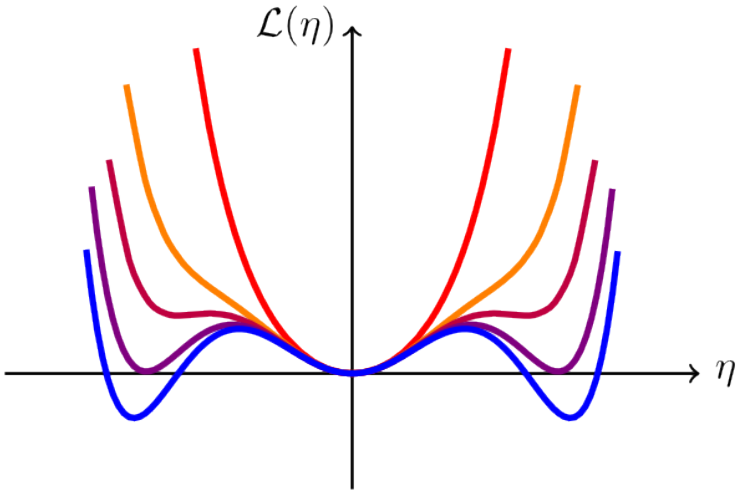


FIG. 16: A ϕ^6 even Landau model exhibiting a first-order transition for $u < 0$ and a second-order transition for $u > 0$ separated by a tricritical point at $t = 0, u = 0$.

minima become degenerate with $\phi = 0$ one) and a spinoidal at $t = 0$, when the curvature at the origin vanishes. Upon increasing u, the nature of transition changes to a second-order transition at a tricritical point, $t = 0, u = 0$, as illustrated in Fig.17.

5. Isotropic-to-Nematic (IN) transition in liquid crystals

Liquid crystals are fascinating systems of anisotropic constituents (typically rode- or plate-like, though there are quantum liquid crystals of even point-like electrons, driven by strong frustrated interactions), that exhibit a rich variety of phases intermediate between a fully-disordered isotropic fluid and fully-ordered crystalline solid. Classical liquid crystals[7,

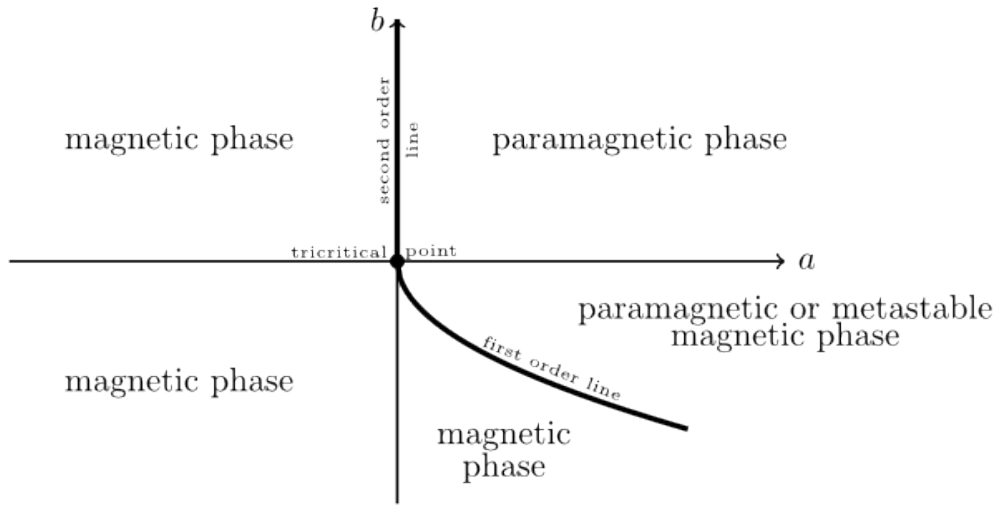


FIG. 17: A ϕ^6 even Landau model exhibiting a first-order transition for $u < 0$ and a second-order transition for $u > 0$ separated by a tricritical point at $t = 0, u = 0$.

