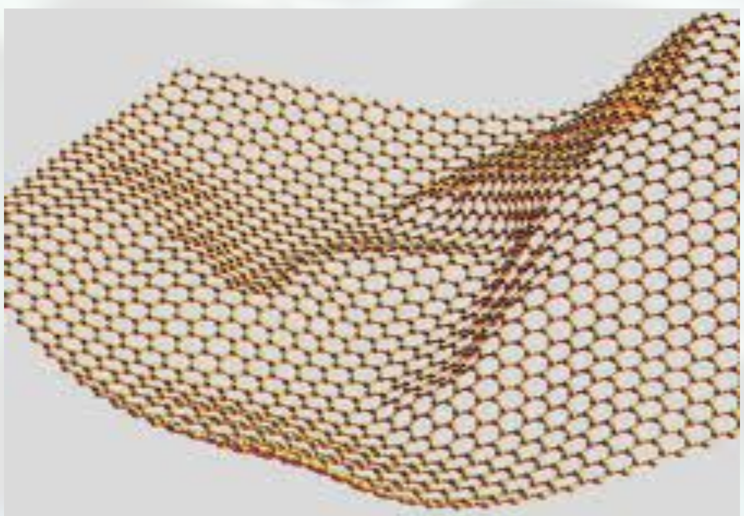
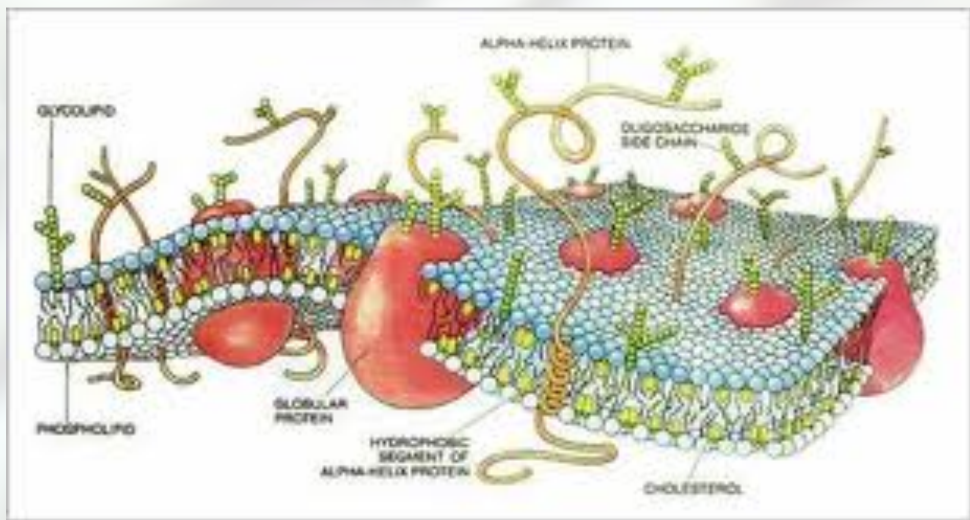
Statistical mechanics of elastic sheets

University of Colorado at Boulder

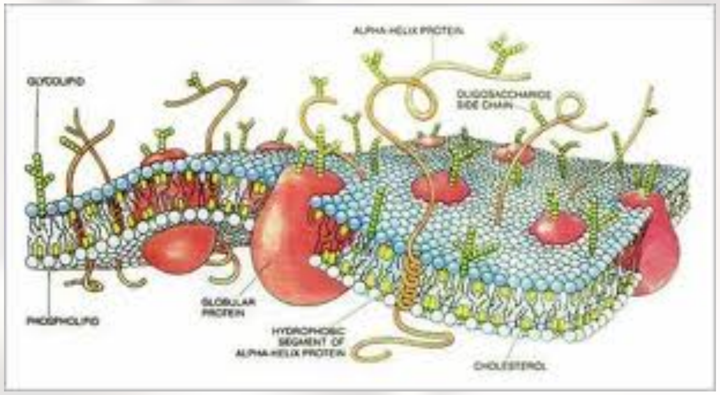




$ ***NSF-MRSEC, Materials Theory***

Thin Sheets, Lorentz Center, Sept, 2012

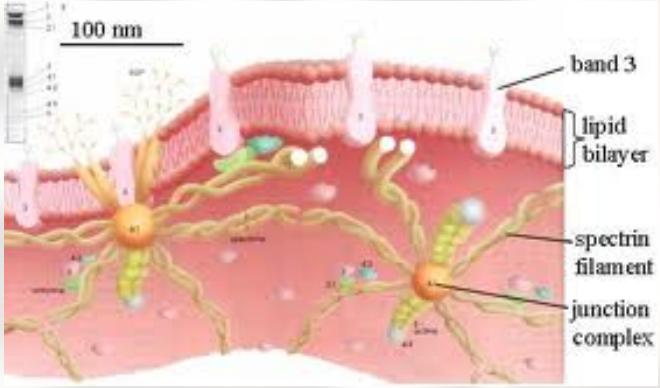
Outline



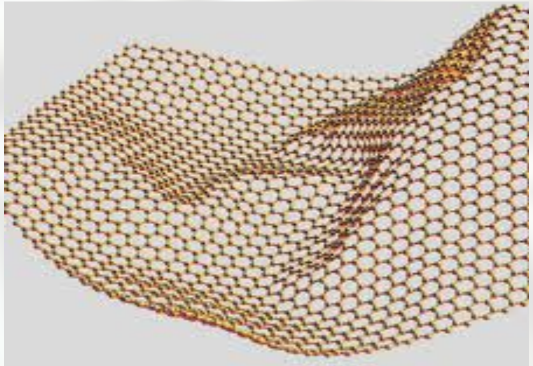
- Model
- Crumpling transition and global phase diagram
- Anomalous elasticity of flat phase
- In-plane (dis-)order: anisotropy and heterogeneity
- Open questions and conclusions

Motivation

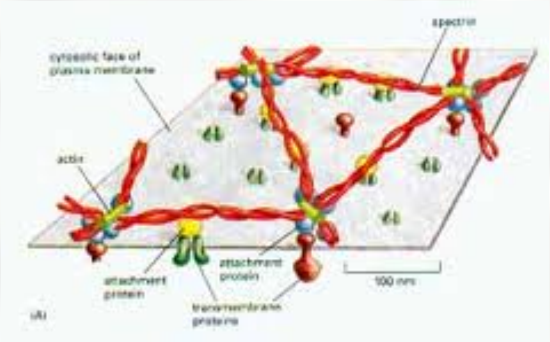
- physical realizations:



  - ✧ *biological membranes*



  - ✧ *graphene*

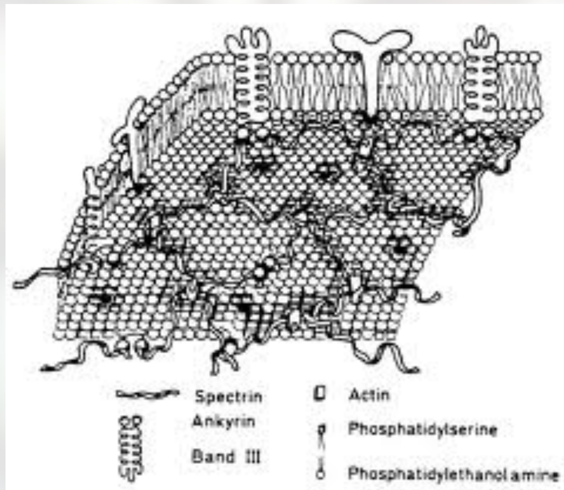


  - ✧ *2d polymers and gels*
  - ✧ *MoS_2, ZrP sheets*

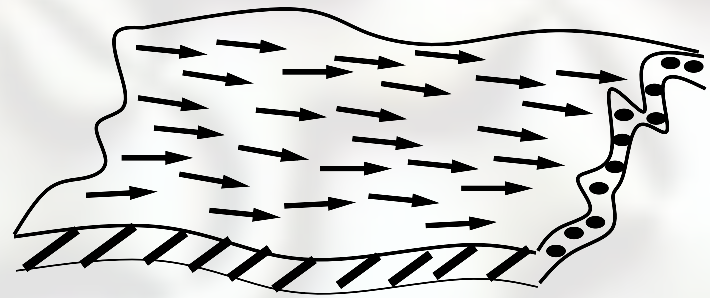
- fascinating interplay of statistical mechanics, field theory and geometry

Ingredients

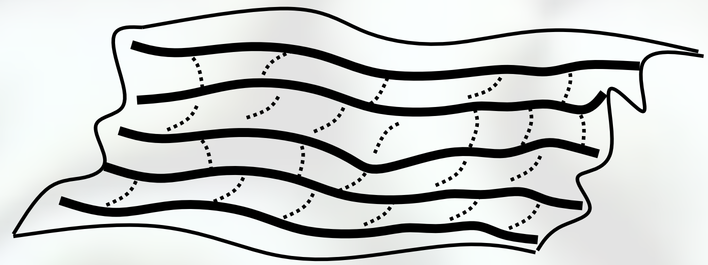
- vanishing surface tension
- bending rigidity



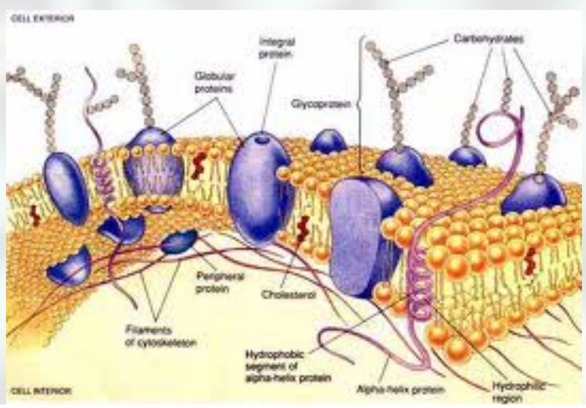
- in-plane elasticity



- in-plane order: anisotropy, hexatic,...

