Lecture 13: Spin

- Recall that found angular momentum algebra (commutation relations)

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

allow $j = \frac{1}{2}n$, $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

So far only looked at $j \in \mathbb{Z}$, implemented via scalar group of fnc's that transform only through their coord. argument.

$$\Psi(\hat{n}) \xrightarrow[R_{\vec{\theta}}]{} e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{L}} \Psi(\hat{n}) = \Psi(R_{\vec{\theta}}\hat{n})$$

implemented via a differential operator acting on single fnc $\Psi(\hat{n})$

$$\approx \left(1 - \frac{i}{\hbar}\vec{\theta}\cdot\vec{L}\right)\Psi(\hat{n})$$

$$\underset{\vec{\theta} = \varepsilon\hat{z}}{=} \left(1 - \frac{i}{\hbar}\varepsilon\,\partial_\varphi\right)\Psi(\theta,\varphi)$$

- How do we realize states with half integer j? Cannot do it with single-valued fnc of coordinate; recall $e^{im\phi} \xrightarrow[2\pi]{} e^{im(\phi+2\pi)} = \left(e^{im2\pi}\right)e^{im\phi}$, $m \in \mathbb{Z}$.

Answer: introduce many-component state:

$$\Psi_\sigma(\theta,\varphi), \quad \sigma = 1, 2, 3, \ldots, n$$

$$\psi(\theta,\varphi) \rightarrow \underbrace{\psi_\sigma(\theta,\varphi)}_{\text{spinor}} = \begin{pmatrix} \psi_1(\theta,\varphi) \\ \psi_2(\theta,\varphi) \\ \vdots \\ \psi_n(\theta,\varphi) \end{pmatrix}$$

- has continuous coordinates θ, φ & discrete coordinate $\sigma = 1, 2, \ldots, n$.

- Generalized unitary operator on $\psi_\sigma(\theta,\varphi)$

$$\psi_\sigma(\theta,\varphi) \xrightarrow[R_{\vec{\varepsilon}}]{} \sum_{\sigma'} U^s_{\sigma\sigma'} \psi_{\sigma'}(R_{\vec{\varepsilon}}\hat{n}) = e^{-\frac{i}{\hbar}\vec{\varepsilon}\cdot\vec{L}} \sum_{\sigma'} U^s_{\sigma\sigma'} \psi_{\sigma'}(\theta,\varphi)$$

$\vec{L}$ only acts on θ, φ coord.

$U^s_{\sigma,\sigma'}$ only acts on σ spin coord.

$$U_{\sigma\sigma'} = \underbrace{U^L \otimes U^s_{\sigma\sigma'}}_{\text{direct product acting on spinor space}} = e^{-\frac{i}{\hbar}\vec{\varepsilon}\cdot\vec{L}}\, e^{-\frac{i}{\hbar}\vec{\varepsilon}\cdot\vec{S}_{\sigma,\sigma'}}$$

act on different spaces & so commute.

$$\psi_\sigma(\theta,\varphi) \in \mathbb{V}_e = \mathbb{V}_o \otimes \mathbb{V}_s$$

$$U_{\sigma\sigma'} \simeq \mathbb{1} - \frac{i}{\hbar}\vec{\varepsilon}\cdot\overbrace{\left(\vec{L}\,\delta_{\sigma\sigma'} + \vec{S}_{\sigma\sigma'}\right)}^{\vec{J}}$$

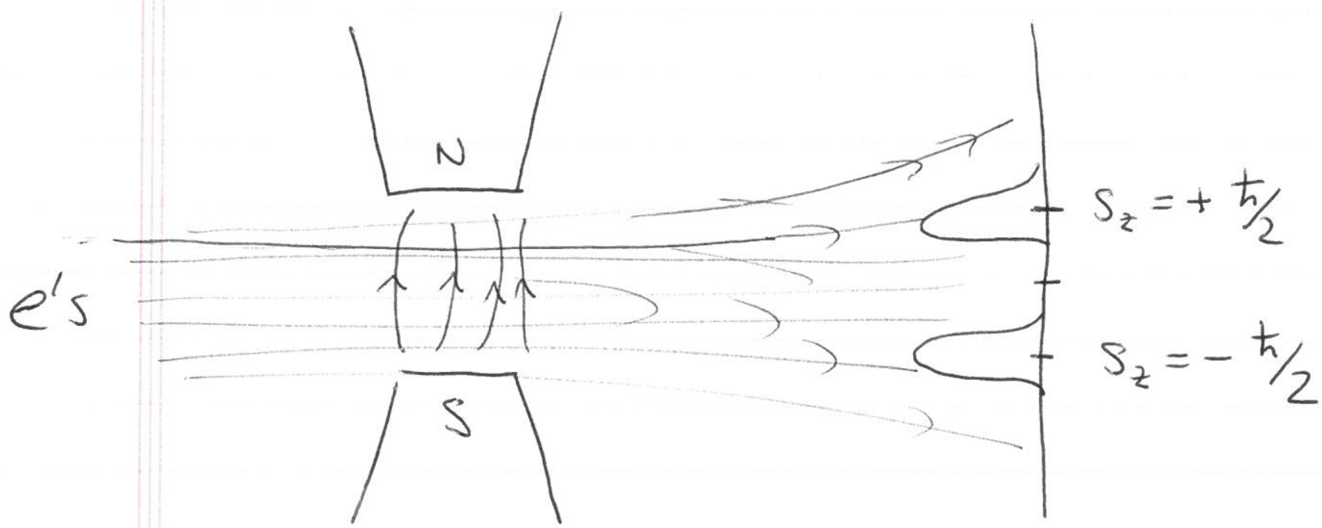
- Recall for $j = 1/2$, $m_j = \pm 1/2 \rightarrow$ just <u>two</u> states

$\Rightarrow \vec{S}_{\sigma\sigma'}$ are 2×2 matrices

$$\left([J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k;\ [L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k;\ [S_i, S_j] = i\hbar\,\epsilon_{ijk} S_k \right)$$

- Why?

A: Stern-Gerlach (and other) experiment demands it! That is electron (and more generally fermions) has $s = \hbar/2$!



$$E = -\gamma \vec{B} \cdot \vec{S} = -\gamma B S_z$$

$$\vec{F} = -\vec{\nabla} E = -\gamma S_z \vec{\nabla} B$$

two spots $\Rightarrow$ S_z takes on only 2 values

$\Rightarrow$ $S_z = \pm \hbar/2$.

- Recall j^{th} representation is implementable via $(2j+1)\times(2j+1)$ matrix

$\Rightarrow$ for $j = s = 1/2 \Rightarrow 2\times 2$ matrix

simple computation (see hw 7) gives:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>with</u> $\psi_a(x,y,z) = \begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix} = \psi_+(r)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_-(r)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\mathbb{1}\vec{L}|\psi\rangle = \begin{bmatrix} \vec{L} & 0 \\ 0 & \vec{L} \end{bmatrix}\begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix}$$

eigenstates of $\hat{S}_z$ with $S_z = \pm \hbar/2$

- <u>basis for V_e</u>: $|\vec{r}, s_z\rangle = |\vec{r}\rangle \otimes |s = 1/2, s_z\rangle$

If $H = H_o + H_s$

$\Rightarrow |\psi(t)\rangle = |\psi_o(t)\rangle \otimes |\chi_s(t)\rangle$

where $H_o|\psi_o\rangle = E_o|\psi_o\rangle$ ← diff. spin components have independent evolution.

