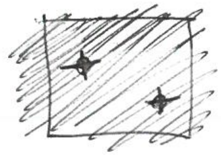
# Lecture 9
## Symmetries & Their Consequences

**1 Examples:** (many others)

- space translational invariance infinitesimal $T_\varepsilon$ $\varepsilon \in \mathbb{R}$, e.g. in a liquid, vacuum all points in space look same.



- discrete $T_a$, translations by $a$ & its integer multiples $n \in \mathbb{Z}$

  e.g. in a crystal


- time translational invariance, e.g. systems whose Hamiltonian does not change in time $t \Leftrightarrow t+\varepsilon$

- time-reversal $t \Leftrightarrow -t$

- parity $\vec{r} \to -\vec{r}$

- continuous rotational invariance $O_\varepsilon$

  e.g. liquid, vacuum
  isotropic

- discrete rot. invnce $O_{\theta,\varphi}$.
  e.g. crystal

(2.) Each **set** of transformations forms a **group** of transformations.

sophisticated mathematics subject "group theory"

**Group $\mathcal{G}$:** $\{ \underbrace{T_1, T_2, \ldots, T_N} \} = g$

could be $\infty$ # of elements and even continuous set like cont. transl. & rotations

$d(d+1)/2$ dimensional
$= d$ - translations generators
$d(d-1)/2$ - rot. generators

Euclidean group $E(d)$

set of elements with some consistency properties.

- $\mathbb{1}$ element $\in g$
- $T_1 T_2 = T_j \in g$ multiplication (not necessarily abelian)
- inverse exists : $T_1^{-1} T_1 = \mathbb{1}$.
- associativity

⋮

Ex's:
- Euclidean group of continuous transl. & rotations
- Point group of discrete rot ~~[illegible]~~ in Euclidean space.
- Parity group

etc.

discrete & continuous (Lie groups)

(3) ~~group~~ Symmetry: group elements commute with Hamiltonian

(a) $[T_i, H] = 0 \Leftrightarrow$ H is symmetric under the transformation.

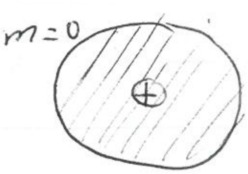
why? $T^{-1} H T = H$

Note: symmetric H or equiv. eqn of motion

$\not\Rightarrow$ symmetric eigenstates / solutions.

e.g. - drum head is rotationally invariant
$\Rightarrow$ symm. when quiet (groundstate)
but is not rot. invnt in its vibrating state!

- Hydogen atom: rot invnt Hamiltonian & ground state $n=0$, $l=0$, $m=0$



but not excited states,

e.g $n=1$, $l=1$, $m=+1$

- translational invnt, but $\psi_k \sim \sin kx$ not

... now lets look at details of some of these symmetries & associated groups.

(b) conserved quantity $\frac{dT_i}{dt} = [T_i, H] = 0$ ✓

- Action on wave fnc:

$$T_\varepsilon |x\rangle = |x+\varepsilon\rangle$$

unitary operator

$$|\psi_\varepsilon\rangle = \hat{T}_\varepsilon |\psi\rangle = \hat{T}_\varepsilon \int_{-\infty}^{\infty} |x\rangle\langle x|\psi\rangle$$

$$= \int_{-\infty}^{\infty} |x+\varepsilon\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} |x'\rangle \underbrace{\langle x'-\varepsilon|\psi\rangle}_{\psi(x'-\varepsilon)}$$

$$\underline{\langle x|\psi_\varepsilon\rangle = \psi(x-\varepsilon)} \Leftrightarrow \psi(x) \xrightarrow[\hat{T}_\varepsilon]{} \psi(x-\varepsilon)$$

- Translational invariance

  - infinitesimal: Taylor expand to $\varepsilon^1$

$$T_\varepsilon = I - i\varepsilon G$$

generator of translations

<u>Note:</u> in 3d $T_{\vec{\varepsilon}} = I - i\vec{\varepsilon}\cdot\vec{G}$

3 generators of translations $T_x, T_y, T_z$

"Algebra" of translations

$[G_i, G_j] = 0$, <u>Abelian</u>

check that $G^\dagger = G$.

What is $G$ in coordinate representation?

$$e \quad \hat{T}_{\vec{\varepsilon}}^{\dagger} \vec{r} \hat{T}_{\vec{\varepsilon}} = \vec{r} + \vec{\varepsilon}$$

$$e^{i\frac{\hat{p}}{\hbar}\cdot\vec{\varepsilon}} \vec{r} e^{-i\frac{\hat{p}}{\hbar}\cdot\vec{\varepsilon}} = \hat{r} + \vec{\varepsilon}$$

$\Rightarrow$ require $[e^{i\frac{\hat{p}}{\hbar}\cdot\vec{\varepsilon}}, \vec{r}] = \vec{\varepsilon} e^{i\frac{\hat{p}}{\hbar}\cdot\vec{\varepsilon}}$

$$\Rightarrow [r_i, p_j] = i\hbar\delta_{ij}$$

$$\langle x|G|x'\rangle = ?$$

$$\langle x|T_\varepsilon|\psi\rangle = \psi(x-\varepsilon)$$

$$\varepsilon \to 0 \Rightarrow \langle x|(I - i\varepsilon \underbrace{G)|\psi\rangle}_{\int |x'\rangle\langle x'|} = \psi(x) - \varepsilon \frac{\partial \psi}{\partial x}$$

$$\psi(x) - i\varepsilon \int dx'\, G_{xx'}\, \psi(x')$$

$$\Rightarrow G_{xx'} = -i \frac{\partial}{\partial x} \delta(x-x')$$

$$\Leftrightarrow \langle x|G|\psi\rangle = -i \frac{\partial \psi}{\partial x} \Rightarrow \underline{\hat{G} = \frac{1}{\hbar}\hat{P}}$$

$$\Rightarrow \boxed{\hat{T}_\varepsilon = e^{-i\frac{\varepsilon}{\hbar}\hat{P}}}$$

as in classical mech. [where p is generator of (infinit) translations] $\hat{p}$ is a q.m. generator of translations.