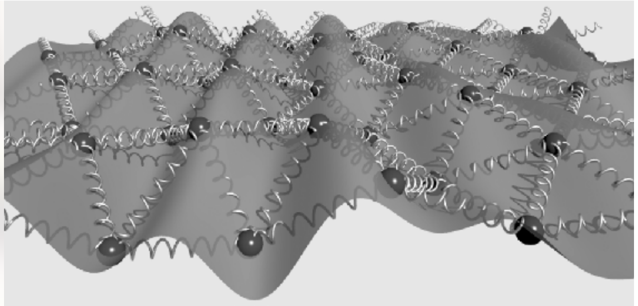


- heterogeneity

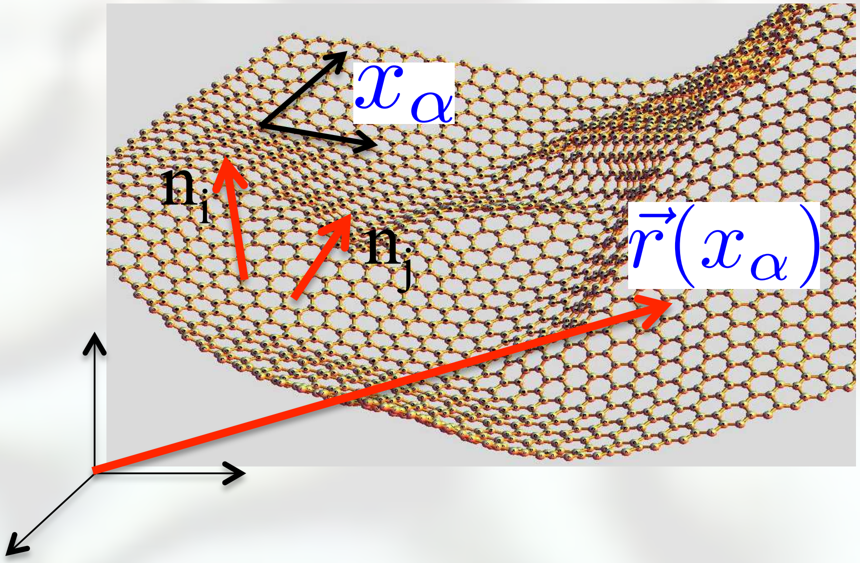


Kantor, Kardar, Nelson '86

Model



$$H = -\kappa \sum_{<ij>} \hat{n}_i \cdot \hat{n}_j + \sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|)$$



- want Landau description
- O(D) x O(d) order parameter:

$$\vec{t}_\alpha = \partial_\alpha \vec{r}$$

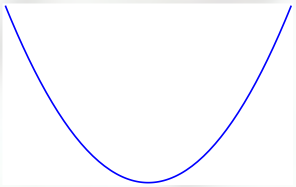
$$F[\vec{r}] = \int d^D x \left[\kappa(\partial^2 \vec{r})^2 + \tau_\alpha(\partial_\alpha \vec{r})^2 + g(\partial_\alpha \vec{r})^4\right] + v \int d^D x d^D x' \delta^{(N)}(\vec{r}(\mathbf{x}) - \vec{r}(\mathbf{x}'))$$

bending rigidity *self-avoidance*

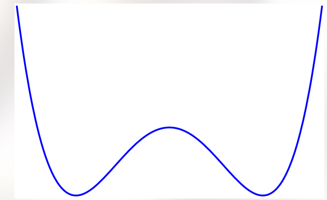
$\vec{t}_\alpha = \partial_\alpha \vec{r}$

Mean-field theory

Kantor, Kardar, Nelson '86
L.R., Toner '97, '99



$$f \sim \tau_\alpha (\vec{t}_\alpha)^2 + g(\vec{t}_\alpha)^4$$



ignore $k_B T$, minimize:

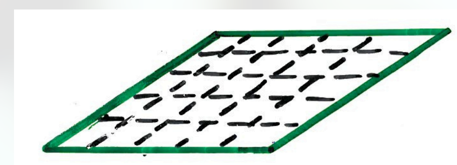
- Crumpled phase ($\tau_x > 0$, $\tau_y > 0$): $\langle \vec{t}_x \rangle = \langle \vec{t}_y \rangle = 0$

$\vec{r}_c = 0$

- Tubule phase ($\tau_x > 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle = 0, \langle \vec{t}_y \rangle > 0$

$\vec{r}_t = (0, t_y y, 0)$

- Flat phase ($\tau_x < 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle > 0, \langle \vec{t}_y \rangle > 0$

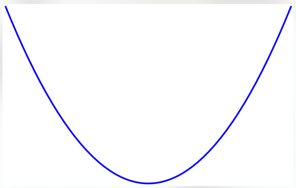


$\vec{r}_t = (t_x x, t_y y, 0)$

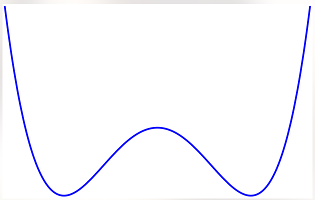
$k_B T$, self-avoidance, heterogeneity, nonlinearities: ???

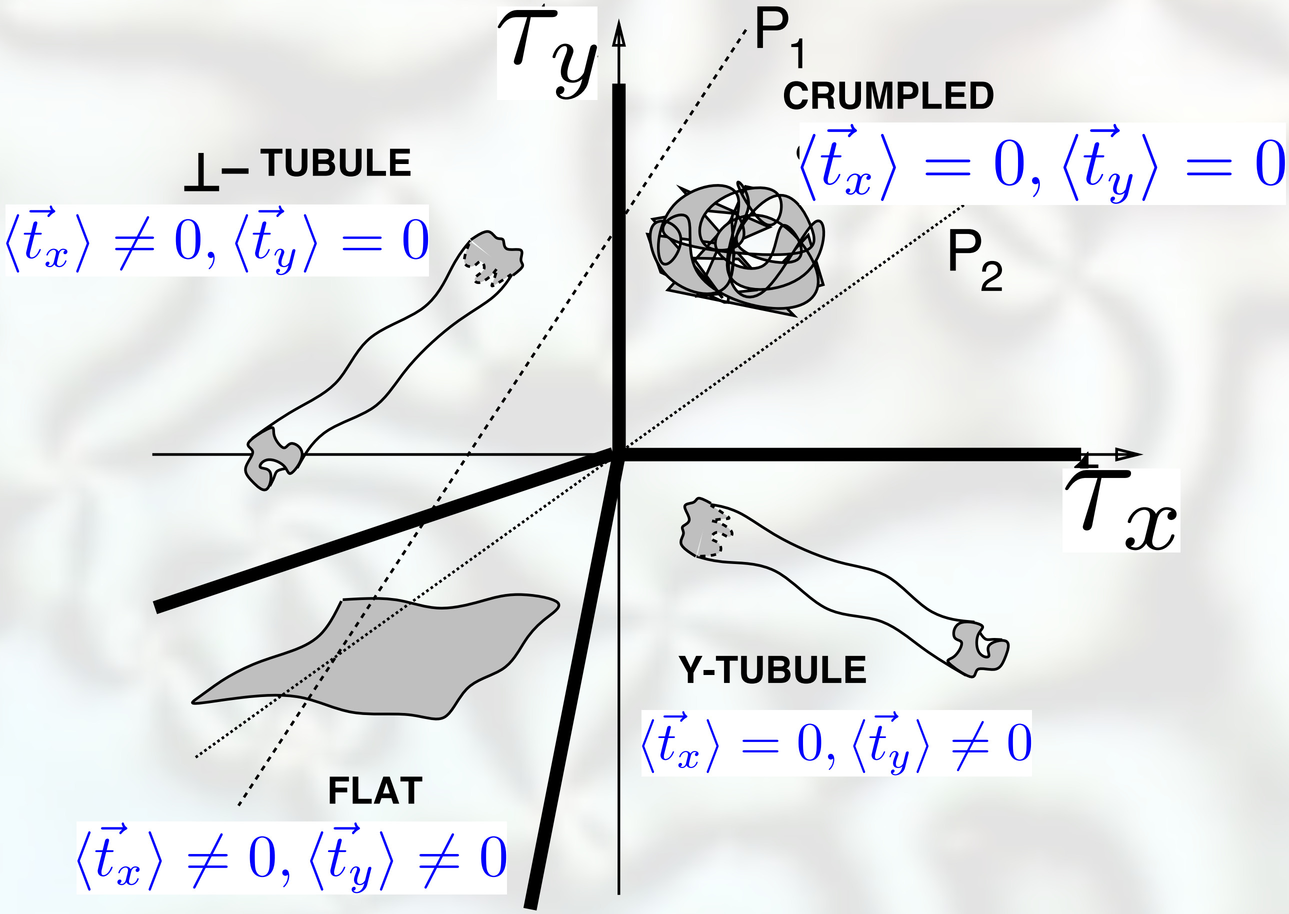
Crumpling transition

L.R., Toner '97, '99

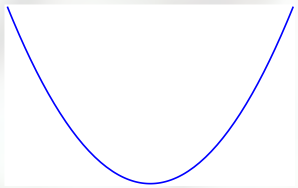


$$f \sim \tau_\alpha (\vec{t}_\alpha)^2 + g(\vec{t}_\alpha)^4$$

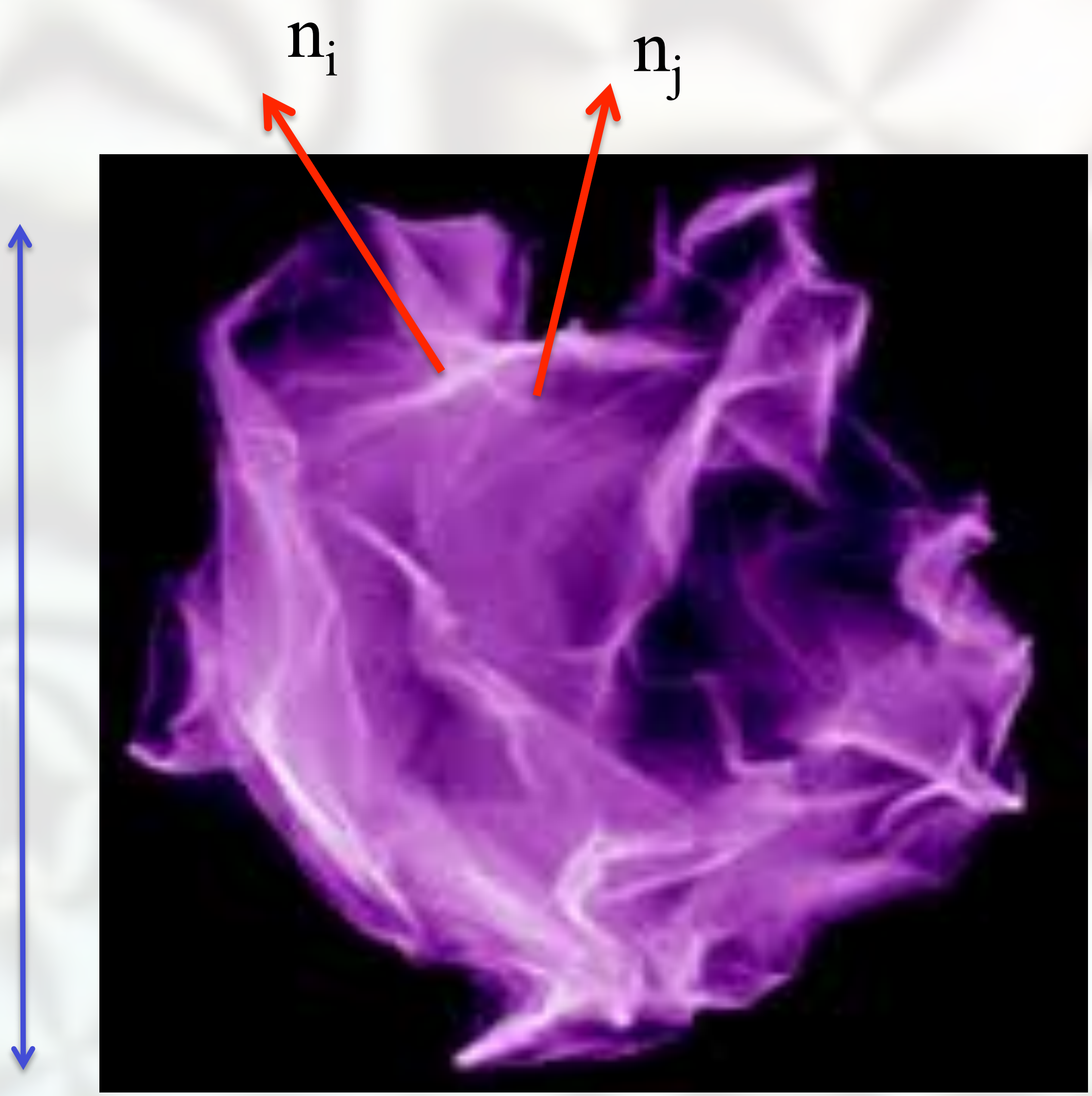




Crumpled phase



$R_G \sim L^{\nu}$



Crumpled phase

$$F_c[\vec{r}] = \tau \int d^D x (\partial_\alpha \vec{r})^2 + v \int d^D x d^D x' \delta^{(N)}(\vec{r}(\mathbf{x}) - \vec{r}(\mathbf{x}'))$$