$H_s|\chi_s\rangle = E_s|\chi_s\rangle$

$E = E_o + E_s$.

- Focus on spinor part for now

$\Rightarrow$ $|s, m\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \xrightarrow[\text{basis}]{\text{in } S_z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|s, m\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow |\chi\rangle = \alpha|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \beta|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\langle\chi|\chi\rangle = 1 = |\alpha|^2 + |\beta|^2$

$\rightarrow$ $\langle \tfrac{1}{2} \pm \tfrac{1}{2} | \vec{S} | \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm \frac{\hbar}{2}\hat{z}$

more generally eigenstates of

$(\hat{n}\cdot\vec{S})|\hat{n}, \pm\rangle = \pm\frac{\hbar}{2}|\hat{n}, \pm\rangle$

$|\hat{n}, \pm\rangle \underset{\text{in } \hat{n} \text{ basis}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underset{\text{in } \hat{z} \text{ basis}}{=} \underbrace{\begin{pmatrix} \cos\theta/2 \\ e^{+i\phi}\sin\theta/2 \end{pmatrix}, \begin{pmatrix} \sin\theta/2 \\ -e^{+i\phi}\cos\theta/2 \end{pmatrix}}_{\text{orthogonal } \varphi_\uparrow^+ \cdot \varphi_\downarrow}$

$\langle\hat{n},+|\hat{n}\,-\rangle = 0$ ✓

$\langle\hat{n} \pm | \vec{S} | \hat{n} \pm\rangle = \pm\hbar/2\ \underbrace{\hat{n}}_{2 \text{ d.o.f. } \theta, \varphi.}$

$|\chi\rangle = \alpha|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \beta|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$\alpha, \beta \rightarrow 2$ d.o.f. (2 complex $\cdot$ 2 $-$ 1 (normal) $-$ 1 overall phase)

$\Rightarrow$ $\hat{n}$ can label a spinor $(s=\tfrac{1}{2})$ as $|\hat{n}, \pm\rangle$.

Hence for every spinor $|\chi\rangle$ (characterized by α, β)
can equivalently characterize
by $\underline{\hat{n}}$ s.t. $(\hat{n}\cdot\vec{S})|\chi\rangle = \frac{\hbar}{2}|\chi\rangle$
(<u>not</u> so for $j > 1/2$)

- <u>Props</u> of $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i = (\sigma_x, \sigma_y, \sigma_z) \ ; \quad \sigma_\alpha = (\sigma_x, \sigma_y, \sigma_z, \mathbb{1})$$

<u>props of σ's</u> :

→ traceless, Hermitian

→ $[\sigma_i, \sigma_j] = i2\epsilon_{ijk}\sigma_k$

→ $[\sigma_i, \sigma_j]_+ = 2\delta_{ij}\mathbb{1}$

→ $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{1} \Rightarrow (\mathrm{Tr}\,\sigma_\alpha\sigma_\beta = 2\delta_{\alpha\beta})$

→ $\mathrm{Tr}\,\sigma_i = 0$

→ $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

$$\Rightarrow (\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1}$$

<u>Note</u>: σ_α is a <u>complete</u> set, i.e. any 2x2 matrix $M = \sum_{\alpha=0}^{3} c_\alpha\sigma_\alpha$

<u>proof</u>? find c_α : $\mathrm{Tr}(M\sigma_\beta) = \sum_\alpha^3 c_\alpha \underbrace{\mathrm{Tr}(\sigma_\alpha\sigma_\beta)}_{2\delta_{\alpha\beta}}$

$$\Rightarrow \underline{c_\alpha = \frac{1}{2}\mathrm{Tr}(M\sigma_\alpha)} \quad \checkmark$$

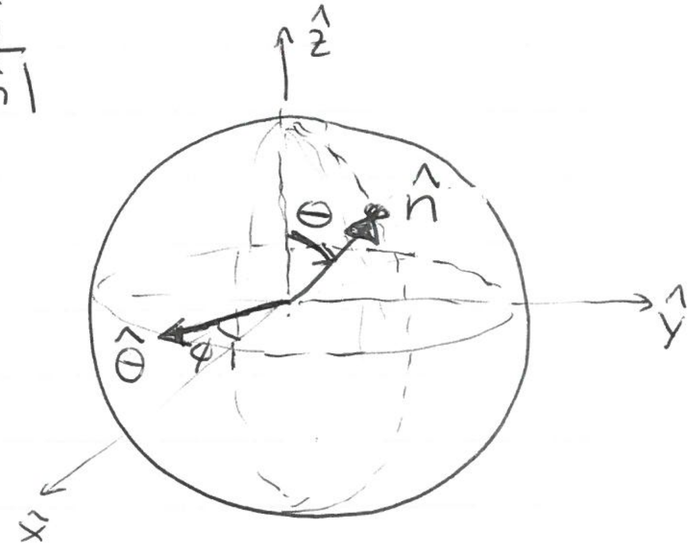
Using props of σ's can show (see Shankar)

$$\underline{U_{R_{\vec{\theta}}} = e^{-\frac{i}{\hbar}\vec{\theta}\cdot\vec{S}} = \left(\cos\frac{\theta}{2}\right)\mathbb{1} - i\left(\sin\frac{\theta}{2}\right)\hat{\theta}\cdot\vec{\sigma}}$$

How does one construct $|\hat{n}, \pm\rangle$?

via $U_{R_{\vec{\theta}}}$ applied to $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$

with $\vec{\theta} = \theta \dfrac{\hat{z}\times\hat{n}}{|\hat{z}\times\hat{n}|}$

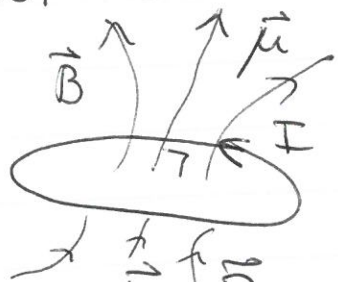


- Spin dynamics : precession

$\vec{T} = \vec{\mu} \times \vec{B}$ ← torque of moment $\vec{\mu}$

$\Rightarrow H = -\vec{\mu} \cdot \vec{B}$

→ • __classically__ think about current loop



$$\underline{\vec{\mu} = \hat{n}\, I A / c}$$

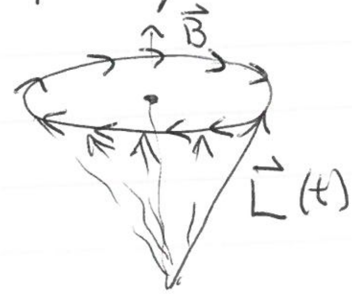
$\underline{H = -\vec{\mu} \cdot \vec{B}}$ is just the magnetostatic interaction of $\vec{B}$ with current (that itself generates B via Ampere's law) I in the loop.