16] are typically driven by competing orientational and positional entropies, with some most common phases illustrated in Fig.18

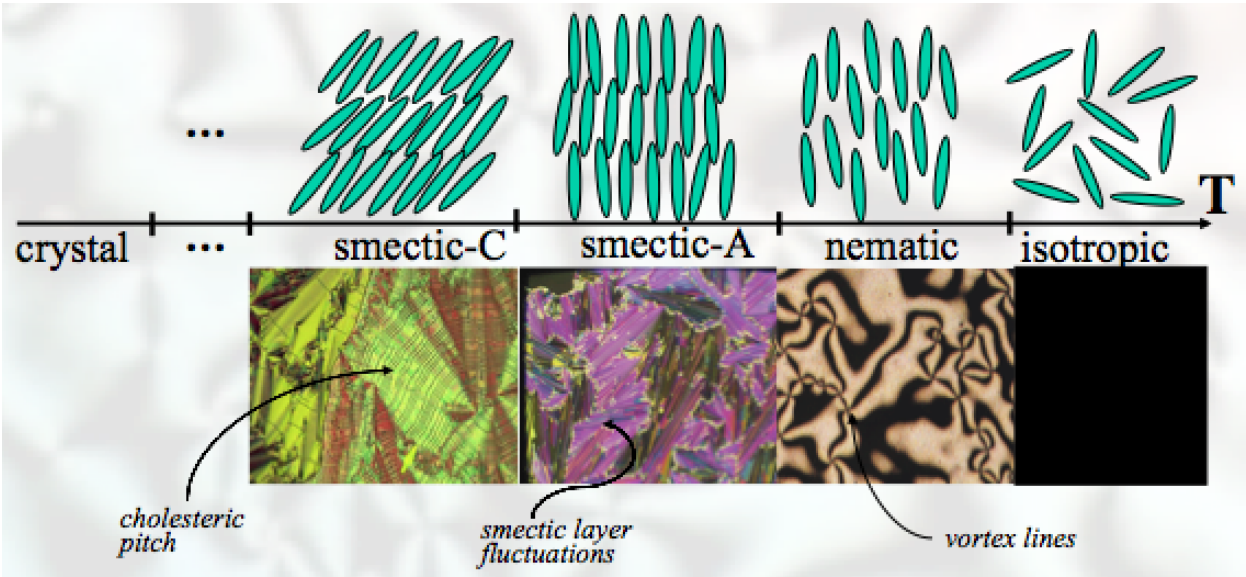


FIG. 18: Most ubiquitous nematic (orientationally ordered uniaxial fluid), smectic-A and smectic-C (one-dimensional density wave with, respectively isotropic and polar in-plane fluid orders) liquid crystal phases and their associated textures in cross-polarized microscopy (N.A. Clark laboratory).

To be concrete we focus on the Isotropic-to-Nematic (IN) liquid crystal transition.[7, 16] To identify the corresponding order parameter I note that the distinction between isotropic

and nematic fluids is that in the former the rod-like molecules are isotropically distributed, while in the latter (by the very definition of the nematic phase) they align along some spontaneously chosen axis (see Fig.18). I note that it is a spontaneous choice of an *axis* (headless arrow) and *not* of a vector (as e.g., in a FM phase) that defines the nematic phase. This is because nematic axis $\hat{n}$ ordering is driven by steric molecular interaction. Thus, in contrast to the Heisenberg model's exchange interaction, the nematic director interaction is given by the Lebwohl-Lasher model (biquadratic interaction)[7, 16],

$$H_N = -\frac{1}{2}K\sum_{ij}(\hat{n}_i \cdot \hat{n}_j)^2, \tag{26}$$

with not only the Hamiltonian but also the nematic phase invariant under the Z_2 symmetry, $\hat{n} \to -\hat{n}$.

More macroscopically, the nematic phase is characerized by anisotropic quadrapolar dielectric tensor ϵ_{ij}, that distinguishes it from an optically isotropic fluid, $\epsilon^I_{\alpha\beta} = \epsilon_0\delta_{\alpha\beta}$. The nematic order parameter is thus a tensor

$$Q_{\alpha\beta} = S\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right) + B\left(m_\alpha m_\beta - \ell_\alpha\ell_\beta\right), \tag{27}$$

with $S \neq 0, B = 0$ describing a uniaxial anisotropy along $\hat{n}$ and $S \neq 0, B \neq 0$ capturing the biaxial anisotropy in the $\hat{m}-\hat{\ell}$ plane, transverse to $\hat{n}$; $\hat{n}, \hat{m}, \hat{\ell}$ form an orthonormal triad. The $Q_{\alpha\beta}$ traceless symmetric tensor generically exhibits three distinct eigenvalues, controlled by strengths of the uniaxial and biaxial orders S and B, respectively. As a check, I note that in the isotropic phase, $\hat{n}_\alpha\hat{n}_\beta$ averages to $\frac{1}{3}\delta_{\alpha\beta}$ and $m_\alpha m_\beta$ and $\ell_\alpha\ell_\beta$ have identical averages. $Q_{\alpha\beta}$ thus vanishes in the isotropic phase, nonzero in the nematic phase, and is therefore the characteristic order parameter of the IN transition.

