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PHYS 5250: Quantum Mechanics - I

**Homework Set 3**

Issued September 16, 2015
Due September 28, 2015

Reading Assignment: Shankar, Ch. 5, 6, 7, Sakurai, Ch.2 (2.1-2.5)

1. Consider a particle in the ground state of a box (infinite square-well) of length $L$.
   (a) Calculate the probability distribution of momentum in this state.
   (b) Imagine *suddenly* the potential is shut off. Write down, but do not evaluate (the resulting complicated integral) the expression for the wavefunction $\psi(x,t)$ at subsequent time.
   (c) Consider *instead* a situation where this box potential expands suddently (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

2. (a) Show that for any normalized state $|\psi\rangle$, $\langle\psi|H|\psi\rangle \geq E_0$, where $E_0$ is the lowest-energy eigenvalue.
   Hint: expand $|\psi\rangle$ in the eigenbasis of $H$.
   (b) Prove the following theorem: Every attractive potential in one dimension has at least one bound state.
   Hint: Since $V(x)$ is attractive, if we define $V(\infty) = 0$, it follows that $V(x) = -|V(x)|$ for all $x$. To show that there exists a bound state with $E < 0$, consider a variational wavefunction $\psi_\alpha(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$ and calculate $E(\alpha) = \langle\psi_\alpha|H|\psi_\alpha\rangle$, $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - |V(x)|$. Show that $E(\alpha)$ can be made negative by a suitable choice of $\alpha$. The desired result follows from the theorem proved above.

3. Consider $V(x) = -U_0\delta(x)$ potential.
   (a) Show that it admits a bound state of energy $E = -mU_0^2/2\hbar^2$, finding the corresponding wavefunction. Are there any other bound states?
   Hint: Solve Schrodinger's equation outside the potential for $E < 0$, keeping only the solution that has the right behavior at infinity and is continuous at $x = 0$. Integrate the Schrodinger's equation over an *infinitesimal* region $\epsilon \to 0$ around

the potential at $x = 0$, to determine the slope discontinuity of a cusp in the wavefunction around the origin, and thereby finding the energy of the bound state.

(b) Use similar proceedure to explicitly determine wavefunctions of all positive energy solutions. Is the spectrum discrete or continuous here?

Hint: Use similar proceedure as in (a) above to determine the slope discontinuity, that should allow you to compute the coefficients appearing in the wavefunction.

(c) Sketch the bound state and a typical odd and an even positive energy solutions, indicating their qualitatively important features.

(d) Consider now a case of the repulsive potential, i.e., $U_0 < 0$. What changes from the attractive case above? Remember to think about both bound and continuum states.

Hints: For problems above, because the potential is an even function, the Hamiltonian commutes with the parity operator and therefore all wavefunctions can be separated into odd and even ones, that should be considered separately.

For continuum solutions, how are the odd eigenfunctions modified by this particularly simple potential?

4. Consider a particle in a finite square-well potential

$$V(x) = \begin{cases} 0, & |x| \leq a \\ V_0, & |x| > a. \end{cases} \tag{1}$$

As discussed in class, a finite $V_0 > 0$ now allows both bound and continuum set of states.

(a) Consider a parity operator $\hat{P}$ whose action on a function is $\hat{P}f(x) = f(-x)$. Show that $\hat{P}$ commutes with the Hamiltonian for this problem. Hence prove that eigenfunctions of the Hamiltonian are also eigenfunctions of the parity operator with eigenvalues $\pm 1$, and can therefore be classified into two classes (labelled by two eigenvalues of $\hat{P}$) corresponding to even and odd eigenfunctions.

(b) Show that even bound state solutions have energies that satisfy the transcendental equation $k \tan ka = \kappa$, while the odd ones have energies that satisfy $k \cot ka = -\kappa$, where $k$ and $i\kappa$ are the real and complex wavenumbers inside and outside the well, respectively. Find $k$ and $\kappa$ and note that they are related by $k^2 + \kappa^2 = 2mV_0/\hbar^2$.

(c) The spectrum of bound states $E_n$ can be found by solving above simultaneous equations for $k$ and $\kappa$ graphically, by looking for intersections of the curves, generated by the transcendental equation and the circle above.

Implement this graphical procedure for three of values of $V_0$, showing how as the strength of the potential $V_0$ increases more bound state eigenvalues appear.

(d) Sketch three lowest bound eigenfunctions.

(e) Show that there is always one even bound state solution (consistent with the theorem you proved above) and that there is no odd solution unless $V_0 \geq \hbar^2\pi^2/8ma^2$. What is $E$ when $V_0$ just meets this requirement?

(f) For this critical value of $V_0 = \hbar^2\pi^2/8ma^2$ compute explicitly the coefficients determining the *normalized* ground state wavefunction; you will need to use Mathematica, or a calculator to actually find the numerical solution.

(g) Verify that in limit of $V_0 \to \infty$, the bound-state eigenfunctions and eigenenergies reduces to those of the infinite square-well potential found in class.