__Note__: $I = \frac{q v}{2\pi r} \Rightarrow \mu = \frac{q v}{2\pi r}\frac{\pi r^2}{c} = \frac{q v r}{2c} = \left(\frac{q}{2mc}\right) m v r$

$$\Rightarrow \underline{\vec{\mu} = \left(\frac{q}{2mc}\right)\vec{L}}$$

$\Rightarrow$ precession: $\frac{d\vec{L}}{dt} = \vec{T} = \vec{\mu} \times \vec{B} = \left(\frac{q}{2mc}\right)\vec{L} \times \vec{B}$

$$\boxed{\vec{\omega}_0 = \frac{q}{2mc}\vec{B}}$$



→ quantum mechanically:

$$i\hbar\, \partial_t \hat{\vec{J}} = [\hat{\vec{J}}, H]$$

$$i\hbar\, \partial_t \hat{J}_i = [\hat{J}_i, -\gamma\, \hat{J}_j B_j]$$

$$= -\gamma B_j\, \epsilon_{ijk}\, i\hbar\, \hat{J}_k$$

$$\Rightarrow \underline{\partial_t \hat{\vec{J}} = \gamma\, \hat{\vec{J}} \times \vec{B}}$$ ← precession of operators

▲ Orbital Zeeman energy:

$$H = \frac{(\vec{p} - q\vec{A}/c)^2}{2m} = \frac{p^2}{2m} - \frac{q}{2mc}(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}) + \frac{q^2}{2mc^2}A^2$$

$$\vec{B} \text{ const} \Rightarrow \vec{A} = -\frac{1}{2}\vec{r}\times\vec{B}, \Rightarrow \vec{\nabla}\cdot\vec{A} = 0$$

$$H = \frac{p^2}{2m} - \frac{q}{mc}\underbrace{\vec{A}\cdot\vec{p}} + \frac{q^2B^2}{8mc^2}r_\perp^2$$

$$-\frac{1}{2}\epsilon_{ijk}\, r_i B_j p_k = \frac{1}{2}(\underbrace{\vec{r}\times\vec{p}}_{\vec{L}})\cdot\vec{B}$$

$$\Rightarrow H = \frac{p^2}{2m} - \underbrace{\frac{q}{2mc}\vec{L}\cdot\vec{B}}_{\text{Zeeman effect}} + \underbrace{\frac{m}{2}\left(\frac{qB}{2mc}\right)^2 r_\perp^2}_{\text{oscillator potential}}$$

oscillator potential with $\omega_0 = \frac{qB}{2mc}$

tiny in atomic environment: diamagn. term.

see hw 7

$$H_{\text{Zeeman}} = -\frac{q}{2mc}\vec{L}\cdot\vec{B} \equiv -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = \underbrace{\frac{q}{2mc}}_{\equiv\gamma}\vec{L}$$, as in classical case.

$\equiv \gamma$ – gyromagnetic ratio

$$\mu_z = \left(\frac{q\hbar}{2mc}\right) m \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

$\frac{q\hbar}{2mc}$ – Bohr magneton of particle with charge q, mass m.

<u>electrons Bohr magneton:</u>

$$\mu_B = \frac{e\hbar}{2mc} \simeq 0.6\times 10^{-8}\ \text{eV}/\text{G}$$

$$\mu_{\text{nucleon}} = \frac{1}{2000}\mu_B \quad , \quad \text{since } M_p = 2000\, m_e .$$

$$H_{\text{zeeman}} = -\vec{\mu}\cdot\vec{B} = \underbrace{-\mu_B B m}$$

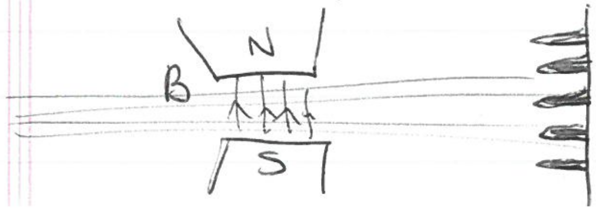
$\vec{B}$ breaks rot. symm.

$\Rightarrow$ splits ℓ levels into $2\ell+1$ equally spaced ones.

(1896 Zeeman)

<u>Zeeman effect</u>

Measured via Stern-Gerlach exp



need inhomogeneous B field such that $\underline{\vec{F} = -\nabla(\vec{\mu}\cdot\vec{B})}$

▲ Spin magnetic moment → Anomalous Zeeman effect

$$\vec{\mu} = \gamma \vec{S}$$

comes from Dirac eqn for spin ½ (or S) charged particle in a $\vec{B}$ field

$$H_{Dirac} = c\left(\vec{p} - \frac{q\vec{A}}{c}\right)\cdot\vec{\alpha} + mc^2\beta$$

matrices

in 2D: $\vec{\alpha} = (\sigma_x, \sigma_y)$

$\beta = \sigma_z$

in 3D: $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}_{4\times 4}$

$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}_{4\times 4}$

$$\boxed{\mathbb{1}\, H_{\text{klein Gordon}} = H_{Dirac}^2}$$

$$H_{schrod.} = \sqrt{H_{KG}} - mc^2$$

Taylor expanded in $\frac{p}{mc}$

see hw 7 & QM-II

$$\Rightarrow H_{sch} = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} - 2\left(\frac{q}{2mc}\right)\vec{S}\cdot\vec{B}$$

twice as big as orbital effect

$\alpha = \frac{e^2}{\hbar c}$

$$g = \underbrace{2\left[1 + \frac{\alpha}{2\pi} + \dots\right]}$$

For electron: $\vec{\mu} = g\left(\frac{-e}{2mc}\right)\vec{S}$

$= -g\,\mu_B\,S_z$

QED theory agrees with exp to $O(\alpha^3)$ 10 signif. digits.

much worse for p, n since there $\alpha_s \simeq 15 \gg 1$!