In d-dim:

$$\hat{T}_{\vec{\varepsilon}} = e^{-\frac{i}{\hbar}\vec{\varepsilon}\cdot\vec{P}}$$

$\vec{P} = (P_x, P_y, P_z)$, $\vec{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z)$

- Conservation law (of p):

$$\langle\psi|H|\psi\rangle = \langle\psi_\varepsilon|H|\psi_\varepsilon\rangle = \langle\psi|T_\varepsilon^+ H T_\varepsilon|\psi\rangle$$

$$= \langle\psi|(I + \tfrac{i}{\hbar}\varepsilon P) H (I - \tfrac{i}{\hbar}\varepsilon P)|\psi\rangle$$

$$= \langle\psi|H|\psi\rangle + \frac{i\varepsilon}{\hbar}\langle\psi|[P,H]|\psi\rangle$$

$$\Rightarrow \boxed{[P,H] = 0} \Rightarrow \underline{\dot{p} = 0} \Rightarrow p = \text{const}$$

conservation of p.

$[\hat{P}, \hat{H}] = 0$

$\hat{P}$ generates translations $\Rightarrow \delta \hat{H} = \frac{i}{\hbar}[\hat{P}, \hat{H}]$,

symmetry $\Leftrightarrow \delta H = 0$ (cf. $\frac{\partial \hat{O}}{\partial t} = \frac{i}{\hbar}[\hat{H}, \hat{O}]$)

$\Rightarrow [\hat{P}, \hat{H}] = 0$, but this also

means that $\frac{\partial \hat{P}}{\partial t} = + \frac{i}{\hbar} \underbrace{[H, P] = 0}_{\Rightarrow \text{ can be diagonalized simult.}} \Rightarrow \dot{P} = 0$

as we saw for free particle

Note precise correspondence:

Unitary transf (preserves $\langle\psi|\psi\rangle$ & $[A,B]$) in Q.M $\Leftrightarrow$ Canonical transf. (preserves $\{A, B\}$) in C.M.

- Finite translational operator $T_a$

$\rightarrow$ already saw plausibility for $\hat{T}_a = e^{-\frac{i}{\hbar}\hat{p}a}$ via unitarity & group property

$(\hat{T}_a^{+} = \hat{T}_a^{-1} = \hat{T}_{-a})$ $(\hat{T}_{a_1}\hat{T}_{a_2} \underset{\uparrow}{=} \hat{T}_{a_1+a_2})$

require only if $a_1 \parallel a_2$ i.e. same generator $G$ (so can add exponentials since $[G, G] = 0$)

$\rightarrow$ "another" way:

$$\hat{T}_a \underset{N\to\infty}{=} \Big(\hat{T}_{\underbrace{a/N}_{\varepsilon}}\Big)^N = \left(I - i\frac{\hat{P}}{\hbar}\frac{a}{N}\right)^N$$

$$\underline{\hat{T}_a = e^{-i\frac{a}{\hbar}\hat{P}}} \quad \left(\text{cf } \hat{U}_t = e^{-\frac{it}{\hbar}\hat{H}}\right)$$

Note: $\langle x|\hat{T}_a|x'\rangle = e^{-a\frac{\partial}{\partial x}}\delta(x-x') \Rightarrow \langle x|\hat{T}_a|\psi\rangle = e^{-a\frac{\partial}{\partial x}}\psi(x)$

$= \psi(x) - a\frac{\partial\psi}{\partial x} + \frac{1}{2!}a^2\frac{\partial^2\psi}{\partial x^2} + \ldots = \psi(x-a)$ ✓

- System of N particles in 3d:

$$\langle \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N | \hat{T}_{\vec{\varepsilon}} | \psi \rangle = \psi(\vec{r}_1 - \vec{\varepsilon}, \vec{r}_2 - \vec{\varepsilon}, \ldots \vec{r}_N - \vec{\varepsilon})$$

$$\langle \vec{r}_1, \ldots, \vec{r}_N | \left( I - \frac{i}{\hbar} \vec{\varepsilon} \cdot \vec{P} \right) | \psi \rangle = \psi(\vec{r}_1, \ldots, \vec{r}_N) - \sum_{i=1}^{N} \vec{\varepsilon} \cdot \vec{\nabla}_i \psi$$

$$\Rightarrow \hat{T}_{\vec{\varepsilon}} = \hat{I} - \frac{i}{\hbar} \vec{\varepsilon} \cdot \vec{P}$$

<u>total momentum</u> $- \vec{P} = \sum_{i=1}^{N} p_i = -i\hbar \sum_{i=1}^{N} \vec{\nabla}_i$

$$\Rightarrow [\hat{P}, \hat{H}] = 0 \Rightarrow P_{total} = \text{const}$$

$\not\Rightarrow$ $\hat{H}$ is trivial with $V(\vec{r}) = 0$

but only means that $V(\vec{r}_i - \vec{r}_j) \Rightarrow$

$$T_{\varepsilon} \rightarrow V((\vec{r}_i + \cancel{\vec{\varepsilon}}) - (\vec{r}_j + \cancel{\vec{\varepsilon}})) = V(\vec{r}_i - \vec{r}_j) \checkmark$$

<u>example</u>: $$H = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i \neq j} \sum \frac{e_i e_j}{|\vec{r}_i - \vec{r}_j|}$$

All known fundamental interactions (strong, weak, gravitational, e&m) are translationally invnt.
$\Rightarrow$ reflects <u>homogeneity</u> of free space (vacuum)
$\Rightarrow$ does <u>not</u> matter where particles in a system are in absolute terms, only relative to each other. In vacuum every point is equivalent to other. $\Rightarrow$ reason why experiments in diff. places are a "same" experiment & can be compared.