- short-range order in normals

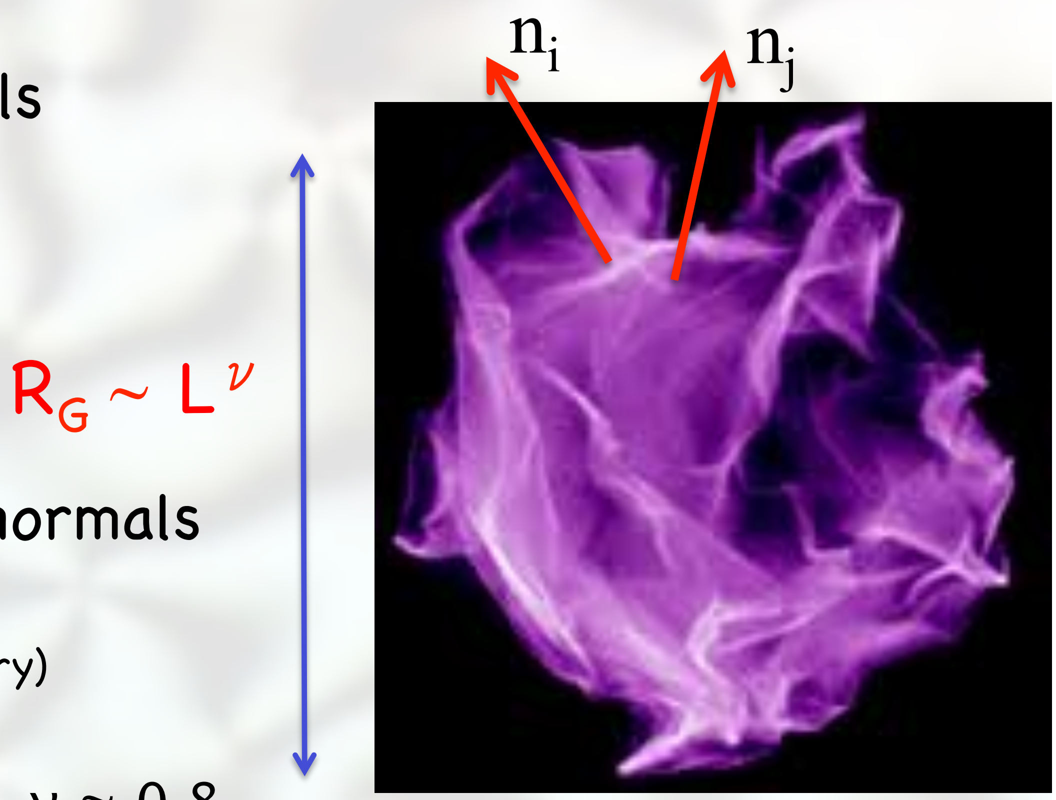
$$n_i \bullet n_j \approx e^{-|i-j|/\xi}$$

- disordered by $k_B T$

- analog of PM state of the normals

- fractal $M \sim R_G^{d_F}$ (Flory)

$d_F = D/\nu \approx D(d+2)/(D+2) = 2.5, \ \nu \approx 0.8$



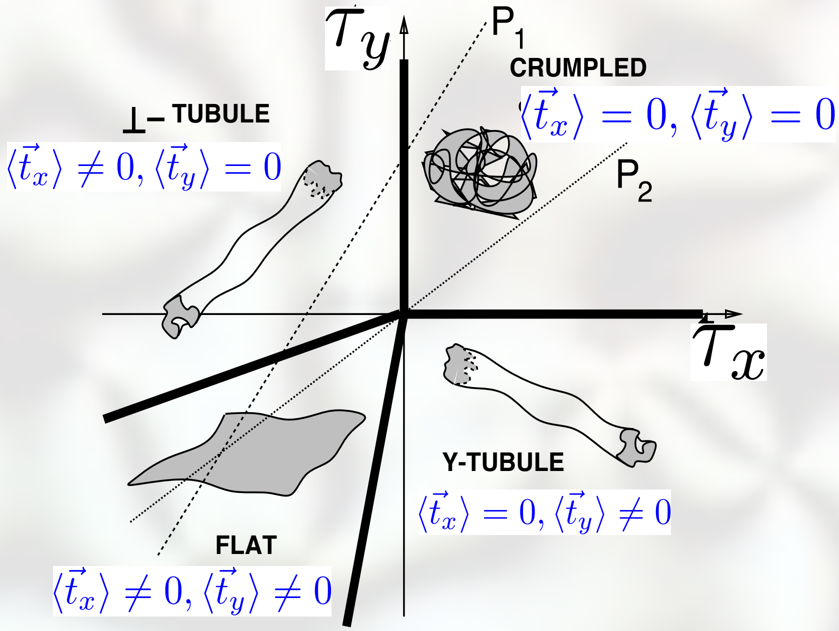
- self-avoiding interaction important: $R_G^0 \sim \sqrt{\ln L} \longrightarrow R_G \sim L^{4/5}$

$\vec{t}_\alpha = \partial_\alpha \vec{r}$

Mean-field theory

Kantor, Kardar, Nelson '86
L.R., Toner '97, '99

$f \sim \tau_\alpha (\vec{t}_\alpha)^2 + g(\vec{t}_\alpha)^4$



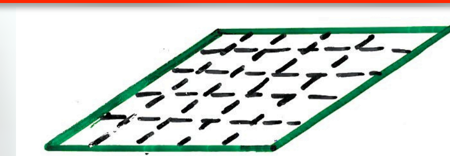
- Crumpled phase ($\tau_x > 0$, $\tau_y > 0$): $\langle \vec{t}_x \rangle = \langle \vec{t}_y \rangle = 0$

$\vec{r}_c = 0$

- Tubule phase ($\tau_x > 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle = 0, \langle \vec{t}_y \rangle > 0$

$\vec{r}_t = (0, t_y y, 0)$

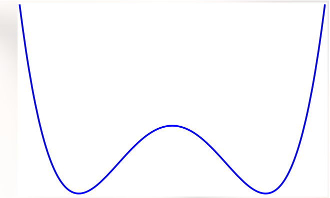
- Flat phase ($\tau_x < 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle > 0, \langle \vec{t}_y \rangle > 0$



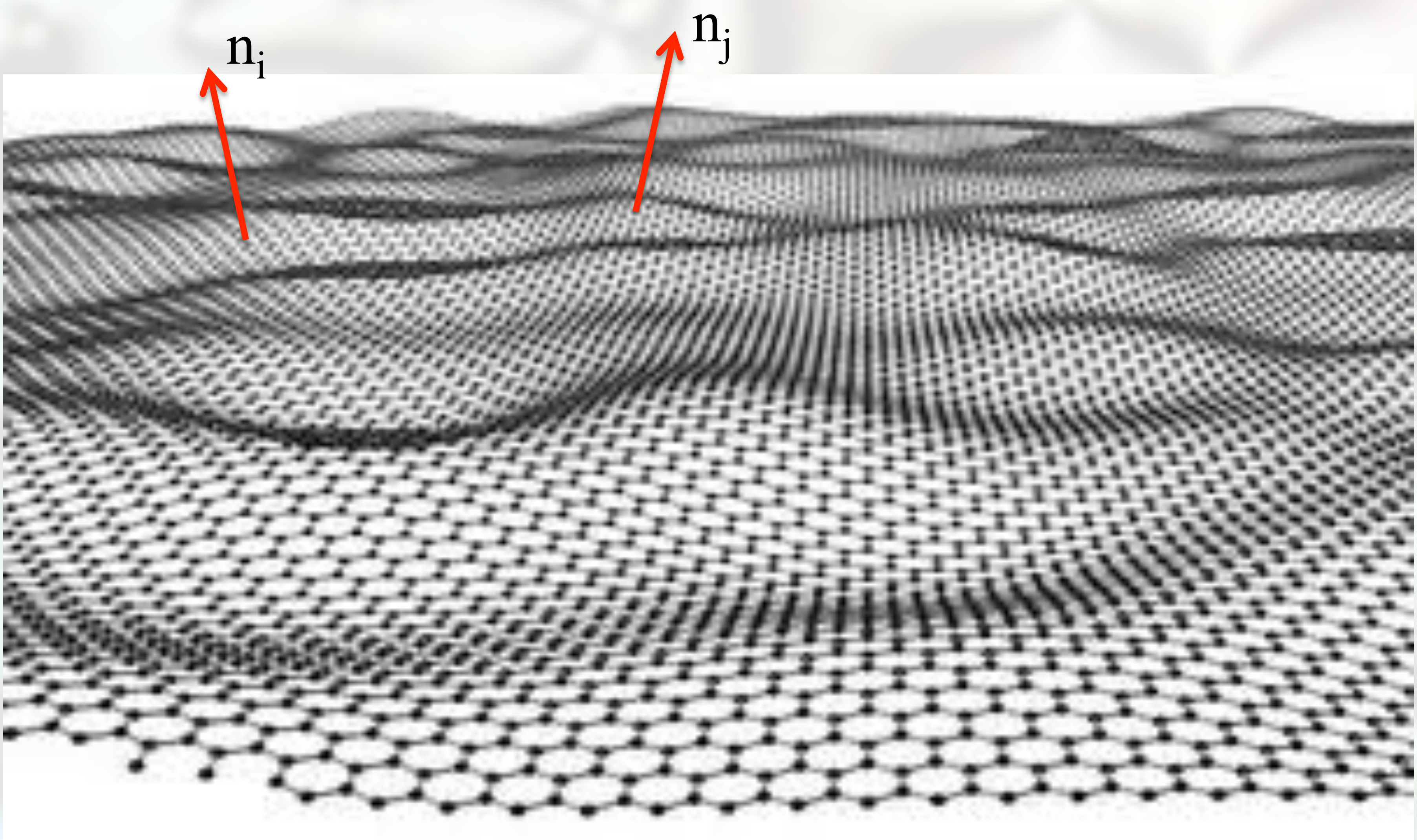
$\vec{r}_t = (t_x x, t_y y, 0)$

$k_B T$, self-avoidance, heterogeneity, nonlinearities: ???