→ Spin precession revisited:

(a) via Heisenberg eqn & angular momentum algebra

$$\Rightarrow \partial_t \hat{\vec{S}} = \gamma \hat{\vec{S}} \times \vec{B}$$

(b) via evolution operator

$$U(t) = e^{-iHt/\hbar} = e^{i\gamma t(\vec{S}\cdot\vec{B})/\hbar}$$

$$\Rightarrow |\chi(t)\rangle = U(t)\,|\chi(0)\rangle$$

$$= \underbrace{e^{-\frac{i}{\hbar}(\underbrace{-\gamma t\vec{B}}_{\equiv \vec{\Theta}(t)})\cdot\vec{S}}}_{t\text{-dependent rotation operator!}} |\chi(0)\rangle$$

$$\vec{\Theta}(t) = -\gamma \vec{B} t$$

$$\underline{\vec{\omega}_0 = -\gamma \vec{B}}$$

More explicitly $\vec{B} = B\hat{z} \Rightarrow U(t) = \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{pmatrix}$

take $|\chi(0)\rangle = |\hat{n},+\rangle = \begin{pmatrix} \cos\theta/2\, e^{-i\phi/2} \\ \sin\theta/2\, e^{i\phi/2} \end{pmatrix}$

$$\Rightarrow |\chi(t)\rangle = \begin{pmatrix} \cos\theta/2\, e^{-\frac{1}{2}i(\phi-\omega_0 t)} \\ \sin\theta/2\, e^{\frac{1}{2}i(\phi-\omega_0 t)} \end{pmatrix}$$

$$\Rightarrow \phi(t) = \phi_0 - \omega_0 t \Rightarrow \dot{\phi} = -\omega_0$$

▲ Applications: Paramagnetic Resonance

Combination of dc + ac B fields, causes transitions between j_z states.
(see Shankar Ch. 14)

▲ Hydrogen atom with spin in $\vec{B}$ field.

$$H = H_0 + H_s = \frac{(\vec{p} + \frac{e}{c}\vec{A})^2}{2m} - \frac{e^2}{r} + 2\frac{e}{2mc}\vec{S}\cdot\vec{B}$$

since $[\hat{\vec{S}}\cdot\vec{B}, H_0] = 0$

$$\Rightarrow |\psi\rangle = |\psi_0\rangle \otimes |\chi_s\rangle$$

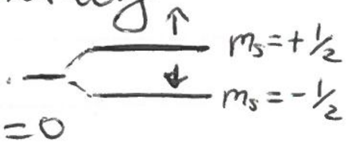
$$\uparrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha \underset{m_s=+1/2}{\begin{pmatrix}1\\0\end{pmatrix}} + \beta \underset{m_s=-1/2}{\begin{pmatrix}0\\1\end{pmatrix}}.$$

$$\Rightarrow |\psi\rangle = |n\,l\,m; m_s\rangle$$

$$\Rightarrow \psi_{nlm}(r,\theta,\varphi)\,\chi_\alpha^{m_s}, \quad m_s = \pm\tfrac{1}{2}$$

$$\Rightarrow H|nlm;m_s\rangle = \Big[-\frac{E_{Ry}}{n^2} + \underbrace{\frac{eB\hbar}{2mc}(m+2m_s)}_{\text{splits the degeneracy}}\Big]|nlm_s\rangle$$

$\Rightarrow$ g.s. $\underline{n=1,\ l=0}$



▲ Spin-orbit interaction

relativistic effect → Dirac eqn, from which spin arises, links spin & orbital spaces ⇒ leads to terms like $\vec{L}\cdot\vec{S}$

"Hand-waving": in proton's reference frame just $\vec{E}_{\text{Coulomb}}$ field. In e's (moving) ref. frame there is also a $\vec{B}_{\text{Coul}}$ field

$$\vec{B}_{\text{Coul}} \approx \underbrace{\frac{-\vec{v}}{c}}_{\gamma\simeq 1} \times \vec{E}_{\text{coul}} = \frac{-\vec{v}}{c} \times \left(-\vec{\nabla} V_{\text{coul}}\right)$$

$$\Rightarrow H^{S-O}_{\text{zeeman}} = -2\left(\frac{-e}{2mc}\right)\vec{S}\cdot\vec{B}_{\text{coul}} \times \frac{1}{2}$$

$$= \frac{+e}{2m^2c^2}\,\vec{p}\times\underbrace{\vec{\nabla} V_{\text{coul}}}\cdot\vec{S}$$

↑ Thomas precession

$$H^{S-O} = \frac{-e^2}{2m^2c^2r^3}\underbrace{(\vec{p}\times\vec{r})}_{-\vec{L}}\cdot\vec{S} \qquad e\vec{\nabla}\frac{1}{r} = -\frac{e}{r^2}\hat{r}$$

$$H^{S-O} = \left(\frac{e^2}{2m^2c^2r^3}\right)\vec{L}\cdot\vec{S}$$ → minimize $\vec{J}$, since $\vec{L}$ & $\vec{S}$ anti parallel. (need to include Thomas precession)

⇒ now $|\Psi\rangle \neq |\psi_0\rangle \otimes |\chi\rangle$ since must diagonalize

$$\vec{L}\cdot\vec{S} = \frac{1}{2}(\vec{L}+\vec{S})^2 - \frac{1}{2}L^2 - \frac{1}{2}S^2$$

<u>very important in spectra of atoms</u>

(In 3D $\vec{p}\cdot\vec{\alpha} + \beta mc^2 + V = H_{\text{Dirac}}$)

$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$; note: no β on V since V is energy & this is $E\Psi \to (E-V)\Psi$

- $H = H_0 + H_s$ ← matrix

e.g. $-\vec{B}\cdot\vec{S}$

e.g. $\frac{p^2}{2m} + V(r)$

$$H|E\rangle = E|E\rangle$$

$$\Rightarrow \quad H_{\sigma\sigma'} \psi_E^{\sigma'}(r) = E\, \psi_E^{\sigma'}(r)$$

2×2 matrix but if separable ⇒

$$\Rightarrow \quad |E\rangle = |nlm\rangle \otimes |m_s\rangle$$

$$\Leftrightarrow \quad \begin{pmatrix} \psi_E^{\uparrow}(r) \\ \psi_E^{\downarrow}(r) \end{pmatrix} = \psi_{nlm}(r) \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}_{m_s}$$

- In most realistic situations spin-orbit coupling, i.e terms like

$$H = \frac{p^2}{2m} + V(r) + \alpha\, \vec{S}\cdot\vec{L}$$

→ origin of SO coupling: Dirac eqn, heuristics.

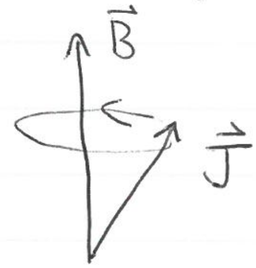
→ could solve by solving from "scratch" two coupled Sch. Eqns. for $\begin{pmatrix} \psi_E^{\uparrow}(r) \\ \psi_E^{\downarrow}(r) \end{pmatrix}$

→ short cut?
addition of angular momenta
$\vec{J}_1 \cdot \vec{J}_2$ diagonalization
trick $\left[\vec{J}_1\cdot\vec{J}_2,\ (J_1+J_2)^2\right] = 0 = \left[\vec{J}_1\cdot\vec{J}_2,\ (J_1+J_2)_z\right]$!

⇒ $\vec{J}_1\cdot\vec{J}_2$ diagonalized by eigenstates of J^2, J_z, J_1^2, J_2^2

Lecture Summary

- $H_J = -\gamma \vec{J} \cdot \vec{B}$
  simplest Hamiltonian
  $\rightarrow$ precession about $\vec{B}$



$[J_z, H_s] = 0 \Rightarrow J_z = m$ - fixed.