## 5 Time translational invariance

All "points" in time are equivalent

transformation: $t \xrightarrow[U_\varepsilon]{} t+\varepsilon$

invariance: $H \xrightarrow[U_\varepsilon]{} H$

conservation: $\dot{H} = 0 \rightarrow$ conservation of energy.

- transformation operator:

$$|\psi(t)\rangle \xrightarrow[\hat{U}_\varepsilon]{} |\psi(t+\varepsilon)\rangle = \hat{U}_\varepsilon |\psi(t)\rangle$$

What is $\hat{U}_\varepsilon$ that "translates" things in time by $\varepsilon$?

$U_\varepsilon$ - Evolution operator, of course!

$$\hat{U}_\varepsilon = e^{-\frac{i}{\hbar}\varepsilon \hat{H}},$$ $\hat{H}$ is the generator of time translations.

Why?

recall: $$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \Leftrightarrow \partial_t\hat{O} = \frac{1}{i\hbar}[\hat{O},\hat{H}]$$

$$\Leftrightarrow \hat{O}(t) = e^{\frac{i}{\hbar}t\hat{H}}\,\hat{O}(0)\,e^{-\frac{i}{\hbar}t\hat{H}}$$

$\Rightarrow$ time evolution (executed by $\hat{U}(t)$, with $\hat{H}$ generator) is nothing but translation in time by $t$.

- <u>invariance</u> $\hat{U}_\varepsilon^+ \hat{H}(t) U_\varepsilon = \hat{H}(t+\varepsilon) = \hat{H}(t)$

$$\Leftrightarrow \frac{1}{i\hbar}[\hat{H}, \hat{H}] = \frac{d\hat{H}}{dt} = 0$$

$\Rightarrow$ $\hat{H}$ independent of $t$.

- <u>conservation principle</u> : Energy is conserved.

We have seen this many times before:

$$i\hbar\, \partial_t |\psi\rangle = H|\psi\rangle$$

if $H$ is $t$-independent $\Rightarrow$ $|\psi(t)\rangle = |E\rangle e^{-iEt/\hbar}$

$\Rightarrow$ TISE: $H|E\rangle = E|E\rangle$

All known laws of nature are space-time invnt!

If it were otherwise (e.g. <u>laws</u> of nature changed in time, <u>not</u> according to Hamiltonian evolution) could <u>not</u> do science/experiments $\rightarrow$ meaningless.

- Parity: $\Pi \vec{E} = \vec{E}' = -\vec{E}$
- some "vectors" are actually pseudo-vectors.

i.e. $\Pi \vec{L} = \vec{L}' = \vec{L}$

ex. $\vec{L} = \vec{r} \times \vec{p} \xrightarrow[\Pi]{} (-\vec{r}) \times (-\vec{p})$

$\vec{B} =$

- terms like $\vec{L} \cdot \vec{p}$ in $H$ violate parity since $\vec{L} \xrightarrow[\Pi]{} \vec{L}$ but $\vec{p} \to -\vec{p} \Rightarrow H_{int} \xrightarrow[\Pi]{} -H_{int}$

- violated only by weak interactions (proposed by Lee & Yang '56 observed by C.S. Wu '57)

# 6. Parity invariance – discrete transformation

- transformation: reflection through origin
  space inversion $\Rightarrow$ $\vec{r} \to -\vec{r}$ $\Rightarrow$ RH $\to$ LH.
  only two elements in a parity group.
  $(I, \Pi)$, because $\Pi^2 = I$

Q.M.: $\hat{\Pi}|\vec{r}\rangle = |-\vec{r}\rangle$ ; $\hat{\Pi}|\vec{p}\rangle = |-\vec{p}\rangle$

on $\psi(\vec{r})$: $\langle \vec{r}|\hat{\Pi}|\psi\rangle = \psi(-\vec{r})$

properties: (a) $\Pi = \Pi^{-1}$
(b) eigenvalues of $\Pi$ are $\pm 1$
(c) $\Pi$ – Hermitian & unitary

$\Rightarrow$ odd & even eigenkets with $-1$ & $+1$ eigenvals.

Equivalently $\Pi^+ \vec{r}\, \Pi = -\vec{r}$, $\Pi^+ \vec{p}\, \Pi = -\vec{p}$

$\Pi$ is a symmetry if $\Pi^+ H \Pi = H(-\vec{r}, -\vec{p})$
$= H(\vec{r}, \vec{p})$

$\Rightarrow [\Pi, H] = 0 \Rightarrow$ simultaneously diagonalized $\Rightarrow |E, \pm\rangle$

(saw this for square well, harmonic oscillator, etc...)

e.g. violated by $\vec{L} \cdot \vec{p}$ and other pseudo scalars.

- Parity conservation: $[\Pi, U_t] = 0 \Rightarrow \dot{\Pi} = 0$
  i.e. state $|\psi\rangle$ that is an eigenstate of $\Pi$, does not change its eigenvalue with t evolution.

