PHYS 5250: Quantum Mechanics - I

Homework Set 6

Issued October 26, 2015 Due November 9, 2015

Reading Assignment: Shankar, Ch. 12; Sakurai, Ch.3.1-3.6

- 1. Show that any function H(x) of a scalar product $x = \mathbf{E}_1 \cdot \mathbf{E}_2$ of two arbitrary vector operators \mathbf{E}_1 , \mathbf{E}_2 commutes with all components of the angular momentum operator \mathbf{L} . This thereby demonstrates explicitly that a rotationally-invariant Hamiltonian (i.e., one that consists only of scalar products of vector operators) commutes with the angular momentum vector.
- 2. Show that the z-component of the angular momentum operator $L_z = -i\hbar(x\partial_y y\partial_x)$ can be written as $L_z = -i\hbar\partial_{\phi}$, where ϕ is the azimuthal angle of the spherical coordinate system.
- 3. Three identical, elementary bosonic particles of mass m are confined to move on a circle of radius r and are rigidly fixed to form an equilateral triangle, so that their only degree of freedom is the common azimuthal angle ϕ . Ignoring spin, compute the wavefunction and the corresponding energy spectrum.

Hint: Think about the symmetry of the eigenfunctions $\psi_E(\phi)$.

- 4. Express the lowest six spherical harmonics $Y_{\ell,m}(\theta,\phi)$ in Cartesian coordinates, x, y, z.
- 5. (a) A particle is described by a wavefunction $\psi(\rho, \phi) = Ae^{-\rho^2/\rho_0^2} \cos^2 \phi$. By expressing ψ in terms of eigenstates of L_z , show that the probability $P(\ell_z)$ of finding the particle with the z component of angular momentum equal ℓ_z is P(0) = 2/3 and $P(\pm 2\hbar) = 1/6$ and zero for all other values of ℓ_z .
 - (b) Another particle is described by a wavefunction $\psi(x, y, z) = A(xy+yz+xz)e^{-r/r_0}$. What is the probability of finding the particle with a particular value of its square of angular momentum? If the particle is found to have $\ell = 2$, what are the relative probabilities of finding this particle with $m = \pm 2, \pm 1, 0$?
- 6. (Shankar 12.5.14) Rotation of $\ell = 1$ eigenstates
- 7. Harmonic Oscillator

- (a) Study a two-dimensional harmonic oscillator in polar (ρ, ϕ) coordinates.
 - i. Find the energy spectrum and the corresponding eigenstates. For the eigenstates it is sufficient to establish the series solution to the corresponding differential equation and to determine the recursion relation for the series-solution coefficients c_i .

Hint: To help guide you, follow the steps outlined in problem 12.3.7 of Shankar.

- ii. Show the spectrum E_n by marking (with a short horizontal dashes) the energy levels for the lowest 10 states labelled by n, m, where n is the principal (radial) quantum number and m the eigenvalue of the z-component of the angular momentum L_z/\hbar . Make the horizontal axis the m-axis so that two states with the same value of n but different value of m are shifted horizontally relative to each other, with each appearing at its correct value of m along the horizontal axis.
- iii. Deduce the degeneracy of energy E_n , confirming that it agrees with that obtained solving this problem via Cartesian coordinates x, y.
- iv. Write down the explicit wavefunctions $\psi_{n,m}(\rho, \phi)$ for the lowest eigenstates $(n = 0, m = 0), (n = 1, m = \pm 1), (n = 2, m = 0, \pm 2)$, and verify explicitly that each of these is a linear combination of corresponding degenerate eigenstates $\psi_{n_x,n_y}(x,y) = H_{n_x}(x)e^{-x^2/2}H_{n_y}(y)e^{-y^2/2}$. Please make this connection explicit.

Hint: In establishing the relation, please do not worry about any normalization constants.

- (b) The above Hamiltonian is a good model for a noninteracting (atoms do not see each other, only the harmonic trap potential) atomic gas confined in a harmonic trap potential. Consider what happens if one rotates such a gas at frequency Ω, as is now routinely done by our colleagues (maybe even some of your classmates) in JILA. The corresponding Hamiltonian for the rotating system is given by an addition of a single term, -ΩL_z to the Hamiltonian above. Find the energy spectrum for such rotating atomic gas and plot it the way you did above, doing it for Ω < ω. What do you think happens when Ω → ω and faster rotation? Hint: Having found the spectrum for the stationary gas, you should simply be able to write down the spectrum for the rotating one without doing any calculations. Regarding Ω → ω limit, think about what happens for a corresponding classical problem of a particle trapped in a quadratic (harmonic) potential with frequency ω, and rotated with angular frequency Ω.
- (c) Guided by the discussion of the last section of Ch. 12 in Shankar (also see problem 12.6.11) for the 3d harmonic oscillator solved in spherical coordinates and characterized by quantum numbers n, ℓ, m , answer the following:
 - i. Compute the degeneracy of the *n*-th energy level of a 3d harmonic oscillator,

confirming that it agrees with the result you found in a previous homework using Cartesian coordinates $D_n^{3d} = (n+1)(n+2)/2$.

ii. Using the results of discussion in Shankar (but without performing any explicit series solution calculations), write down the four lowest eigenfunctions $\psi_{n,\ell,m}(r,\theta,\phi)$ in spherical coordinates, and then demonstrate (in a way very similar to that for 2d case, above) an explicit connection of these states to wavefunctions in Cartesian coordinates,

 $\psi_{n_x,n_y,n_z}(x,y,z) = H_{n_x}(x)e^{-x^2/2}H_{n_y}(y)e^{-y^2/2}H_{n_z}(z)e^{-z^2/2}.$ Hint: For $\ell = n$ all expansion coefficients c_i (in the series solution) vanish except the lowest one, c_0 . In establishing the relation, please do not worry about any normalization constants.