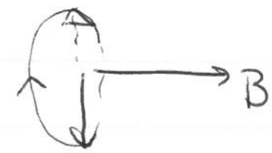
How to make $\vec{J}$ undergo transitions between m's?
Apply $\vec{B}$ field in different direction $\rightarrow$ transverse field.
so as to make $|j\, m_J\rangle$ not an eigenstate

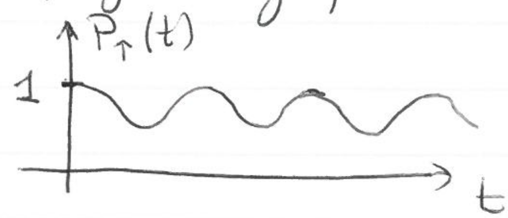
For $\vec{J} = \vec{S} \Rightarrow j = 1/2$

$H_s = -\gamma \vec{B} \cdot \vec{S} = -\gamma B_z S_z - \gamma B_x S_x$

$\Rightarrow |\hat{z}, -1/2\rangle$ not eigenstate of H_s $\rightarrow$ will evolve in time, oscillating between $|\hat{z}, -1/2\rangle$ & $|\hat{z}, +1/2\rangle$

$\rightarrow$ physically precesses around new $\vec{B}$





Rabi oscillations

- So far considered only $H = H_0 + H_s$
  $\rightarrow$ in general spin-orbit coupling $\Rightarrow \psi_\sigma(r) = \psi(r)\chi_\sigma$
  $H_{SO} = \alpha \vec{S} \cdot \vec{L}$
  - derive
  - $[H_{SO}, S_z] \neq 0$, $[H_{SO}, L_z] \neq 0$

  but $[H_{SO}, J_z] = 0$, $[H_{SO}, J^2] = 0$
  $\Rightarrow$ Look at eigenstates of $\vec{J} = \vec{S} + \vec{L}$ (J^2, J_z)
  $|j, m_J; \ell, s\rangle$ instead of $|s\, m_s\rangle \otimes |\ell\, m_L\rangle$
  $\Rightarrow$ Addition of angular momenta problem.

Summary on Spin:

- $\Psi(\vec{r}) \Rightarrow \Psi_\sigma(\vec{r}) \leftarrow$ spinor

$$|\Psi_\sigma\rangle \xrightarrow[R]{} e^{-\frac{i}{\hbar}\vec{\theta}\cdot(\vec{L}+\vec{S})} |\Psi_\sigma\rangle$$

- $S = 1/2 \Rightarrow \vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots$

→ $\hat{S}^2 |\frac{1}{2}, S_z\rangle = \hbar^2 \frac{3}{4} |\frac{1}{2} S_z\rangle, \quad S_z |\frac{1}{2}, \pm\frac{1}{2}\rangle = \pm\frac{\hbar}{2} |\frac{1}{2}, \pm\frac{1}{2}\rangle$

→ $|\frac{1}{2} \pm \frac{1}{2}\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

→ quant. axis along $\hat{n}$: $\hat{n}\cdot\vec{S}|\hat{n}, \pm\frac{1}{2}\rangle = \pm\frac{\hbar}{2}|\hat{n}, \pm\frac{1}{2}\rangle$

$|\hat{n}, \pm\frac{1}{2}\rangle \longrightarrow \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}, \begin{pmatrix} \sin\theta/2 \\ -e^{i\phi}\cos\theta/2 \end{pmatrix}$, by diagonalizing $\hat{n}\cdot\vec{S}$ or by rotating $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

→ $e^{-\frac{i}{\hbar}\theta\hat{\theta}\cdot\vec{S}} = \cos\theta/2 + i\,\hat{\theta}\cdot\vec{\sigma}\sin\theta/2 = \begin{pmatrix} & \\ & \end{pmatrix}$

$\hat{\theta} = \frac{\hat{z}\times\hat{n}}{|\hat{z}\times\hat{n}|}$

→ props of $\vec{\sigma}$'s: $[\ \ ]$, $[\ \ ]_+$, $\mathrm{Tr}\,\sigma$, $\sigma^\dagger = \sigma$, $\sigma_i\sigma_j = \dots$

$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \dots$

- physical consequences of spin: huge!
  - chemistry → life
  - materials / metals, insulators, magnetism, etc.

  → mainly stemming from $2\times$ as many states for each orbital states for fermions.

  ... but more in detail:

  → Zeeman effect (1896)
  - ▲ orbital → splitting of atomic spectra $\quad -\vec{\mu} = \frac{q}{2mc}\vec{L}, \quad E = -\vec{\mu}\cdot\vec{B}$
  - ▲ spin → anomalous Z. effect.
    - → no classical intuition
    - → origin of spin: Dirac eqn. in B field. $\Rightarrow \vec{\mu} = 2\left(\frac{q}{2mc}\right)\vec{S}$, $g = 2 + \dots$

  → $\vec{J}$ precession ⇒ paramagnetic resonance: flipping spin with $B_\perp$

- Spin-orbit interaction:
  - → L·S coupling origin:
    - ▲ Dirac eqn.
    - ▲ semiclassics via Lorentz transf./or Biot-Savart B at e desktop.
  - → L_z, S_z not good q.#'s

    $\vec{J}^2 = (\vec{L}+\vec{S})^2$, & J_z

- Addition of J's.

Lecture 14:
Addition of Angular Momenta

Hilbert space of 2 spins spanned by

$|s_1 m_1\rangle \otimes |s_2 m_2\rangle \equiv |s_1 m_1 ; s_2 m_2\rangle$

4 states — eigenstates of S_i^2, S_{iz}

... but often convenient to work with eigenstates of total $\vec{S} = \vec{S}_1 + \vec{S}_2$ i.e.

S^2, S_z

$$S_z |++\rangle = \hbar |++\rangle$$
$$\left.\begin{aligned} S_z |+-\rangle &= 0 |+-\rangle \\ S_z |-+\rangle &= 0 |-+\rangle \end{aligned}\right\} \text{two-fold degenerate}$$
$$S_z |--\rangle = -\hbar |--\rangle$$

but $|m_1 m_2\rangle$ not eigenstates of S^2 since

$$S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2$$
$$\Rightarrow [S^2, S_{1z}] = [S^2, S_{2z}] \neq 0 \text{ because of}$$

$$\underset{\text{product basis}}{S^2 =} \hbar^2 \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Diagonalize S^2 by

triplet (symmetric)
$$\begin{cases} |S=1, m=1, S_1=\tfrac{1}{2}, S_2=\tfrac{1}{2}\rangle = |++\rangle \\ |S=1, m=0, S_1=\tfrac{1}{2}, S_2=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |S=1, m=-1, S_1=\tfrac{1}{2}, S_2=\tfrac{1}{2}\rangle = |--\rangle \end{cases}$$

singlet (antisymm)
$$\left\{ |S=0, m=0; S_1=\tfrac{1}{2}, S_2=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\underbrace{\left(|+-\rangle - |-+\rangle\right)}_{\substack{\text{singlet} \\ \text{(antisymmetric)}}} \right.$$