- Parity violated by weak interactions:
  e.g. $\pi \rightarrow \mu + \nu$

case 1

$S_z = +\frac{1}{2} \uparrow$   $\bullet\ \nu$   $\uparrow p_z > 0$

$S_z = 0$   $\bullet\ \pi$ , $p_z = 0$

$S_z = -\frac{1}{2} \downarrow$   $\bullet\ \mu$   $\downarrow p_z < 0$

———————————————— mirror

case 2

$S_z \downarrow$   $\mu$   $p_z \uparrow$

$\circ$   $\pi$

$\uparrow$   $\nu$   $\downarrow$

case 1 & 2 are mirror images of each other
but do <u>not</u> occur with equal prob. in exp.

- C - charge conjugation particle $\leftrightarrow$ antiparticle
  observed to be violated by weak interactions.
  in above $\pi^+ \rightarrow \mu^+ + \nu_\mu$ go via case 2 &
  $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ go via case 1 only
- CP also violated $\simeq 1\%$ (1964 by Cronin, Fitch, et al)
- CPT conserved always (essential to QFT)
- $\Rightarrow$ T is violated!

B. <u>Time reversal</u> ("reversal of motion", E. Wigner '32)

→ Newtons eqn: invnt under $t \to -t$

$\Longleftrightarrow$ $x(t)$ & $x(-t)$ both solns.

→ Maxwell's eqn: $t \to -t$ & $\vec{E} \to \vec{E}$, $\vec{B} \to -\vec{B}$

$\rho \to \rho$, $j \to -j$

→ Sch. Eqn: $t \to -t$ & $\psi(r,t) \to \psi^*(r,t)$

$\hat{T}$ – time reversal operator must be <u>antilinear</u>

i.e. $T(c_1|\alpha\rangle + c_2|\beta\rangle) = c_1^* T|\alpha\rangle + c_2^* T|\beta\rangle$

but also require $|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|$

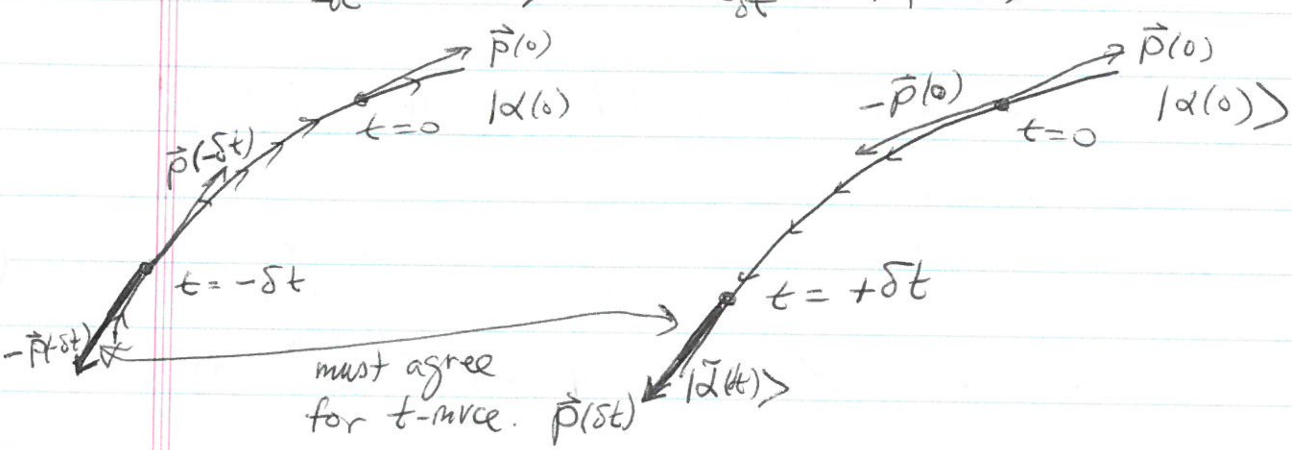
$\Rightarrow$ $\hat{T}$ – antiunitary, i.e. $\underline{\langle\tilde{\beta}|\hat{\alpha}\rangle = \langle\beta|\alpha\rangle^*}$

$\Rightarrow$ $\hat{T} = \hat{U}\hat{K}$ complex conjugation op.

Why?

Want T-reversal invnce to satisfy:

$$\hat{T}\,\hat{U}_{-\delta t}|\psi(0)\rangle = U_{\delta t}\,\hat{T}|\psi(0)\rangle$$



Note: need $T$ to be antilinear based on:

$$[x, p] = i\hbar$$

$$T[x,p]T^{-1} = T(i\hbar)T^{-1}$$

$$[\tilde{x}, -\tilde{p}] = \hbar T(i)T^{-1}$$

$$[\tilde{x}, \tilde{p}] = -\hbar T(i)T^{-1} \underset{\text{require}}{=} i\hbar$$

$$\Rightarrow T i T^{-1} = -i$$

similarly: $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

with $T\vec{J}T^{-1} = -\vec{J}$

consistent with $\vec{J} = \vec{r} \times \vec{p}$

$$|\psi\rangle = \int_{x'} \langle x'|\psi\rangle |x'\rangle \Rightarrow T|\psi\rangle = \int_{x} \langle x'|\psi\rangle^* T|x'\rangle \Rightarrow \underline{T\psi(x) = \psi^*(x)}$$

$$\psi(x,0) \xrightarrow[U_t]{} e^{-iHt/\hbar}\psi(x,0) = \psi(x,t) \xrightarrow[T]{} e^{iH^*t/\hbar}\psi^*(x,0)$$

$$\psi(x,0) \xrightarrow[T]{} \psi^*(x,0) \xrightarrow[U_{-t}]{} e^{iHt/\hbar}\psi^*(x,0)$$