The corresponding Landau Hamiltonian density must be rotationally invariant, with all the indices contracted,

$$\begin{aligned}
\mathcal{H}[Q_{\alpha\beta}] &= \frac{1}{2}tQ_{\alpha\beta}Q_{\beta\alpha} - \frac{1}{3}wQ_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\alpha} + \frac{1}{4}u(Q_{\alpha\beta}Q_{\beta\alpha})^2, &(28)\\
&= \frac{1}{2}t\mathrm{Tr}(\mathbf{Q}^2) - \frac{1}{3}w\mathrm{Tr}(\mathbf{Q}^3) + \frac{1}{4}u[\mathrm{Tr}(\mathbf{Q}^2)]^2, &(29)\\
&= \frac{1}{2}\tilde{t}S^2 - \frac{1}{3}\tilde{w}S^3 + \frac{1}{4}\tilde{u}S^4, &(30)
\end{aligned}$$

where the last expression was worked out for the uniaxial nematic with $B = 0$ and I leave it to you to work out tilde'd coefficients. I note that in fact there appears to be a second

distinct quartic invariants of $\mathbf{Q}$ that is possible, namely $Q_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\delta}Q_{\delta\alpha} = \mathrm{Tr}(\mathbf{Q}^4)$. However, in 3d these can be shown to be proportional to each other, as I encourage the reader to verify for herself.

I further note, that, because there is no symmetry under $\mathbf{Q} \to -\mathbf{Q}$, (with this corresponding to oblate [disks] to prolate [rods] uniaxial order transformation), crucially in 3d, above Landau theory allows a cubic invariant, $\mathrm{Tr}(\mathbf{Q}^3) \propto S^3$. Thus, based on earlier discussion of the liquid-gas transition, our analysis generically predicts a first-order IN phase transition, that is only continuous if w is fine-tuned to zero. In contrast in two-dimensions the cubic term is not allowed (i.e., vanishes, as I invite the reader to demonstrate for herself) and the 2d IN transition is thus generically second order, isomorphic to the XY universality class.

Finally, I note, that by G/H counting a 3d Nematic is characterized by two Goldstone modes. However, although locally the coset space looks like a two-dimensional sphere, globally it is the $RP^2 = S^2/Z_2$ manifold, i.e., a two-dimensional sphere with antipodal points identified due to headless arrow ($\hat{n}_\alpha \hat{n}_\beta$) nature of the nematic order parameter.

6. Liquid-to-crystal (L-Cr) phase transition

A feature that distinguishes a 3d crystal from a fluid is the existence of 3d periodicity of the density, n. Thus, a crystal can be thought of as a liquid that develops a three dimensional mass density wave. Complementary to this, is a reciprocal, momentum (Fourier) space description, in which the distinguishing crystal feature is the appearance of nontrivial Fourier coefficients $n_{\mathbf{G}}$ of the mass density,

$$n(\mathbf{x}) = \sum_{\mathbf{G}} n_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{x}}, \tag{31}$$

where $\mathbf{G}$ span the reciprocal lattice of the crystal. $n_{\mathbf{G}}$ are thus a set of order parameters for crystalization.

The corresponding Landau Hamiltonian that describes the L-Cr transition must be translationally invariant and is thus given by

$$\mathcal{H}[n_{\mathbf{G}}] = \frac{1}{2}t|n_{\mathbf{G}}|^2 - \frac{1}{3}w \sum_{\{Gv_i\}}{}' n_{\mathbf{G}_1} n_{\mathbf{G}_2} n_{\mathbf{G}_3} + \frac{1}{4}u \sum_{\{Gv_i\}}{}' n_{\mathbf{G}_1} n_{\mathbf{G}_2} n_{\mathbf{G}_3} n_{\mathbf{G}_4} + \ldots, \tag{32}$$

where the sums can be limited to a set of fundamental reciprocal lattice vectors, and prime denotes a constraint, $\sum_i \mathbf{G}_i = 0$ of momentum conservation. Because of the cubic invariant, allowed in three dimensions (but not in 2d, see KTHNY theory), generic crystallization transition is first-order.