Addition of L's:

change basis from $(S_1^2, S_2^2, S_{1z}, S_{2z}) \longrightarrow$
$$\longrightarrow (S^2, S_z, S_1^2, S_2^2)$$

$$\underbrace{\tfrac{1}{2} \otimes \tfrac{1}{2}}_{2\times 2 = 4 \text{ states}} = \underset{3 \text{ states}}{1} \oplus \underset{1 \text{ state}}{0}$$

$2 \times 2 = 4$ states $=$ 3 states $+$ 1 state

<u>Group theory jargon</u>:

$|S_{1z}\rangle \otimes |S_{2z}\rangle$ is a <u>reducible</u> representation

break it down to two <u>irreducible</u> representations
$$|S=1, S_z\rangle \oplus |S=0, 0\rangle$$

Fermion total ψ must be antisymmetric

$\Rightarrow |\psi\rangle = |O_A\rangle \otimes |S=1, S_z\rangle$

or $|\psi\rangle = |O_S\rangle \otimes |S=0, 0\rangle$

Ex. He atom 2 e's

both e's are in $|n=1, l=0, m=0\rangle$

$$|\psi_{He}^{ground}\rangle = \underset{\underset{\underbrace{|100\rangle \otimes |100\rangle}_{l=0,\ \text{symmetric}}}{\uparrow}}{|\psi_0\rangle} \otimes |\chi_S\rangle$$

singlet $S=0$ (antisymm)

Which basis ($|m_1 m_2\rangle$ vs $|S, m\rangle$) to use depends on H.

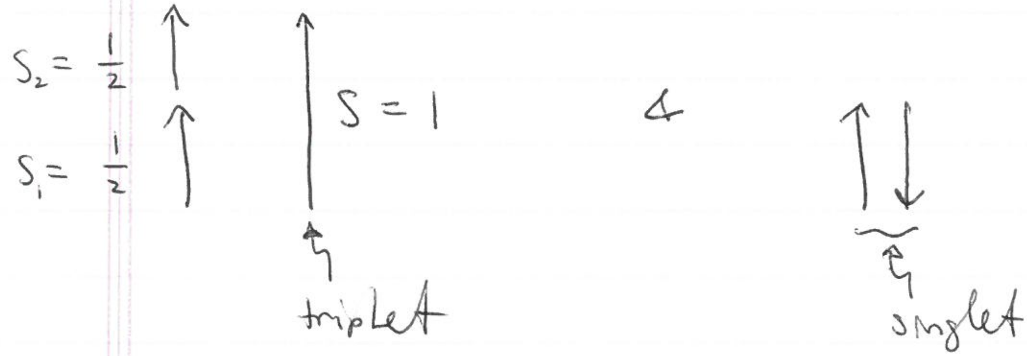
$H = -(\gamma_1 S_1 + \gamma_2 S_2) \cdot \vec{B} \rightarrow$ best $|m_1 m_2\rangle$

on the other hand

$H = A S_1 \cdot S_2 = \frac{A}{2}(S^2 - S_1^2 - S_2^2)$

total S basis diagonalizes H.

Graphical addition:



$|1, 1\rangle, |1, 0\rangle, |1, -1\rangle$ $\qquad$ $|0, 0\rangle$

General problem of addition $\vec{J}_1$, $\vec{J}_2$

$(2j_1+1)(2j_2+1)$ – dimensional reducible tensor product space

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = |j_1 m_1; j_2 m_2\rangle$

$\vec{J} = \vec{J}_1 + \vec{J}_2$

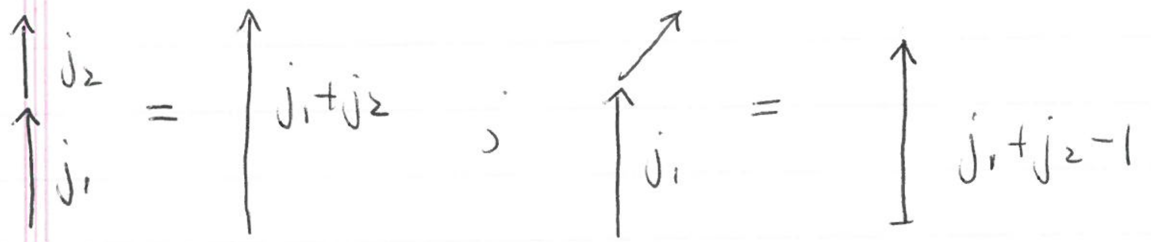
J_z is diagonal in product basis but J^2 is not degen.

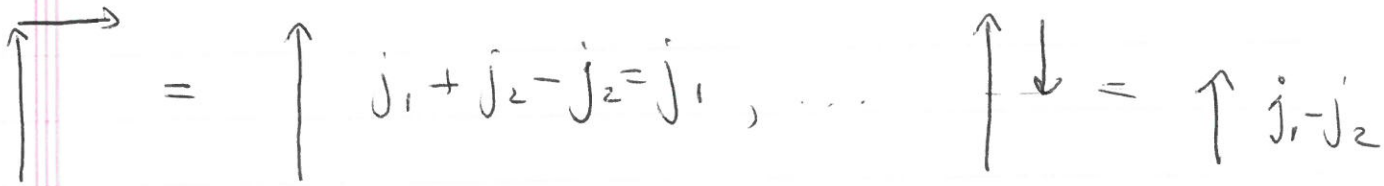
highly degenerate, except for $|j_1 \pm j_1; j_2 \pm j_2\rangle$

e.g. $m = j_1 + j_2 - 2$ is a linear combination of

$|m_1 = j_1, m_2 = j_2 - 2\rangle$; $|m_1 = j_1 - 1, m_2 = j_2 - 1\rangle$, $|j_1 - 2, j_2\rangle$

Graphically: $j_1 + j_2$





$$\boxed{\underbrace{j_1 \otimes j_2}_{(2j_1+1)(2j_2+1)} = \underbrace{(j_1+j_2)\oplus(j_1+j_2-1)\oplus\dots\oplus(j_1-j_2)}}$$

check dimensionality: $D = \sum_{j=j_1-j_2}^{j_1+j_2}(2j+1) = \sum_{j=0}^{j_1+j_2}(2j+1) -$

$$- \sum_{j=0}^{j_1-j_2-1}(2j+1) = \left[\frac{\cancel{2}(j_1+j_2)(j_1+j_2+1)}{\cancel{2}} + j_1+j_2+1\right]$$

$$-\left[(j_1-j_2-1)(j_1-j_2) + j_1-j_2\right]$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1 = (2j_1+1)(2j_2+1) \quad ✓$$