"Flat" phase

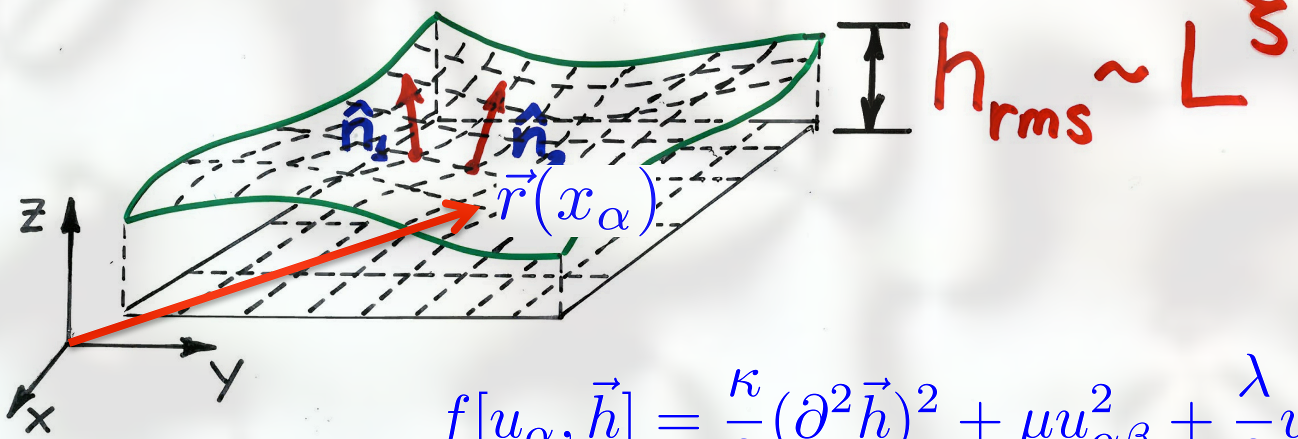


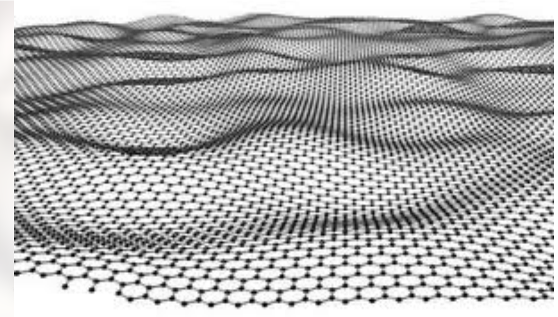
long range 2D orientation order: *impossible ?*



Nelson, Peliti '87

"Flat" phase





Aronovitz, Lubensky '88
David, Guitter '88

$$f[u_\alpha, \vec{h}] = \frac{\kappa}{2}(\partial^2 \vec{h})^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2$$

- long-range order in the normals, breaks O(3) symmetry
  - geometric analog of 2D ferromagnet in the normals
  - "circumvents" Mermin-Wagner-Coleman theorem, via order-from-disorder
- characterized by power-law roughness, $\zeta \approx 0.59$

LeDoussal, L.R.'92

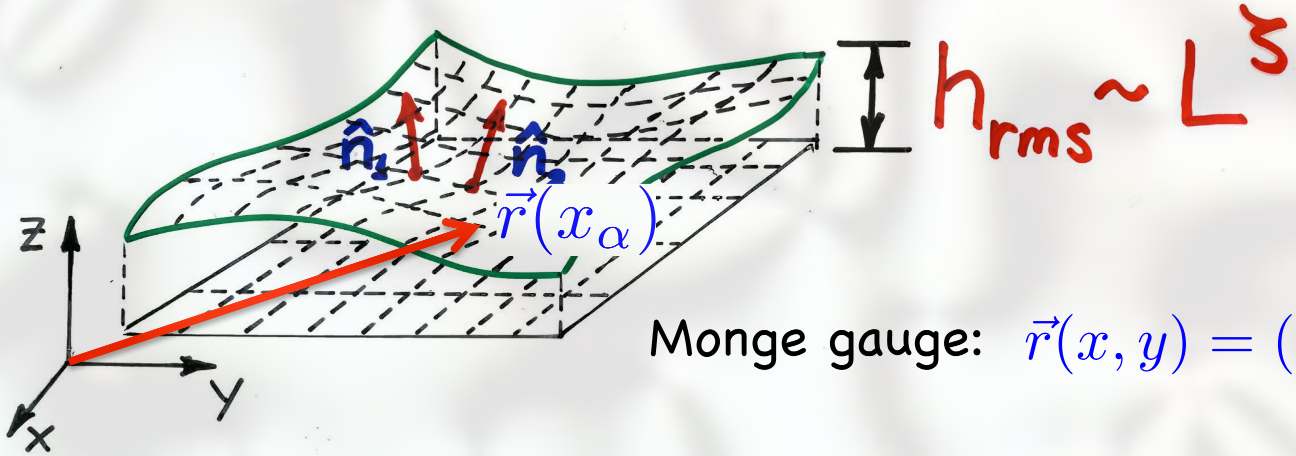
- "critical phase" with universal anomalous elasticity:

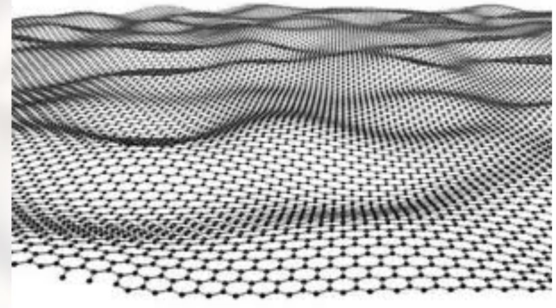
  $\kappa(L) \sim L^{\eta}$, $\mu(L) \sim L^{-\eta_u}$, $\sigma = -1/3$ $\quad (\eta_u = 4 - D - 2\eta)$

  (agrees well with MC simulations by Bowick, Falcioni et al, '96, '97)

Nelson, Peliti '87

"Flat" phase model





Monge gauge: $\vec{r}(x,y) = (x + u_x, y + u_y, h(x,y))$

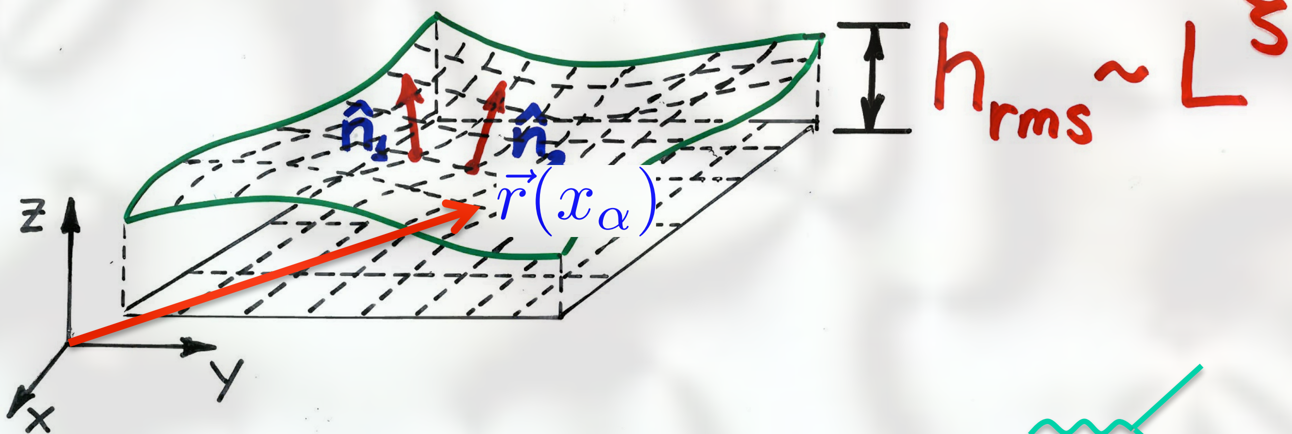
- free-energy density: $f[u_\alpha, \vec{h}] = \frac{\kappa}{2}(\partial^2 \vec{h})^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2$

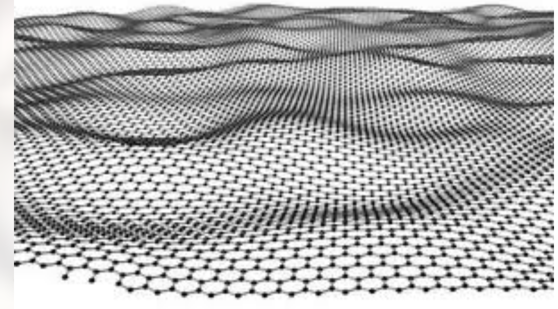
- nonlinear strain: $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \vec{h} \cdot \partial_\beta \vec{h}) = \frac{1}{2}(g_{\alpha\beta} - \delta_{\alpha\beta})$

- integrate out u_α: $f_{\text{eff}}[\vec{h}] = \frac{\kappa}{2}(\partial^2 \vec{h})^2 + \frac{1}{4}\underbrace{(\partial_\alpha \vec{h} \cdot \partial_\beta \vec{h}) K_{\alpha\beta,\gamma\delta} (\partial_\gamma \vec{h} \cdot \partial_\delta \vec{h})}$

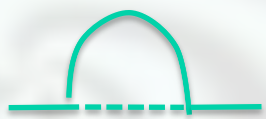
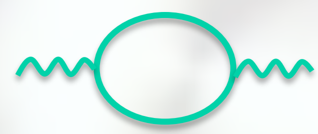
Gaussian curvature interaction: $R\frac{1}{\nabla^4}R$

$k_B T$ + nonlinearities





- PT in elastic nonlinearities: $\partial u \partial h \partial h + \frac{1}{4}(\partial h \partial h)^2$

$\delta\kappa \sim \frac{\mu T}{\kappa^2} L^{4-D} =$  $\delta\mu, \delta\lambda \sim -\frac{\mu T}{\kappa^2} L^{4-D} = -$ 

- diverges for $L > \xi_{\rm NL} \equiv \sqrt{\frac{\kappa^2}{\mu T}}$ *≈ 10Å for graphene → electronic physics*

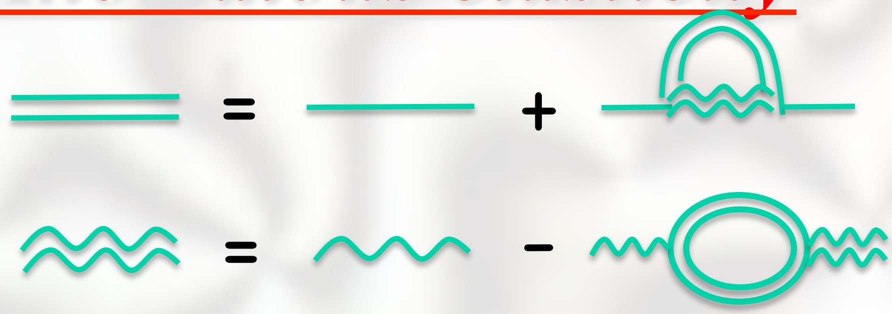
- need a fully nonlinear treatment; physical interpretation?

Nelson, Peliti '87

Anomalous elasticity

LeDoussal, L.R.'92

- SCSA theory:

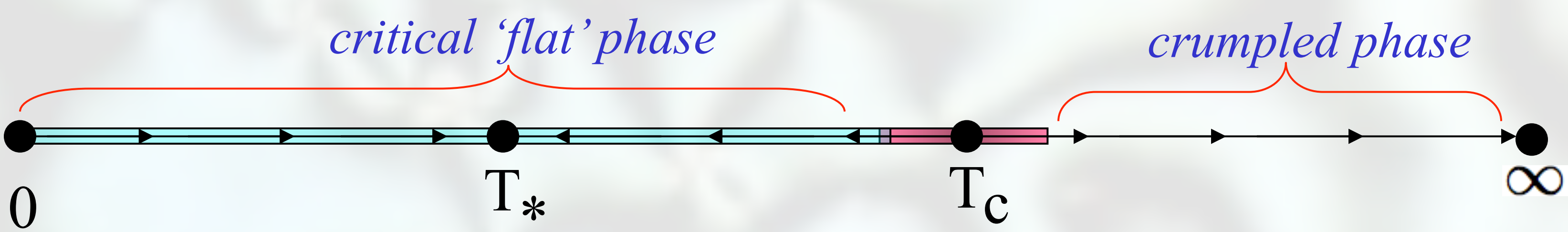


- length-scale dependent moduli: $\kappa(k) \sim k^{-\eta}, \quad \mu(k), \lambda(k) \sim k^{\eta_u}$

- Ward identity O(3) symmetry $(\partial u + \frac{1}{2}\partial h \partial h)$ → $\eta_u = 2 - 2\eta$