7. *Liquid-to-"stripe" states (CDW, SDW, PDW) phase transition*

Another important set of a states that break translational symmetry of a liquid are "striped" phases of matter and their generalizations. Physical examples of these include spin- , charge- , Cooper-pair density waves (respectively SDW, CDW, PDW), that exhibit periodic stripe-like order. In these systems the spatial ordering of the corresponding density takes place at a *colinear*, commensurate set of wavevectors, $\mathbf{q}_n$, thus the density wave is one dimensional "crystallization" in a D-dimensonal "fluid". Thus, in a prominant example is that of a charge-density wave, where electron charge density develops a 1D modulation. The charge density is a real scalar, $n(\mathbf{x})$, that can be expressed as a discrete Fourier series,

$$n(\mathbf{x}) = \sum_{\mathbf{q}_n} n_{\mathbf{q}_n} e^{i\mathbf{q}_n\cdot\mathbf{x}}, \tag{33}$$

$$\approx 2Re\left[\psi_{\mathbf{q}_0} e^{i\mathbf{q}_0\cdot\mathbf{x}}\right] = 2|\psi_{\mathbf{q}_0}|\cos[\mathbf{q}_0\cdot\mathbf{x} + \theta(\mathbf{x})], \tag{34}$$

where in the second line we limited the characterization of the 1D periodic CDW in terms of the leader complex order parameter $\psi_{\mathbf{q}_0}(\mathbf{x}) = |\psi_{\mathbf{q}_0}(\mathbf{x})|e^{i\theta(\mathbf{x})}$, the dominant Fourier coefficients of the charge density at wavevectors $\mathbf{q}_{\pm 1} = \pm\mathbf{q}_0$. It should be physically clear that the phase $\theta(\mathbf{x}) = q_0 u(\mathbf{x})$ describes a local translation of the CDW and thus is the corresponding 1D CDW phonon $u(\mathbf{x})$.

Given that the order parameter is complex scalar, $\psi(\mathbf{x}) = \psi_{\mathbf{q}_0}(\mathbf{x})$, it is not surprising that the corresponding Landau theory is simply that of an XY-model, given by

$$\mathcal{H}[\psi] = K|\boldsymbol{\nabla}\psi|^2 + t|\psi|^2 + \frac{1}{2}u|\psi|^4 + \dots. \tag{35}$$

We note that in the absence of a commensurate substrate the energy of the CDW must exhibits a global translational invariance, corresponding to $\theta \to \theta + \delta\theta$ and thus requires the Landau Hamiltonian to be invariant under phase rotation of ψ, i.e., exhibit global U(1) symmetry. This forbids cubic invariants and thus generically predicts a 2nd order liquid to CDW phase transition, in stark contrast of the first-order crystallization. We observe,

however, that above CDW system has rotational symmetry broken explicitly, namely the stripes ($\mathbf{q}_0$) cannot be freely reoriented, which is a standard situation in solid state (but not soft matter) physics due to an underlying atomic substrate.

If this last condition is relaxed, allowing the stripe state to also *spontaneously* break the underlying rotational symmetry of the disordered liquid, the description changes significantly, and corresponds to what is conventionally referred to as the smectic liquid crystal state, captured by a variant of the deGennes model. We will explore its significantly richer behavior in latter lectures.

Above arguments straightforwardly generalize to spin-density wave (SDW), where the observable is a spin density, $\vec{S}(\mathbf{x})$ that orders into periodic unidirectional stripes, represented again by,

$$\vec{S}(\mathbf{x}) = \sum_{\mathbf{q}_n} S_{\mathbf{q}_n} e^{i\mathbf{q}_n\cdot\mathbf{x}}, \tag{36}$$

$$\approx 2Re\left[\vec{\psi}e^{i\mathbf{q}_0\cdot\mathbf{x}}\right] \tag{37}$$

$$\mathcal{H}[\vec{\psi}] = K|\boldsymbol{\nabla}\vec{\psi}|^2 + t\vec{\psi}^*\cdot\vec{\psi} + \frac{1}{2}\lambda_1(\vec{\psi}^*\cdot\vec{\psi})^2 + \frac{1}{2}\lambda_2|\vec{\psi}\cdot\vec{\psi}|^2 + \ldots, \tag{38}$$

characterized by two quartic coupling constants.

III. BREAKDOWN OF LANDAU THEORY

One of the crucial ingredients of Landau Theory is that it neglects fluctuations of the order parameter, simply minimizing the Landau functional, rather than tracing over the fluctuating order parameter. Indeed as the transition is approached e.g., from below T_c, the correlation length $\xi(T)$ diverges and thermal excitations become larger and larger. A natural, crucial question is on the limit of validity of this MF approximation that neglects these excitations. Namely, what is the criterion for failure of Landau MFT theory? That it can indeed fail sufficiently close to a critical point is clear from the contrast between experimental and MFT exponents, as indicated in Fig.11.