$\Rightarrow$ require $\underline{H^* = H}$ — real $H$.

e.g. violated by $\vec{B}$ field $\frac{1}{2m}(\vec{p} - i\frac{e}{c}\vec{A})^2 + \ldots$
unless $\vec{B} \to -\vec{B}$.

$$T e^{ikx} = e^{-ikx} \qquad T\psi(x) = \psi^*(x)$$

$$k \to -k \quad \checkmark \quad T\tilde{\psi}(p) = \tilde{\psi}^*(-p)$$

(E)

$$|\tilde{\alpha}(\delta t)\rangle = U_{\delta t}\, T\, |\alpha(0)\rangle \quad \text{vs.} \quad \widetilde{|\alpha(-\delta t)\rangle} = T\, U_{-\delta t} |\alpha(0)\rangle$$

$$U_{\delta t} \simeq \mathbb{1} - \frac{i}{\hbar} H \delta t$$

$\Rightarrow$ require $\quad -i H T |\alpha(0)\rangle = T i H |\alpha(0)\rangle$

$$\Rightarrow \quad (iH)T + T(iH) = 0$$

but also must have $\quad [H, T] = 0 \quad$ since must have $\quad |E\rangle$ & $T|E\rangle$ <u>degenerate</u>

i.e. $\quad H|E\rangle = E|E\rangle \Rightarrow TH|E\rangle = E(T|E\rangle)$

want $\quad H(T|E\rangle) = E(T|E\rangle)$

Hence to have $\quad HT - TH = 0 \quad$ <u>and</u>

$$(iH)T + T(iH) = 0$$

$$\Rightarrow \quad T(iH) = -iTH$$

<u>antilinear</u> $T = UK$.

<u>Note</u>: could not have $\quad HT + TH = 0$

otherwise $\quad T^{-1} \frac{p^2}{2m} T = -\frac{p^2}{2m} \underset{\text{but want}}{=} \frac{(-p)^2}{2m}$ !

important property: $\quad \underline{\langle\beta|\Theta|\alpha\rangle = \langle\tilde{\alpha}|T\Theta^{\dagger}T^{-1}|\tilde{\beta}\rangle}$

$\Rightarrow$ for Hermitian ops: $\quad \langle\beta|H|\alpha\rangle = \langle\tilde{\alpha}|THT^{-1}|\tilde{\beta}\rangle$

$$\Rightarrow \quad THT^{-1} = \pm H$$

$$TpT^{-1} = -p \qquad TrT^{-1} = r$$

$$T|p\rangle = |-p\rangle$$

T-invariance: evolve system for time $t$
$|\psi(0)\rangle \rightarrow |\psi(t)\rangle$; reverse time $t \rightarrow -t$ i.e.
$|\psi(t)\rangle \underset{T}{\rightarrow} |\psi^*(t)\rangle$, evolve for time $t$ & check

if returned to $|\psi(0)\rangle$

more explicitly: $\psi(x,0) \rightarrow e^{-iHt/\hbar}\psi(x,0) \rightarrow$
$\psi(x,0) \rightarrow \underbrace{e^{iH^*t/\hbar}\psi^*(x,0)}_{(\psi(t))^*} \rightarrow e^{-iHt/\hbar}e^{iH^*t/\hbar}\psi^*(x,0)$

$\Rightarrow$ require $H = H^*$ i.e. real Hamiltonian.

note: $H \neq H^*$ for finite $\vec{B}$ field (unless $\vec{B} \rightarrow -\vec{B}$ is implemented)

- Symmetry: $|E_n\rangle$ is an eigenstate,

$T|E_n\rangle$ is also an eigenstate with same energy $E_n$.

$$|E_n\rangle \underset{U_t}{\longrightarrow} |E_n\rangle e^{-\frac{i}{\hbar}E_n t} \underset{T}{\longrightarrow} T\left(e^{-\frac{i}{\hbar}E_n t}|E_n\rangle\right)$$

compare to

to be same need to be antilinear

$$|E_n\rangle \underset{T}{\longrightarrow} T|E_n\rangle \underset{U_{-t}}{\longrightarrow} e^{-\frac{i}{\hbar}E_n(-t)}\, T|E_n\rangle$$

$\Rightarrow$ $T$ not a linear but an antilinear operator:

$$T(a_1\psi_1 + a_2\psi_2) = a_1^* T\psi_1 + a_2^* T\psi_2 \qquad [T,H]=0.$$

Note: $i\hbar\partial_t\psi = H\psi \Rightarrow -i\hbar\partial_t(T\psi) = TH\psi = H(T\psi)$

$\Longleftrightarrow$ $t \rightarrow -t$ in S. Eqn.

$T = K$ $\leftarrow$ antiunitary op. $\Rightarrow$ $T\psi$ evolves according to Sch. Eqn where $t \rightarrow -t$
complex conjugation $\qquad$ $i\partial_t \rightarrow -i\partial_t$

(Proposed by Lee & Yang '56, confirmed by C.S Wu et al '57)

All but <u>weak interactions</u> are parity invariant

Start out with $|\psi(0)\rangle$ & $\Pi|\psi(0)\rangle$ states that are mirror images. After time t because $[\Pi, U_t] \neq 0 \Rightarrow U(t)|\psi(0)\rangle = |\psi(t)\rangle$ & $U(t)\Pi|\psi(0)\rangle$ are <u>not</u> mirror images of each other.