(h) Now consider continuum states (with $E > V_0$) write down the general form of the wavefunctions and derive the expressions for the coefficients entering these wavefunctions. Note, you will need to consider even and odd eigenfunctions separately.

(i) Show that in the limit $a \to 0$ and $V_0 \to \infty$, with $V_0 2a = U_0 = constant$, the lowest bound state and the continuum states reduce to that of the attractive $\delta$-function potential analyzed above. Please verify this for *both* the eigenvalues and eigenfunctions (determined by the constant coefficients) of both bound and continuum states.

Hint: Note that before taking these limits, it is convenient to first shift the zero of energy by a constant amount $V_0$, namely take the zero energy to be outside the potential-well. In such convention it is obvious that $\kappa \to const.$ but $k \to \sqrt{V_0} \sim \sqrt{U_0/a} \to \infty$. Your job should be simplified by the fact that you know what you are looking for, given your solution to problem 3.

5. Consider $\psi = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$ in one dimension. Show that the current $j = (|A|^2 - |B|^2)p/m$, namely that it is the velocity in state $p$ times the difference in probabilities of the particle in the $p$ state moving to the right and to the left. The absence of the cross terms between the right- and left-moving pieces in $\psi$ allows us to associate the two parts of $j$ with the corresponding parts of $\psi$.

6. Harmonic oscillator

(a) Wavepacket oscillations

Consider an initial wavepacket state $\psi(x,0) = \left(\frac{1}{\pi x_0^2}\right)^{\frac{1}{4}} e^{-(x-a)^2/2x_0^2}$, corresponding (what's called) a "coherent state" of the oscillator, displaced from its equilibrium position at $x = 0$ to position $x = a$ (think of a stretched spring) and width $x_0 = \sqrt{\hbar/m\omega_0}$ identical to that of the ground state (corresponding to $a = 0$). Since for $a \neq 0$ the state is not a ground (or any eigen-) state, for $t > 0$ the system will evolve nontrivially. Below we study its evolution.

i. Write down a formal expression for the wavefunction $\psi(x,t)$ at subsequent time $t > 0$.

ii. Write down the expression for the coefficients $c_n \equiv \langle E_n|\psi(x,0)\rangle$ entering $\psi(x,t)$ in terms of the eigenfunctions of the 1d harmonic oscillator $\psi_n(x) = N_n H_n(x/x_0)e^{-x^2/2x_0^2}$, ($N_n = (x_0\pi^{\frac{1}{2}}2^n n!)^{-\frac{1}{2}}$ is the normalization constant found in class). Use the generating function for Hermite polynomials ($Z(x,s) = e^{-s^2+2sx} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}s^n$) and Gaussian-integrals calculus to show that the initial state expansion coefficients are given by $c_n = \frac{1}{\sqrt{2^n n!}}(a/x_0)^n e^{-(a/2x_0)^2}$.

iii. Substituting this $c_n$ into your formal expression for the wavefunction $\psi(x,t)$, and taking advantage of the generating function $Z(x,s)$, show that

$$\psi(x,t) = \left(\frac{1}{\pi x_0^2}\right)^{\frac{1}{4}} e^{-\frac{1}{2}(\hat{x}-\hat{a}\cos(\omega_0 t))^2}e^{-i\phi}, \tag{2}$$

where $\phi = \frac{1}{2}\omega_c t + \hat{x}\hat{a}\sin\omega_c t - \frac{1}{4}\hat{a}^2\sin 2\omega_c t$ and $\hat{x} = x/x_0$, $\hat{a} = a/x_0$. This shows that, as expected from correspondence principle, an initially displaced wavepacket evolves in time with fixed shape, but having its center oscillate with an amplitude $a$ according Newton's law.

(b) Using our results for a 1d harmonic oscillator write down explicitly eigenfunctions $\psi_{\mathbf{n}}(\mathbf{r})$, eigenvalues $E_{\mathbf{n}}$ and deneracy factor $g(E)$ of energy $E$ for the

i. 2d,

ii. 3d

isotropic harmonic oscillators.

(c) Charged particle in a magnetic field: Landau levels

Consider a charged particle of charge $q$ moving in 3d in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$.

i. Recalling that the Hamiltonian for a charged particle is given by $\hat{H} = (\hat{\mathbf{p}} - q\mathbf{A})^2/2m$ and using a convenient (so-called) Landau gauge with the vector potential $\mathbf{A} = Bx\hat{\mathbf{y}}$, write down explicitly (in Cartesian component notation) the corresponding 3d Schrodinger equation for this particle.

ii. Solve the Schrodinger equation, finding its eigenfunctions and eigenenergies. These eigenenergies are called Landau levels, after a famous brilliant Russian physicist Lev Landau who first solved this problem.

Hint: Note that for our clever choice of the vector potential $\mathbf{A}$, above, the $y$ and $z$ dependence of $\psi(x,y,z)$ is easily separated out, reducing the problem to (what by now should be) a familiar effective 1d problem along $x$.