<u>Example</u> with $j_1 = 1/2,\ j_2 = 1/2$

$$|1/2, 1/2\rangle \otimes |1/2, 1/2\rangle = |1, 1\rangle \quad \text{nondegenerate}$$

with $S^2|1\,1\rangle = \hbar^2 2\,|1\,1\rangle$, $S_z|1\,1\rangle = \hbar\,|1\,1\rangle$

$$|1, 1\rangle = |1/2, 1/2; 1/2, 1/2\rangle$$

check: $(S_1^2 + S_2^2 + 2\,S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2-})|\tfrac12\tfrac12;\tfrac12\tfrac12\rangle$

$$= \hbar^2\left(3/4 + 3/4 + 2\cdot 1/2 \cdot 1/2 + 0 + 0\right)|1/2\ 1/2; 1/2\ 1/2\rangle$$

$$S^2|1/2\ 1/2; 1/2, 1/2\rangle = \hbar^2 2\,|1/2\ 1/2; 1/2\ 1/2\rangle \quad \checkmark$$

$|1, 0\rangle = ?$ expect linear combination of

$$|1/2\ \underbrace{-1/2;\ 1/2\ +1/2}_{m=0}\rangle \ \& \ |1/2\ \underbrace{1/2;\ 1/2\ -1/2}_{m=0}\rangle$$

$$S_-|1, 1\rangle = \hbar\sqrt{2}\,|1, 0\rangle$$

$$= (S_{1-} + S_{2-})|1/2\ 1/2; 1/2\ 1/2\rangle = \hbar|1/2\ -1/2; 1/2\ 1/2\rangle + \hbar|1/2\ 1/2; 1/2\ -1/2\rangle$$

$$\Rightarrow |1, 0\rangle = \frac{1}{\sqrt{2}}\left(|-1/2, 1/2\rangle + |1/2\ -1/2\rangle\right)$$

$$|1, -1\rangle = |-1/2, -1/2\rangle$$

$j=1$ states
$$|1\,1\rangle = |++\rangle$$
$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$|1\,-1\rangle = |--\rangle$$

$j=0$ state $\rightarrow$ require to be orthogonal to $|1\,0\rangle$

$$\Rightarrow |0\,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

Convention choice: coeff $+1$ of the $|m_1 = j_1\rangle$

General j_1, j_2

$$|j_1+j_2, j_1+j_2\rangle = |j_1\, j_1; j_2\, j_2\rangle$$

$$J_-|j_1+j_2, j_1+j_2\rangle = \hbar\sqrt{2(j_1+j_2)}\,|j_1+j_2; j_1+j_2-1\rangle$$

$$\Rightarrow |j_1+j_2; j_1+j_2-1\rangle = \frac{1}{\hbar\sqrt{2(j_1+j_2)}}\left(J_{1-} + J_{2-}\right)|j_1\, j_1; j_2\, j_2\rangle$$

$$= \frac{1}{\sqrt{2(j_1+j_2)}}\left(\sqrt{2j_1}\,|j_1\, j_1-1; j_2\, j_2\rangle + \sqrt{2j_2}\,|j_1\, j_1; j_2\, j_2-1\rangle\right)$$

$$|j_1+j_2; j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1\, j_1-1; j_2\, j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1\, j_1; j_2, j_2-1\rangle$$

continue until $|j_1+j_2; -(j_1+j_2)\rangle$

Now look at $|j_1+j_2-1, j_1+j_2-1\rangle = \alpha|j_1 j_1-1; j_2 j_2\rangle + \beta|j_1 j_1; j_2 j_2-1\rangle$

pick α, β s.t. $|\alpha|^2 + |\beta|^2 = 1$ & orthogonal to $|j_1+j_2; j_1+j_2-1\rangle$

$$|j_1+j_2-1;\, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1 j_1;\, j_2 j_2-1\rangle - \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1 j_1-1;\, j_2 j_2\rangle$$

want this coeff be > 0
(convention)

Clebsch-Gordon Coefficients

$$|jm,\, j_1 j_2\rangle = \sum_{m_1}\sum_{m_2} |j_1 m_1,\, j_2 m_2\rangle \underbrace{\langle j_1 m_1,\, j_2 m_2 | j m, j_1 j_2\rangle}_{\text{C-G coeff}}$$

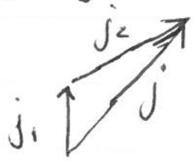
C-G coeff for expansion of $(J_1+J_2)^2$ eigenstates in terms of J_{1z}, J_{2z} eigenstates

Props of C-G coeff:

(1) $\langle j_1 m_1, j_2 m_2 | j m\rangle \neq 0$

only if $|j_1 - j_2| \leq j < j_1 + j_2$

(triangle inequality)



(2) $\langle j_1 m_1, j_2 m_2 | j m\rangle \neq 0$ only if $m_1 + m_2 = m$

(3) C-G coeff are real (by convention)

$< (j_1+j_2)$

(4) $\langle j_1 j_1, j_2 (j - j_1) | j j\rangle$ is positive (by convention)

(5) $\langle j_1 m_1, j_2 m_2 | j m\rangle = (-1)^{j_1+j_2-1} \langle j_1 (-m_1), j_2(-m_2) | j(-m)\rangle$

Find CG coeff for (a) $\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$

(b) $1 \otimes 1 = 2 \oplus 1 \oplus 0$

(6) $\langle jm | j_1 m_1, j_2 m_2 \rangle = \langle j_1 m_1, j_2 m_2 | j m \rangle$
orthogonal matrix, since real CG coeff.

Ex $\frac{1}{2} \otimes \frac{1}{2}$ prob.

$$\begin{array}{c} |jm\rangle \\ \begin{pmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{pmatrix} \end{array} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{array}{c} |m_1 m_2\rangle \\ \begin{pmatrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{pmatrix} \end{array}$$

can invert the relation.

Application: strong $L \cdot S$ coupling in atoms $\Rightarrow$ eigenstates are better approximated by total $\vec{J} = \vec{L} + \vec{S}$ eigenstates

$$|j = l + \tfrac{1}{2}, m_J\rangle = \alpha |l, m_J - \tfrac{1}{2}; \tfrac{1}{2} \tfrac{1}{2}\rangle + \beta |l, m_J + \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|j = l - \tfrac{1}{2}, m_J\rangle = \alpha' |l, m_J - \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle + \beta' |l, m_J + \tfrac{1}{2}; \tfrac{1}{2} -\tfrac{1}{2}\rangle$$

orthonormality $\Rightarrow$ $\alpha^2 + \beta^2 = \alpha'^2 + \beta'^2 = 1$

$$\alpha\alpha' + \beta\beta' = 0$$

AND: $J^2 |j = l + \tfrac{1}{2}, m_J\rangle = \hbar^2 (l + \tfrac{1}{2})(l + \tfrac{3}{2}) |j = l + \tfrac{1}{2}, m_J\rangle$

$\Rightarrow$

$$|j = l \pm \tfrac{1}{2}, m_J\rangle = \frac{1}{(2l+1)^{1/2}} \Big[\pm (l + \tfrac{1}{2} \pm m_J)^{1/2} |l, m_J - \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle + \\ + (l + \tfrac{1}{2} \mp m_J)^{1/2} |l, m_J + \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle \Big]$$

Note: In adding j_1 & j_1, state of total $\vec{J}$ is labelled by $j = 2j_1, \; 2j_1 - 1, \; 2j_1 - 2, \; \ldots \; 0$

symm antisym symm. ...