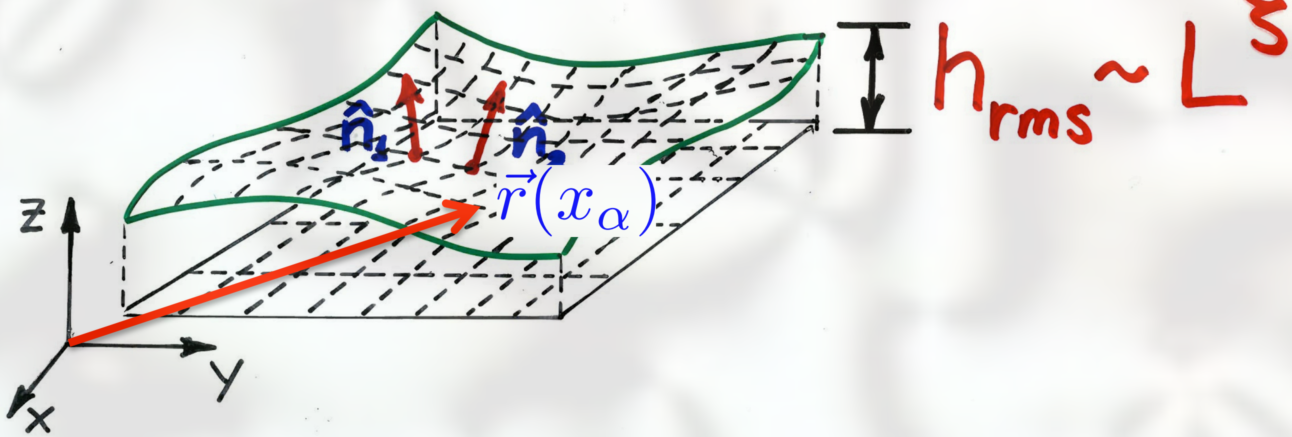
  $\kappa(L) = \mu(L) h_{\mathrm{rms}}(L)^2$

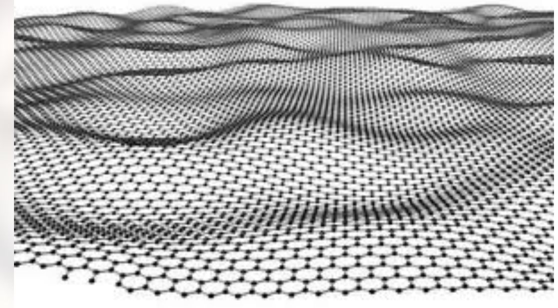
- RG with ε = 4–D, 1/d expansions *(Aronovitz-Lubensky, David-Guitter, '88)*



- SCSA exact: O(ε,d), O(1/d,D), at d=D: $\eta = 0.82, \ \zeta = 0.59, \ \sigma = -1/3$

Order-from-disorder





($\sim 10^{10}$ graphene → crumpling is irrelevant)

- *unstable* harmonically for $L > ae^{4\pi\kappa/3k_BT} \equiv \xi_{\mathrm{crump.}}$

$$\langle\theta^2\rangle \approx \frac{k_BT}{\kappa}\int\frac{d^2q}{(2\pi)^2}\frac{q^2}{q^4} \approx \frac{3k_BT}{4\pi\kappa}\ln L/a \longrightarrow \infty$$

- *stabilized* anharmonically by k_BT: $\theta_{\mathrm{rms}} \sim L^{-\eta/2}, h_{\mathrm{rms}} \sim L^{1-\eta/2}$

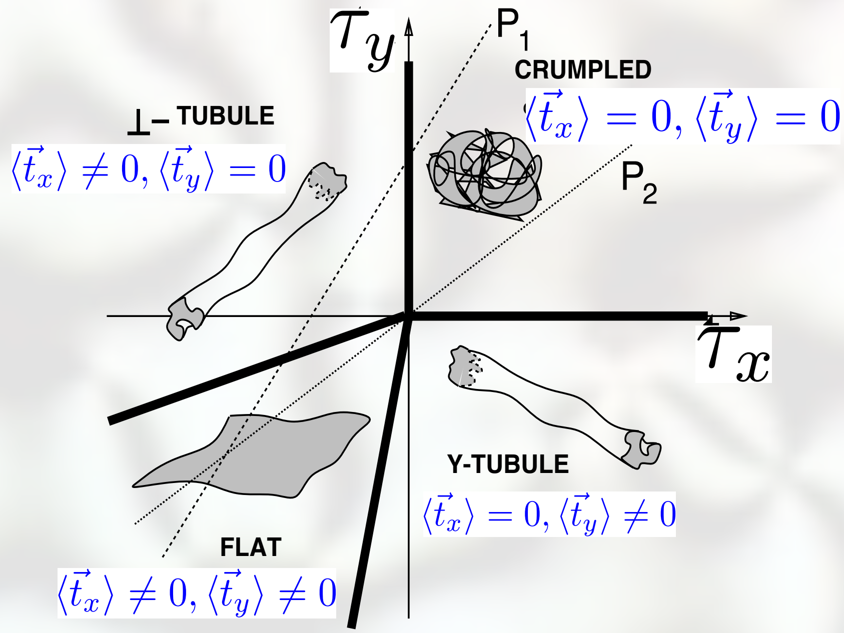
$$\langle\theta^2\rangle \approx k_BT\int\frac{d^2q}{(2\pi)^2}\frac{q^2}{\kappa(q)q^4} \sim TL^{-\eta} \longrightarrow 0$$

$\vec{t}_\alpha = \partial_\alpha \vec{r}$

Mean-field theory

Kantor, Kardar, Nelson '86
L.R., Toner '97, '99

$f \sim \tau_\alpha (\vec{t}_\alpha)^2 + g(\vec{t}_\alpha)^4$



- Crumpled phase ($\tau_x > 0$, $\tau_y > 0$): $\langle \vec{t}_x \rangle = \langle \vec{t}_y \rangle = 0$

$\vec{r}_c = 0$

- Tubule phase ($\tau_x > 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle = 0, \langle \vec{t}_y \rangle > 0$

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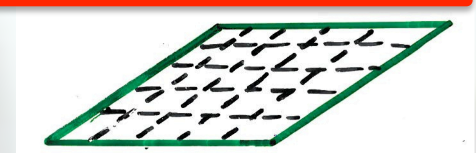
- Flat phase ($\tau_x < 0$, $\tau_y < 0$): $\langle \vec{t}_x \rangle > 0, \langle \vec{t}_y \rangle > 0$



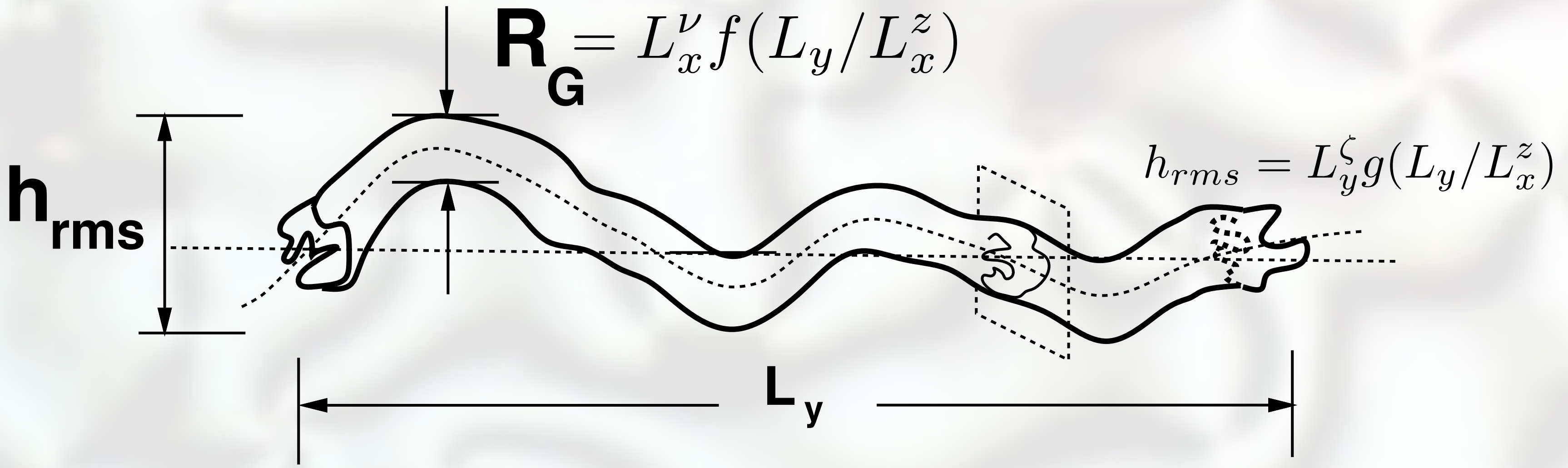
$\vec{r}_t = (t_x x, t_y y, 0)$

$k_B T$, self-avoidance, heterogeneity, nonlinearities: ???

Tubule phase

L.R., Toner '97, '99

$$f_y = \frac{\kappa}{2}(\partial_y^2 \vec{h})^2 + \frac{t}{2}(\partial_\alpha^\perp \vec{h})^2 + \frac{g_\perp}{2}(\partial_\alpha^\perp u)^2 + \frac{g_y}{2}\left(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2\right)^2 + f_{\rm SA}$$



- long-range orientational order in 1d, breaks O(3) symmetry:

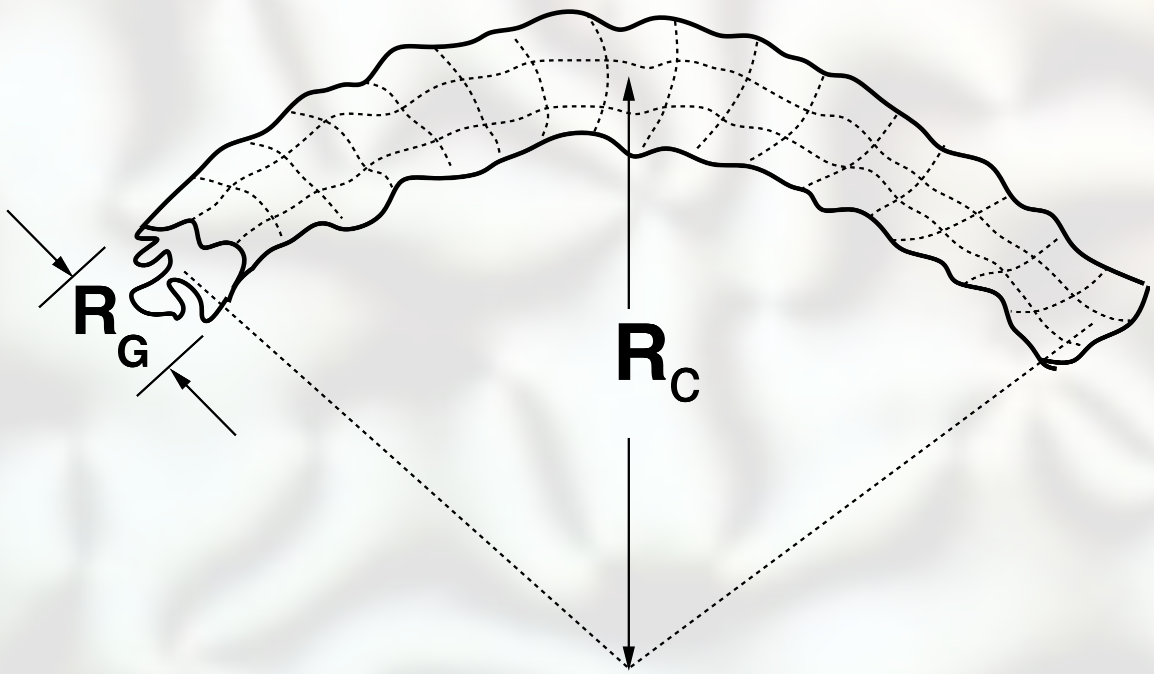
$$\langle\theta^2\rangle \sim L^{-\eta} \ll 1 \longrightarrow \text{stable to } \; k_B T > 0$$