Look at $\beta$-decay ${}^{60}Co \to {}^{60}Ni + e^- + \bar{\nu}$ discussion in Shankar
outcome depends on pseudoscalar $\vec{S}\cdot\vec{p}$

<u>Note</u>: parity $\vec{r} \to -\vec{r} \iff$ mirror + 180° rotation.

## (7) <u>Time-reversal symmetry</u>

(antiunitary op) $\langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\alpha|\beta\rangle^*$

"reversal of motion" symmetry (E. Wigner '32)

Classically if $x(t)$ is soln $x(-t)$ is also a soln of eom
see e.g. Newton's Eq. $m\ddot{\vec{r}} = -\vec{\nabla}V$ (if without "friction")
also Maxwell's Eqns.

B-field breaks (explicitly) T-invariance, but not if currents $j$ are also reversed, thereby reversing $\vec{B} \to -\vec{B}$, $\vec{v} \to -\vec{v}$, $\vec{r} \to \vec{r}$, $\vec{E} \to \vec{E}$.

Q.M. $i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi$

Note: $\psi(\vec{r}, -t)$ is not a soln but $\psi^*(\vec{r}, -t)$ is. $\Rightarrow$ $\underline{T\psi(\vec{r}, t) = \psi^*(\vec{r}, -t)}$ ✓

(for system without spin.)

but $T\psi(\vec{p}, t) = \psi^*(-\vec{p}, -t)$

$$|\psi(0)\rangle \xrightarrow{T} T|\psi(0)\rangle \xrightarrow{U_t} \underbrace{U_t T|\psi(0)\rangle}$$

$$|\psi(t)\rangle = U_t|\psi(0)\rangle \xrightarrow{T} \underbrace{T\,U_t|\psi(0)\rangle} \overset{?}{=} e^{\frac{iEt}{\hbar}}\psi^*(0)$$

$$= T\left(e^{-\frac{iEt}{\hbar}}\psi(0)\right)$$

$E^* = E$ i.e $E$-<u>reel</u>

B

## 3 Discrete transformations.

A. Parity $\Pi$ : inverts coord. system

$\Leftrightarrow \vec{r} \to -\vec{r}, \; \vec{p} \to -\vec{p}, \; \vec{V} \to -\vec{V}$

$\Leftrightarrow$ mirror operation + rot by $\pi$ around $\perp$ axis


$\langle \psi | \Pi^+ \vec{r}\, \Pi | \psi \rangle = -\langle \psi | \vec{r} | \psi \rangle \Leftrightarrow \Pi^+ \vec{r}\, \Pi = -\vec{r}$

$\Pi^+ \vec{p}\, \Pi = -\vec{p}$

$\Pi^2 = \mathbb{1} \Rightarrow \lambda_\Pi = \pm 1$ odd or even.

$\Rightarrow \Pi^+ = \Pi$ Hermitian & unitary.

- Vectors: $\Pi^+ \vec{V}\, \Pi = -\vec{V}$ (e.g. $\vec{r}, \vec{p}, \vec{J}, \vec{E}$)

  $\{\Pi, \vec{V}\} = 0$
- pseudovectors: $\Pi^+ \vec{A}\, \Pi = \vec{A}$ (e.g. $\vec{J} = \vec{r} \times \vec{p}$, $\vec{B}$, $\vec{S}$)

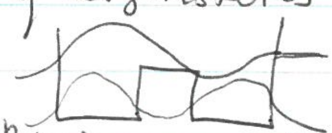
  ($\nabla \times B = J$)
- scalars: $\Pi^+ S\, \Pi = S$ (e.g. $\vec{r} \cdot \vec{p}$)
- pseudoscalars: $\Pi^+ a\, \Pi = -a$ (e.g. $\vec{r} \cdot \vec{J}$, $\vec{E} \cdot \vec{B}$, $\vec{p} \cdot \vec{S}$)

  if present in H will violate parity symm, but not rotation.

Applic. $[\Pi, H] = 0 \Rightarrow \Pi |E\rangle = \lambda_\Pi |E\rangle, \; \lambda_\Pi = \pm 1$

$\Rightarrow |E, \pm\rangle$ are parity eigenstates

ex. - double well



- h.o. $|n\rangle = (a^+)^n |0\rangle \Rightarrow \lambda_\Pi = (-1)^n$ $(a^+ = x + ip)$

odd odd odd.

degeneracy in rot. invnt probs $|E, J^2, J_z\rangle$

& free particle $e^{-ik\cdot r}$ not eigenstate of $\Pi$, $|k\rangle, |-k\rangle$ degenerate.

- If symmetry is more complicated (e.g. rot's) with more than one generator ($J_x, J_y, J_z$) all of which commute with $H$
  $[\vec{J}, H] = 0$, but $[J_i, J_j] \neq 0$.

  $\Rightarrow$ degeneracy $\Rightarrow |E, J_z\rangle$ eigenstates of $H, J_z$ but not of $J_x, J_y$ $\rightarrow$ take between degen. set.
  more on this later.

- Lee & Yang '56 proposed violation of $\Pi$ by weak interaction
  C.S. Wu '57 observed $\vec{S}\cdot\vec{p} > 0$, $< 0$ } $\underline{\text{different}}$
  decays different

  $[H_{weak}, \Pi] \neq 0$

  pseudo scalar.

  $\xrightarrow{S} \xrightarrow[\vec{p}]{}$ vs $\xleftarrow{S} \longrightarrow \vec{p}$

  $U_t \Pi|\psi\rangle \neq \Pi^+ U_t|\psi\rangle$

- selection rules: $\langle \Pi_1 | O_{odd} | \Pi_2 \rangle \neq 0$ only if $\Pi_1 = -\Pi_2$
  $\langle \Pi_1 | O_{even} | \Pi_2 \rangle \neq 0$ only if $\Pi_1 = \Pi_2$

  e.g. $\langle \pm | \vec{r} | \pm \rangle = 0$, $\langle \pm | \vec{r} | \mp \rangle \neq 0$
  $\rightarrow$ dipole selection rules in atomic transitions.