Lowering operator $J_- = J_{1-} + J_{2-}$ does not change symmetry

Ex. $\frac{1}{2} \otimes \frac{1}{2} = 1 + 0$

symm antisymm.

Ex. $1 \otimes 1 = 2 \oplus 1 \oplus 0$

symm antisym symm.

Notation: in atomic physics.

In absence of spin we used s, p, d, f, ... to denote l.

with spin new spectroscopic notation:

$^{2S+1}L_J$ ⇒ j value

total spin

capital letter denoting total orbital L, i.e. S, P, D, ..

$^2P_{3/2}$ denotes: $L = 1$, $S = \frac{1}{2}$, $J = \frac{3}{2}$

Multi-electron atoms

$\vec{L}_1, \vec{L}_2, \vec{L}_3, \ldots ; \vec{S}_1, \vec{S}_2, \vec{S}_3, \ldots$

$\vec{J}_1 = \vec{L}_1 + \vec{S}_1 , \vec{J}_2 = \vec{L}_2 + \vec{S}_2 , \ldots$

$\vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \ldots ; \vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \ldots$

Two competing effects:

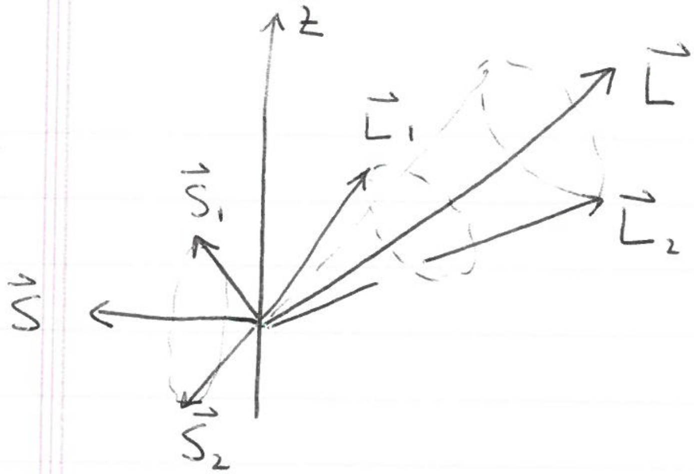
1. (Exchange) Coulomb interaction (beyond Hartree)
2. Spin-orbit.

- Exchange Coulomb couples S_i's & L_i's into $\vec{S}$ & $\vec{L}$ with max value having lowest energy; this is due to the fact that in largest $\vec{S}$ & $\vec{L}$ e's stay away from each other thereby lowering Coulomb repulsion.

$\Rightarrow$ S_{iz}, L_{iz} is not conserved, but S^2, S_z & L^2, L_z are

- Spin orbit leads to $\vec{J}_i = \vec{S}_i + \vec{L}_i$, with $\vec{S}_i$ & $\vec{L}_i$ precessing about total $\vec{J}_i$.

In most atoms (except for largest Z) Exchange dominates $\Rightarrow$ $\vec{L}_i \to \vec{L}$; $\vec{S}_i \to \vec{S}$ weakly coupled by $\vec{L} \cdot \vec{S}$. This is <u>LS coupling</u> (Russell-Saunders)
(JJ coupling for large Z: $\vec{S}_i, \vec{L}_i \to \vec{J}_i$ & then $\vec{J}_i \to \vec{J}$



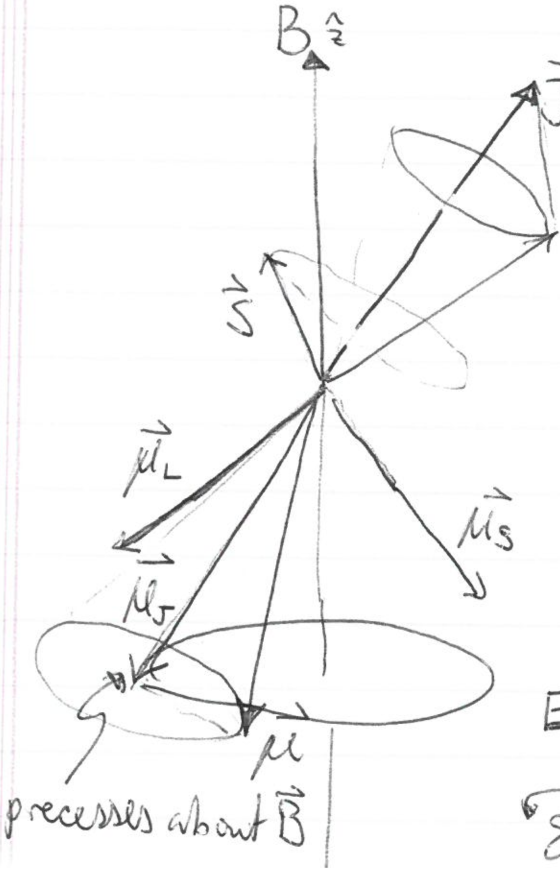
Implications for Zeeman effect: (see Eisenberg & Resnick)

$$\vec{\mu} = -\frac{\mu_B}{\hbar}\vec{L}_1 - \frac{\mu_B}{\hbar}\vec{L}_2 - \dots$$

$$-g\frac{\mu_B}{\hbar}\vec{S}_1 - g\frac{\mu_B}{\hbar}\vec{S}_2 - \dots$$

$$\vec{\mu} = -\frac{\mu_B}{\hbar}(\vec{L}+2\vec{S}) = -\frac{\mu_B}{\hbar}(\vec{J}+\vec{S})$$

$\Rightarrow$ $\vec{\mu}$ <u>not</u> antiparallel to $\vec{J}$!



$$\mu_J = \mu\,\frac{\vec{\mu}\cdot\vec{J}}{\mu J}$$

$$= -\frac{\mu_B}{\hbar}\,\frac{(\vec{L}+2\vec{S})\cdot(\vec{L}+\vec{S})}{J}$$

$$E = -\vec{\mu}\cdot\vec{B} = -\mu_J B \cos\Theta \rightarrow \frac{J_z}{J}$$

$$= +B\frac{\mu_B}{\hbar}\,\frac{(\vec{L}+2\vec{S})\cdot(\vec{L}+\vec{S})}{J}\left(\frac{J_z}{J}\right)$$

$$E = \frac{\mu_B}{\hbar}B\,\frac{(L^2+2S^2+3\vec{L}\cdot\vec{S})}{J^2}J_z$$

$$E = \frac{\mu_B}{\hbar}\underbrace{\frac{(3J^2+S^2-L^2)}{2J^2}}_{g_{\text{Landé}}}J_z B$$

$$g_{\text{Landé}} = 1 + \frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}$$