- nontrivial anomalous fixed point (with SA):

$$h_{rms} \sim L^{1/4}, R_G \sim L^{3/4}, \kappa(L) \sim L^{3/2}$$

Tubule anomalous elasticity

L.R., Toner '97, '99



$$h_{rms} = L_y^{\zeta} g(L_y/L_x^z)$$

$$R_G = L_x^{\nu} f(L_y/L_x^z)$$

- Length-scale dependent bending rigidity:

$$\kappa(L) \sim \mu(L) R_G(L)^2$$

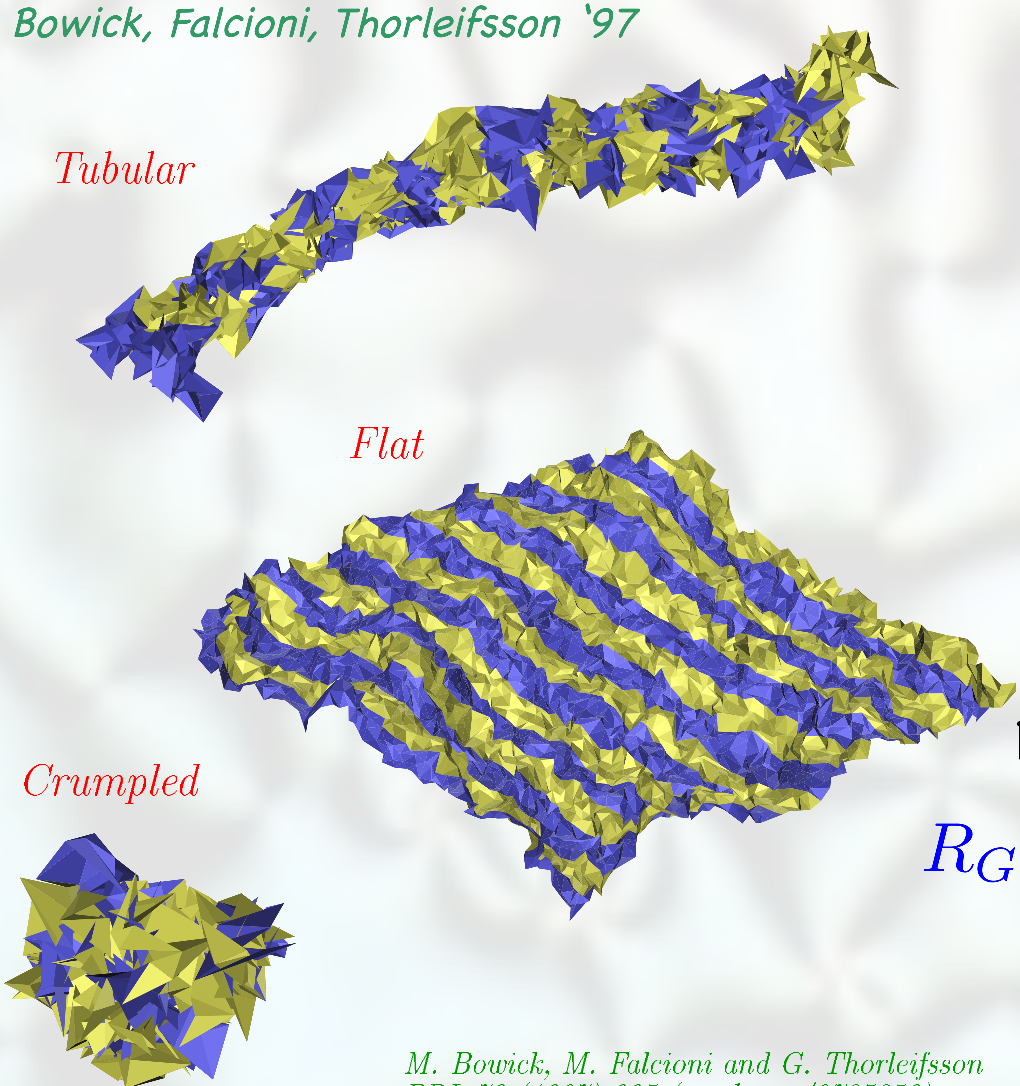
$$\rightarrow 2\nu = z(\eta_\kappa + \eta_\mu)$$

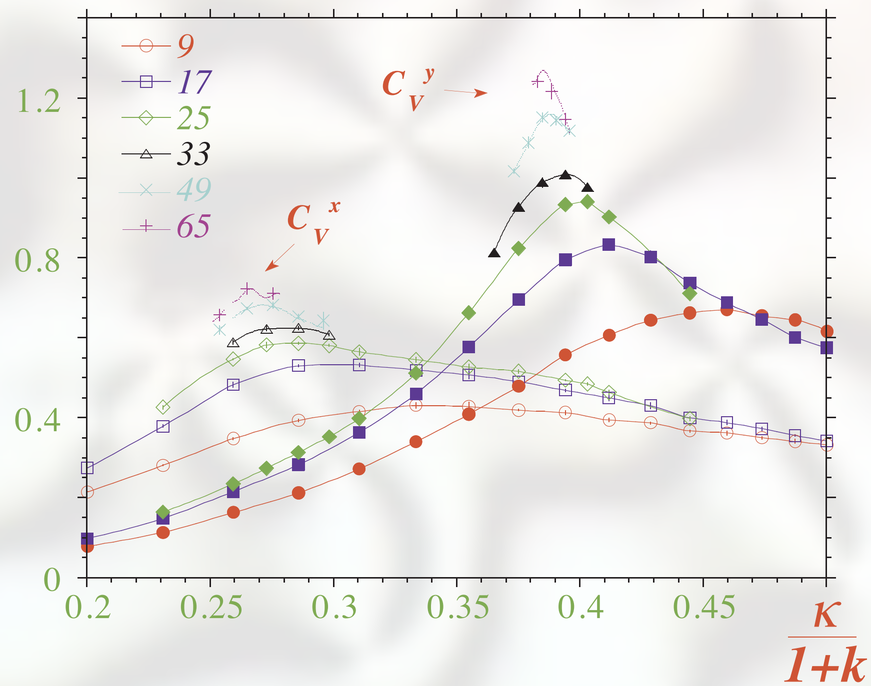
phantom

Monte-Carlo simulations

$$h_{rms} = L_y^{\zeta} g(L_y/L_x^z)$$
$$R_G = L_x^{\nu} f(L_y/L_x^z)$$

Bowick, Falcioni, Thorleifsson '97





Excellent agreement with R.T.:

$$R_G \sim L^{1/4},\ h_{\rm rms} \sim L\ \ (z = 1/2)$$

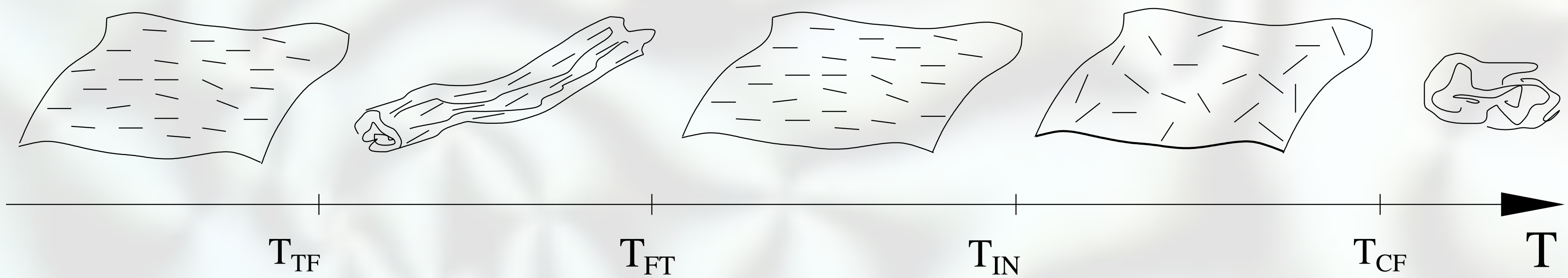
zero (ribbon) mode

M. Bowick, M. Falcioni and G. Thorleifsson
PRL 79 (1997) 885 (cond-mat/9705059)

Tunable spontaneous anisotropy

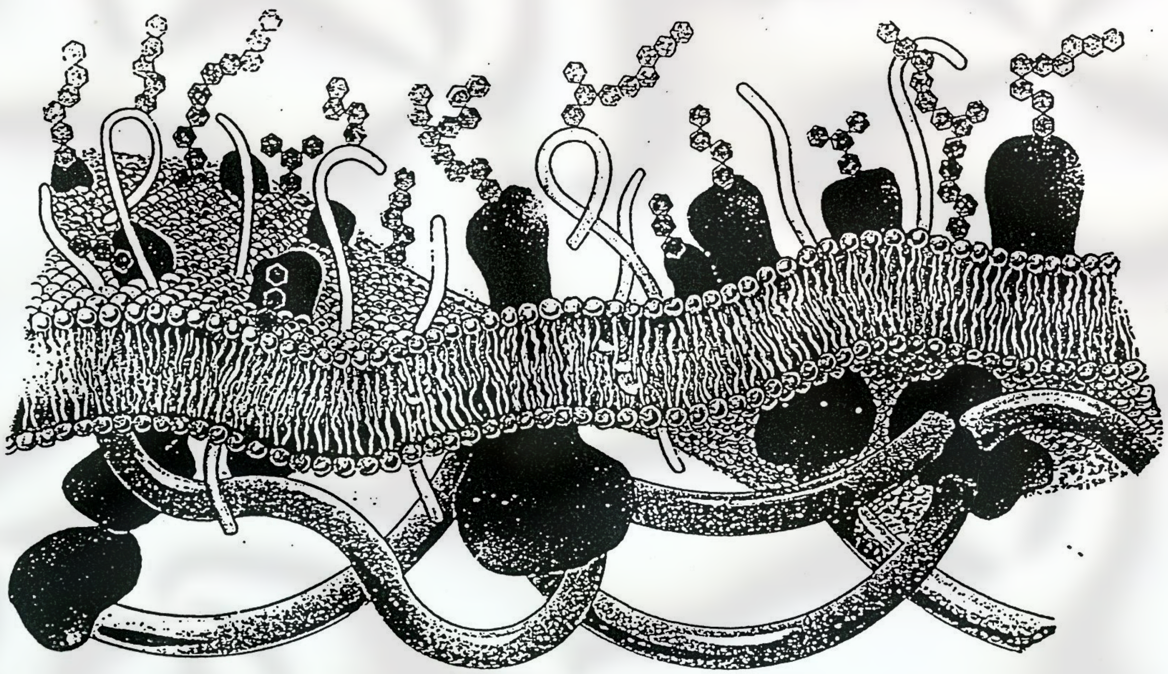
X. Xing, L.R. '04

- *spontaneous* in-plane nematic order (e.g., nematic elastomer membrane) → reentrant flat phase:



Local heterogeneity

L.R., Nelson '91, '92
Bensimon, et al '91
L.R., LeDoussal '91, '92
Morse, Lubensky '92