  $\rightarrow$ $\vec{d} = \langle \psi | \vec{r} | \psi \rangle = 0$
  if non degenerate so
  $\Pi|\psi\rangle = \lambda_\Pi |\psi\rangle$.

# Symmetry in QM

(particularly useful when explicit soln not available)

1 Group of transformations $g = \{T_1, T_2, \ldots T_{..}\}$

- Invariance of system $T^{+} H T = H \Leftrightarrow [T, H] = 0$
- conserv. law $\Rightarrow \dot{T} = \dot{G} = 0$

2 so far looked at continuous groups (Lie groups) $\infty$ # of elements $T_{\vec{\theta}}$ labelled by $\vec{\theta}$

unitary: $\underbrace{T_{\vec{\delta}}}_{\text{group element}} = e^{-i\vec{\delta}\cdot\vec{G}} \simeq \mathbb{1} - i\vec{\delta}\cdot\vec{G}$ ($\leftarrow$ generator), $|\vec{\delta}| \ll 1$ ; $\underbrace{T^{+} = T^{-1}}_{\text{unitary}} \Leftrightarrow \underbrace{\vec{G}^{+} = \vec{G}}_{\text{Hermitian}}$

<u>exs</u>: $\vec{r}$ – translations: $\vec{G} = \vec{P}/\hbar$, $\vec{\delta} = \vec{r}$, $\vec{p} = \text{const}$

$t$ – translations: $G = H/\hbar$, $\delta = t$, $H = \text{const}$.

(later) rotations: $\vec{G} = \vec{J}/\hbar$, $\vec{\delta} = \theta\hat{n}$, $\vec{J} = \text{const}$.

<u>Note</u>: symmetry $[T, H] = 0 \Rightarrow$

→ <u>nondegenerate</u>: $H|E\rangle = E|E\rangle \Rightarrow T|E\rangle = t|E\rangle$
$\Rightarrow |E, t\rangle$ T eigenstates too. (e.g. parity)

→ <u>degenerate</u>: $T|E\rangle$ or the same state

$\Pi|E\rangle = \pm|E\rangle$

## 4. Translational invariance (continuous)
(space)

→ invariance of a system under spatial translations

Quantum mechanically:

transform: $\langle x \rangle \xrightarrow[T_\varepsilon]{} \langle x \rangle + \varepsilon$ ; $\langle p \rangle \to \langle p \rangle$

Invariance: $\langle H \rangle \xrightarrow[T_\varepsilon]{} \langle H \rangle$

conservation: $\langle \dot{p} \rangle = 0$

Two points of view:

A. Active (analog of Schrödinger's picture)

move particle to right by ε $\quad |\psi\rangle \xrightarrow{T_\varepsilon} |\psi_\varepsilon\rangle = \hat{T}_\varepsilon |\psi\rangle$

$$\langle x \rangle_\varepsilon = \langle \psi | X | \psi \rangle \to \langle \psi_\varepsilon | X | \psi_\varepsilon \rangle = \overbrace{\langle \psi | X | \psi \rangle}^{\langle X \rangle} + \varepsilon$$

$$\langle \psi_\varepsilon | P | \psi_\varepsilon \rangle = \langle \psi | P | \psi \rangle$$

B. Passive (analog of Heisenberg picture)

$$\langle \psi | \hat{T}_\varepsilon^+ \hat{X} \hat{T}_\varepsilon | \psi \rangle = \langle \psi | X | \psi \rangle + \varepsilon$$

$$\Rightarrow \quad \hat{X} \to \hat{T}_\varepsilon^+ \hat{X} \hat{T}_\varepsilon = \hat{X} + \varepsilon \hat{I}$$

move coordinate system to left by ε.
$$\hat{P} \to \hat{T}_\varepsilon^+ \hat{P} \hat{T}_\varepsilon = \hat{P}$$

Want $\langle \psi | \psi \rangle = \langle \psi_\varepsilon | \psi_\varepsilon \rangle = \langle \psi | \hat{T}_\varepsilon^{+} \hat{T}_\varepsilon | \psi \rangle$

for any $|\psi\rangle \Rightarrow \hat{T}_\varepsilon^{+} \hat{T}_\varepsilon = \mathbb{1}$

$\Rightarrow \hat{T}_\varepsilon^{+} = \hat{T}^{-1} \Rightarrow \hat{T}_\varepsilon$ – unitary!

$\Rightarrow \hat{T}_\varepsilon = e^{-i\varepsilon G}$

must be Hermitian so that $\hat{T}_\varepsilon$ is unitary

$G^{+} = G$.

why linear in $\varepsilon$?