As we will see (particularly when we proceed with a full field theoretic description) there are number of ways to assess the validity of Landau MF theory. The simplest is to note, that, clearly a MFT approximation fails when the condensation free-energy, F_{excit} (Fig.20) of

the excitation "bubble" of size set by the correlation length $\xi \sim |t|^{-\nu}$, (illustrated in Fig.19) is small compared to thermal energy, $k_B T$.

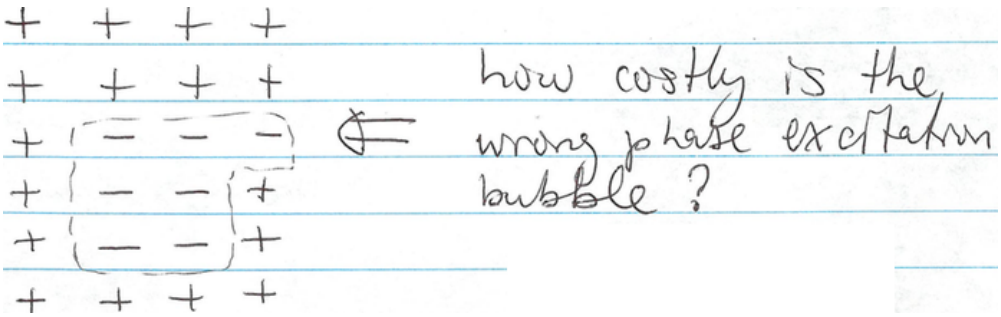


FIG. 19: Excitation "bubble" in the FM ordered state consisting of a region of overturned spins.

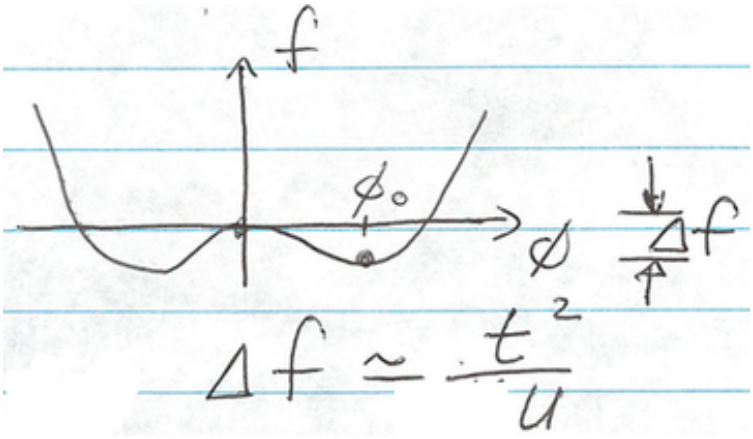


FIG. 20: Condensation free energy, Δf at a second-order phase transition.

To this end I note that the excitation free energy is given by

$$F_{excit}(t) \simeq \xi(t)^d \Delta f(t) \sim \xi_0^d |t|^{-d\nu} t^2/u, \tag{39}$$

exhibiting a competion as $|t| \to 0$. Namely, as T_c is approached, the size of the excitation bubble $\xi(t)$ grows, making it more costly, but at the same time the condensation free-energy density vanishes as t^2. To obtain the Ginzburg criterion on $|t_*|$, below which Landau MFT fails, we equate above excitation free energy to thermal energy, $\xi_0^d |t_*|^{2-d\nu}/u = k_B T$, taking $\nu = 1/2$ and obtaining,

$$t_* \approx \left(\frac{k_B T_c u}{\xi_0^d} \right)^{2/(4-d)}. \tag{40}$$

Thus, we conclude that for $d > d_{uc} = 4$, sufficiently close to T_c the excitation free-energy dominates over thermal energy, fluctuations can be ignored, and MFT approximaton is valid.

In contrast, for $d < d_{uc} = 4$ (where d_{uc} denotes the dimension below which MFT fails), the excitation free-energy vanishes and Landau MFT fails sufficiently close to T_c, within the Ginzburg region given by $|t| < |t_*$. I note that, as expected, the size of the Ginzburg region, t_* increases with increasing interaction strength u and strength of fluctuations $k_B T_c$.

For $d < d_{uc}$, within the Ginzburg region, superuniversality of MFT fails, and singularities crossover to those characterized by the universality class specific to the system. The latter is determined by the dimension of space, symmetry class, nature of the order parameter and in particular the number components, N. Our next goal is to utilize methods of field theory and renormalization group, introduced by Widom, Kadanoff, Migdal, and developed as a full-fledged calculational tool by Ken Wilson and Michael Fisher, to calculate the nontrivial critical behavior beyond Landau's MFT.

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