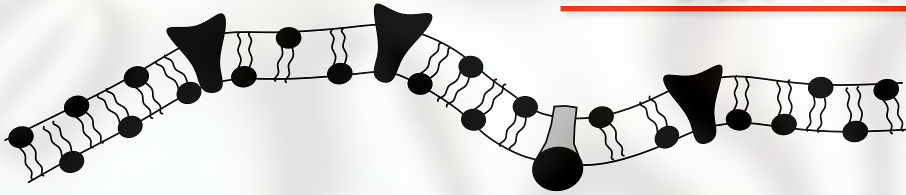


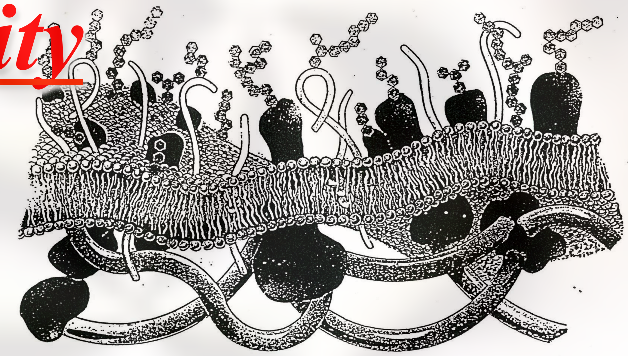
- proteins, nano-pores, holes, network defects, …

  → random distribution of interstitials, dislocation, disclinations, grain-boundaries, …

L.R., LeDoussal '91, '92

Local heterogeneity





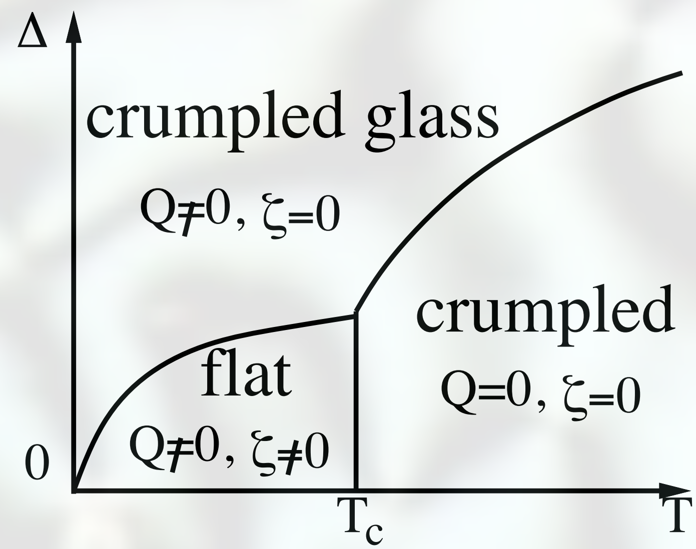
- random stresses, preferred curvature:

$$f = \frac{\kappa}{2}(\partial^2 h - c(\mathbf{x}))^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2 - u_{\alpha\beta}\sigma_{\alpha\beta}(\mathbf{x})$$

- ***"flat glass"*** ground state, anomalous elasticity:

$$\eta = 0.45$$
$$\zeta = 0.775$$

- ***"crumpled glass"*** ground state

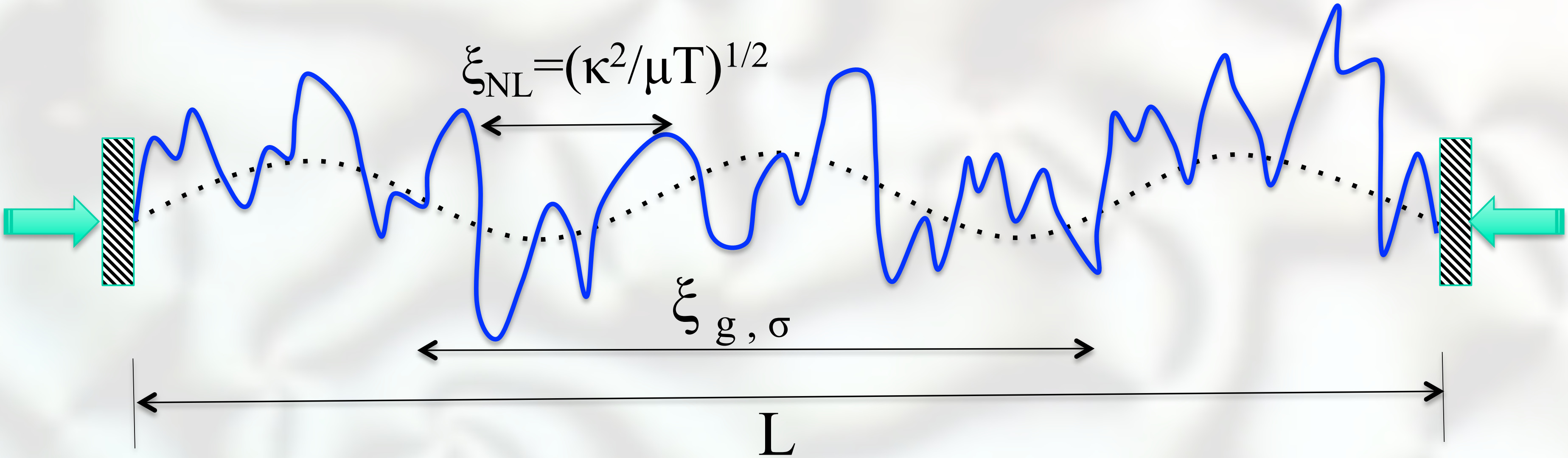


Open questions and implications

- systematic quantitative measurements e.g., graphene
- realization of the crumpling transition
- sheets with tunable anisotropy
- nature of glassy phases
- statistical mechanics of membranes with nontrivial background strain and topology *(see e.g., vesicles: Nelson, et al.)*
- *redoing deformation analysis (Euler, Lame, crumpling,...) for free energy*

Buckling of "flat" phase

- want: $e^{-F/T} = Tr_{h,u} e^{-H[h,u]/T}$
- "poor man's" scaling theory → nonlinear elasticity = no linear response



$$H = \frac{1}{2}\int_x \left[\kappa_R(\nabla^2 h)^2 + \rho g h^2 - \sigma(\nabla h)^2 + \ldots\right]$$

$$\xi_L = L$$

$$P_c^{(L)} = \frac{\kappa}{L^2} \to \frac{\kappa}{L^{2-\eta}}$$

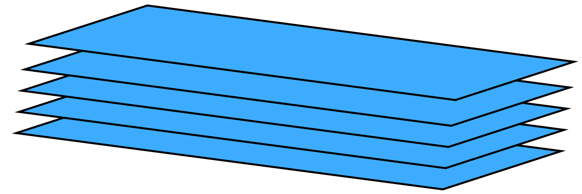
$$\xi_g = \left(\frac{\kappa}{\rho g}\right)^{1/4} \to \left(\frac{\kappa}{\rho g}\right)^{1/(4-\eta)}$$

$$P_c^{(g)} = (\rho g \kappa)^{1/2} \to (\rho g \kappa)^{(2-\eta)/(4-\eta)}$$

"Soft" elastic systems: critical phases

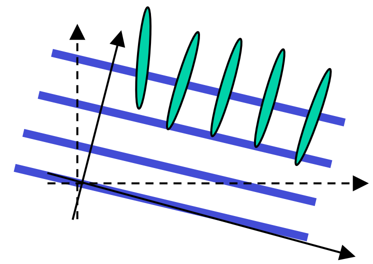
guiding principle: ***partial breaking of spatial symmetry***

- **Smectic phase** (*Grinstein + Pelcovits*)



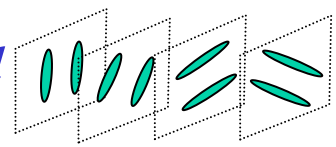
$$H = K(\nabla^2 u)^2 + B(\partial_z u + \frac{1}{2}(\nabla u)^2)^2$$

harmonic | *rotational invariance* | nonlinear

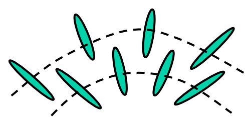


$$H = B_\perp(\nabla_\perp u - \delta n)^2 + B_z(\partial_z u)^2 + K_s(\nabla \cdot n)^2 + K_{tb}(\nabla \times n)^2$$

Higgs mechanism ⟶ ***twist of $\delta\hat{n}$ expelled***

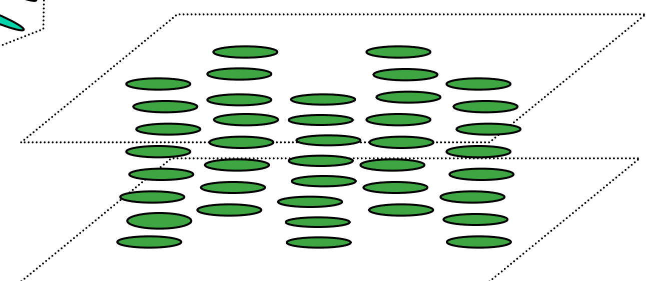


but not splay



- **Columnar phase** *(L.R. + Toner)*

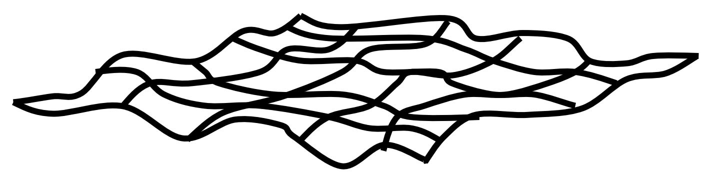
(spontaneous vortex lattice in FM superconductor)



- **Tensionless polymerized membrane** *(Nelson+Peliti, Aronovitz +Lubensky, Le Doussal + L.R.)*

$$H = \frac{\kappa}{2}(\nabla^2 h)^2 + \mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2$$

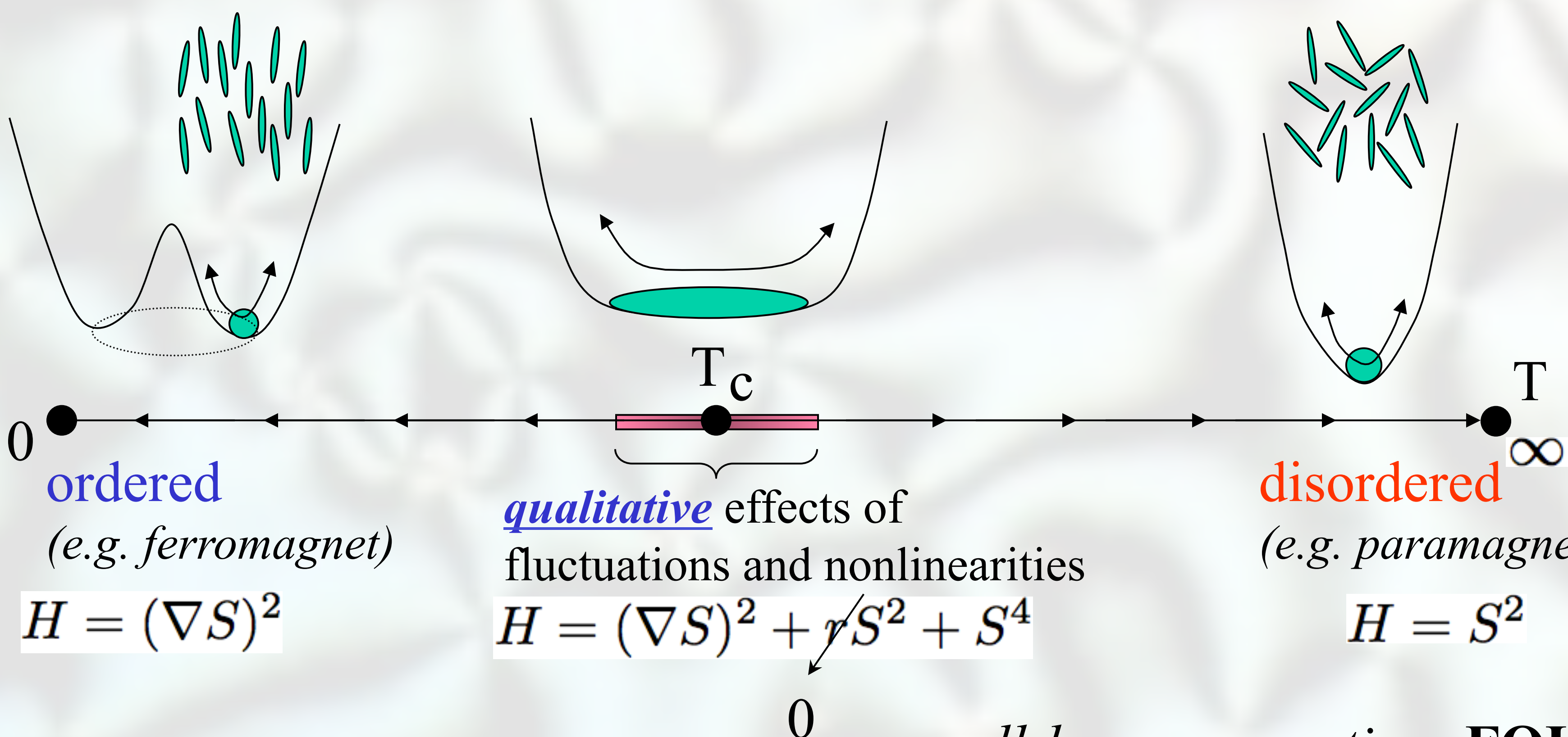
$$u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i h \partial_j h)$$



