Leo Radzihovsky, Fall 2015

PHYS 5250: Quantum Mechanics - 1

Homework Set 1

Issued August 24, 2015
Due September 4, 2015

Reading Assignment: Shankar, Ch. 1, 2, 3; Sakurai, Ch.1.1; Schiff Ch. 1

1. For Hermitian operators, show that eigenvalues are real and (in the absence of degeneracy) eigenvectors are orthogonal.

2. Consider a two-dimensional (2d) isotropic harmonic oscillator

   (a) Derive Newton's equations using (i) Lagrangian, (ii) Hamiltonian formalism, showing that they are consistent with each other.

   (b) Using Poisson brackets form for evolution of an operator, $dO/dt = \{O, H\}$ verify that the canonical (angular) momenta L_ϕ, associated with the polar coordinate angle ϕ is a constant of motion, i.e., is conserved.

3. Using the Lagrangian for a 1d harmonic oscillator, its action functional $S[x(t), \dot{x}(t)]$, and the explicit general solution $x_{cl}(t)$ of the corresponding equation of motion, derive the action $S(x_f, x_i, T)$, as a function of the initial ($x_i \equiv x(t_i)$) and final ($x_f \equiv x(t_f)$) locations of the particle and time duration $T = t_f - t_i$.

   As we will see shortly, the harmonic oscillator evolution operator is then directly given by $U(x_f, x_i, t) \sim e^{\frac{i}{\hbar} S(x_f, x_i, t)}$, a very useful result.

4. Compute the de Broglie wavelength of

   (a) a 1 eV electron

   (b) a thermal neutron, defined as a neutron whose kinetic energy is $3k_B T/2$, with $T = 300^o$K.

5. Consider the Gaussian probability distribution

$$P(x) = Ae^{-(x-a)^2/x_0^2},$$

   where constants A, a, x_0 are positive constants. It of course arises in a ground state of a harmonic oscillator as the square of the ground state wavefunction $P(x) = |\psi_0(x)|^2$.

To compute properties of such distribution we need to derive the Gaussian integral calculus

(a) Derive standard moments of Gaussian distribution

$$Z_0(a) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}, \tag{1}$$

$$Z_1(a) = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{1}{2}ax^2} = \frac{1}{a}\sqrt{\frac{2\pi}{a}} = \frac{1}{a}Z_0, \tag{2}$$

$$Z_n(a) = \int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2} = \frac{(2n-1)!!}{a^n}Z_0, \tag{3}$$

(b) Show that these moments can also be obtained from the generating function $Z(a,h)$ by differentiating with respect to h:

$$Z(a,h) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+hx} = Z_0(a)e^{\frac{1}{2}h^2/a}, \tag{4}$$

$$= \sum_{n=0} \frac{h^{2n}}{(2n)!}Z_n(a). \tag{5}$$

(c) Now use Gaussian integral calculus above to compute:

i. normalization A

ii. averages $\langle x\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$ (where $p = -i\hbar\partial/\partial x$)

iii. standard deviation $\sqrt{\langle(x-\langle x\rangle)^2\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, also demonstrating the last equality.

6. Using Bohr-Sommerfeld quantization, whose simplified version is that angular momentum in a circular orbit is quantized according to $L = mvr = n\hbar$, estimate the spectrum E_n and radius r_n of a hydrogenic atom with atomic number Z.

   Make the units of E_n and r_n explicit by expressing your answer in terms of the electron's Compton wavelength $\lambda_e \equiv h/mc$ and the fine structure constant $\alpha \equiv e^2/\hbar c$.

   Hint: ignore electron-electron interaction and only consider circular stationary "orbits" of a single electron moving in a Coulomb potential of the nucleus. We are looking for a rough qualitative estimates, not worrying about prefactors of order 1 or π. The key focus is on dimensional analysis and precise functional dependence on parameters $\hbar$, m, etc. and the principle quantum number n.

7. Along the same lines as in the previous problem using a combination of Heisenberg uncertainty principle for the nth eigenstate ($p_n r_n \approx n\hbar$) estimate the spectrum E_n and the extent r_n of the corresponding eigenstates for the following bound state problems:

   (a) a harmonic oscillator with $V(x) = \frac{1}{2}m\omega^2x^2$,

(b) a "quartic" oscillator with $V(x) = \frac{1}{4}ax^4$,

(c) a hydrogenic atom with Coulomb potential $V(r) = -Ze^2/r$,

(d) an electron in a potential $V(x) = \frac{1}{s}V_0(x/x_1)^s$. In this latter case also study the spectrum for large $s \gg 1$ and explain the result to which E_n reduces.

Hint: Minimize the corresponding total energy $E[r_n]$ obtained from the Hamiltonian $H[r_n, p_n] = \frac{p_n^2}{2m} + V(r_n)$ when subjected to the above quantum mechanical constraint, $p_n r_n \approx n\hbar$.

8. Show (as expected, given that it represents conserved matter) that the probability distribution $P(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$ satisfies a continuity equation $\partial_t P + \nabla \cdot \mathbf{J} = 0$.

   What is the corresponding particle current $\mathbf{J}$?

   Hint: Take advantage of the fact that $\psi(\mathbf{r}, t)$ satisfies the time-dependent Schrodinger's equation.