Fluctuations, nonlinearities and phase transitions

Upshot of 40 years of research on fluctuations and critical phenomena:

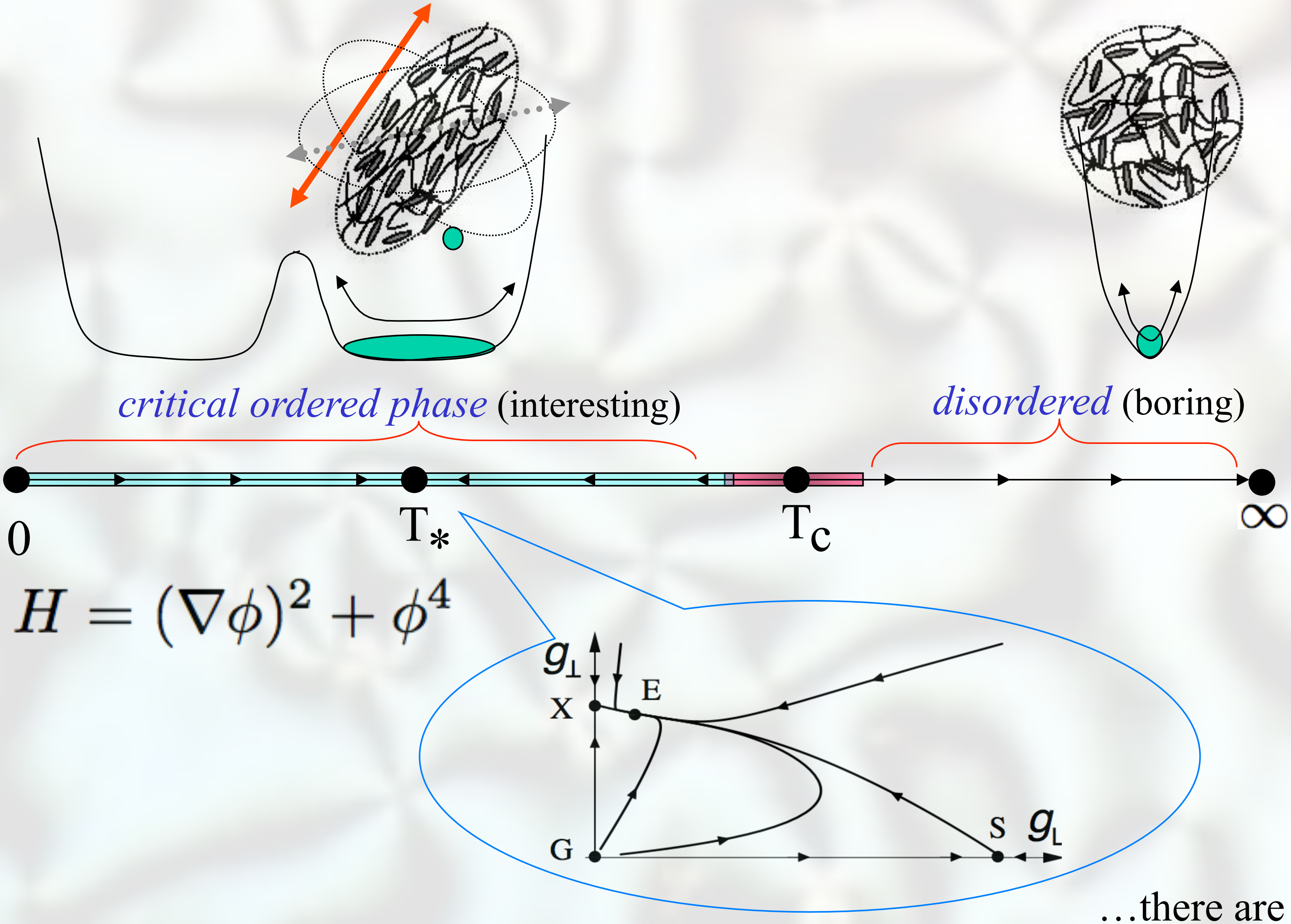
Fluctuations <u>and</u> nonlinearities (*usually*) ***are only important near isolated critical points (continuous phase transition)***



well-known exception: **FQHE**

Critical phases

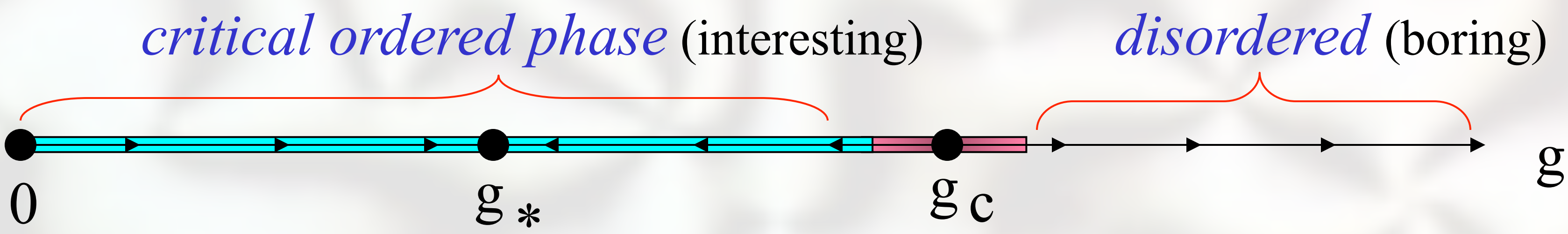
Xing + L.R., PRL (2003)



$$H = (\nabla\phi)^2 + \phi^4$$

…there are others

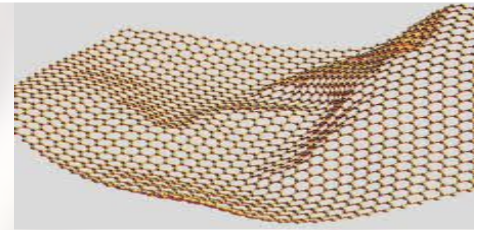
Properties of critical phases



$$H = (\nabla^2 u)^2 + (\partial_z u + (\nabla u)^2)^2$$

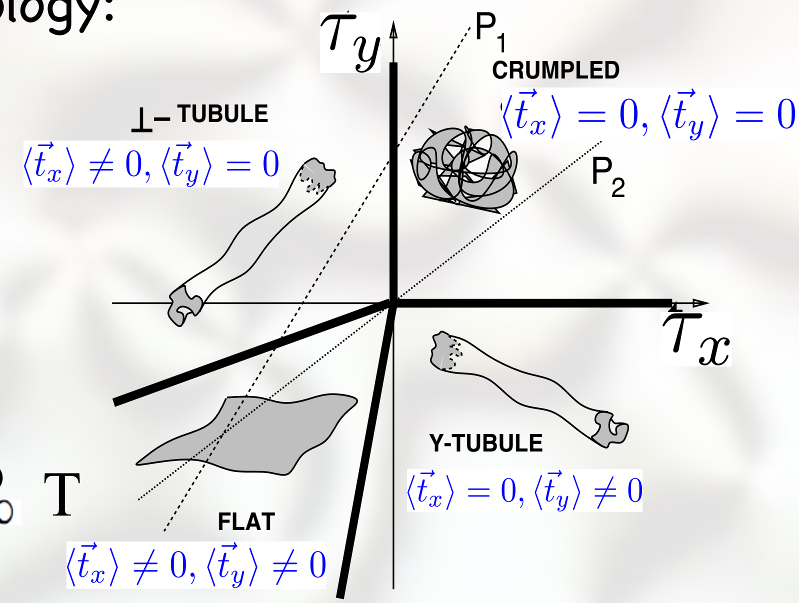
- *spontaneously broken continuous symmetry*
- *nontrivial fixed point of strongly interacting Goldstone modes (c.f. nonlinear O(N) sigma-model)*
- *universal power-law correlation functions and amplitude ratios (throughout the phase)*
- *no fine-tuning to a critical point required*
- *quantum analogs? road to 3d "Luttinger liquids"?*

Summary

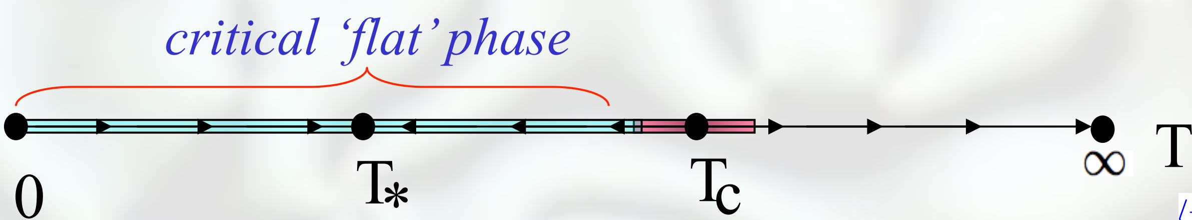


- statistical mechanics of elastic sheets
- rich thermal fluctuations-driven phenomenology:

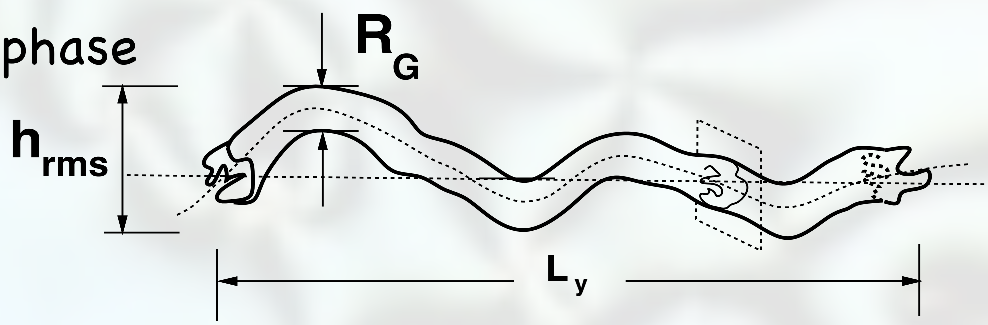
✧ crumpling transition



✧ anomalous elasticity (critical phases)



✧ tubule phase



✧ flat and crumpled glasses