$\rightarrow \hat{T}_\varepsilon^{+} = \hat{T}_\varepsilon^{-1} = e^{i\varepsilon G^{+}} = e^{i\varepsilon G} = e^{-i(-\varepsilon)G}$

$\Rightarrow \hat{T}_\varepsilon^{+} = \hat{T}_{-\varepsilon} = e^{-i(-\varepsilon)G}$

note: any odd power of $\varepsilon$ will do, but also require abelian group property

$\hat{T}_{\varepsilon_1} \hat{T}_{\varepsilon_2} = \hat{T}_{\varepsilon_1 + \varepsilon_2} \Rightarrow \hat{T}_\varepsilon = e^{-i\varepsilon G}$

# Gauge Invariance in Q.M.

## I. Classical E&M.

$$\nabla\cdot B = 0 \qquad \nabla\times B = \frac{4\pi}{c} j + \frac{1}{c}\partial_t E$$
$$\nabla\cdot E = 4\pi\rho \qquad \nabla\times E = -\frac{1}{c}\partial_t B$$

6 - fncs (3d)
8 eqns
but 2 dependent ones
$\nabla\cdot(\nabla\times B) = 0$
$\nabla\cdot(\nabla\times E) = 0$

$$\Rightarrow \nabla\cdot B = 0 \Rightarrow B = \nabla\times A$$

$$\Rightarrow \nabla\times\left(E + \frac{1}{c}\partial_t A\right) = 0 \Rightarrow E = -\frac{1}{c}\partial_t A - \nabla V$$

$\vec{E}, \vec{B}$ field expressed in terms of $\underbrace{\vec{A}, V}_{\text{4 field comp.}}$

... but only $\vec{E}$ & $\vec{B}$ are classically physical

relativistically: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0$$

$$B_i = \epsilon_{i\mu\nu 0} F_{\mu\nu}$$

clearly $A_\mu \longrightarrow A_\mu + \partial_\mu \chi$

leaves $\vec{E}, \vec{B}$, i.e., $F_{\mu\nu}$ unchanged

$$(\partial_\mu\partial_\nu\chi - \partial_\nu\partial_\mu\chi = 0)$$

Gauge freedom to change $A_\mu$ by $\partial_\mu\chi$, leave physical field, $E, B$ unchanged.

$\vec{A}$, $V$ satisfy 4 eqns but only 3 independent ones, due to gauge freedom/symmetry.

<u>Note:</u> Gauge <u>redundancy</u> not symmetry
(extra, unphysical d.o.f.)

Additional constraint ⇒ "pick a gauge"

$$\nabla^2 V + \frac{1}{c}\partial_t \nabla\cdot A = -4\pi\rho$$

Many choices:

$$\nabla\times(\nabla\times A) + \frac{1}{c^2}\partial_t^2 A + \frac{1}{c}\nabla\partial_t V = \frac{4\pi}{c} j$$

1. Coulomb (or transverse $\vec{k}\perp\vec{A}_k = 0$) $\nabla\cdot A = 0$ gauge.

$\Rightarrow \nabla^2 V = -4\pi\rho \Rightarrow V$ responds instantaneously to $\rho$ i.e. $V \propto A_0$ $\underline{\underline{\text{not}}}$ a dynam d.o.f. (not retarded)

only transverse fields $\vec{A}\perp\vec{k} \Rightarrow 2$ d.o.f.

$$\Rightarrow \nabla^2 A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j + \frac{1}{c}\nabla\partial_t V$$

$$= -\frac{4\pi}{c} j_{\text{transverse part only}}$$

2. Lorentz gauge $\partial_\mu A_\mu = 0 = \nabla\cdot A + \frac{1}{c}\partial_t V$

$$\Rightarrow \nabla^2 V - \frac{1}{c^2}\partial_t^2 V = -4\pi\rho \quad \Big| \quad \nabla^2 A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c} j$$

3. $V = 0$ gauge. $\Rightarrow \nabla\cdot A \neq 0 \propto \rho$.

## II. Gauge principle

All "fundamental" interactions in nature are of gauge type, i.e., mediated by gauge field (like $\vec{A}$, V in E&M) and couple to matter via gauge / minimal coupling.

"neutral" particle: $H(\vec{p}, \vec{r})$

$\Rightarrow$ "charged particle: $H(\underbrace{\vec{p} - \frac{q}{c}\vec{A}}_{m\dot{\vec{r}}}, \vec{r})$

not physical   not physical

$m\dot{\vec{r}}$ – kinematic mechanical momentum

Ex's:

1. E & M (QED): $\vec{A} \Rightarrow \vec{E}$ & $\vec{B}$   $U(1)$
   one photon

2. Weak: $\vec{W}_1, \vec{W}_2, \vec{W}_3$ ($\Rightarrow \vec{W}_+, \vec{W}_-, Z_0$)   $SU(2)$

3. Strong (QCD): $\vec{g}_{ab}$ – 8 gluons from traceless 3x3 matrix.

---

## III. Non-relativistic QM

$$H = \frac{p^2}{2m} \implies H = \frac{(\vec{p} - \frac{q}{c}\vec{A})^2}{2m} + V(\vec{r})$$

(note $j = \frac{\hbar}{m}\mathrm{Im}(\psi^* \nabla \psi) - \frac{q}{mc} A|\psi|^2$) (in coulomb gauge)

Maxwell's Eqns + Schrodinger's Eqn.

gauge invnce:

$$\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 |\psi\rangle = (i\hbar\partial_t - V)|\psi\rangle$$

g. transf: $\vec{A} = \vec{A}' + \vec{\nabla}\chi$ , $V = V' - \frac{q}{c}\partial_t \chi$

$$\Rightarrow \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}' - \frac{q}{c}\vec{\nabla}\chi\right)^2 |\psi\rangle$$

$$= \left(i\hbar\partial_t - V' + \frac{q}{c}\partial_t \chi\right)|\psi\rangle$$

clearly $|\psi\rangle$ no longer satisfies Sch. Eqn w/ $\vec{A}'$, $V'$ but $|\psi'\rangle$ does!

$$|\psi\rangle = e^{i\frac{q}{\hbar c}\chi(\vec{r},t)}|\psi'\rangle$$

substitute & use $e^{-i\phi(\vec{r})}\hat{p}\,e^{i\phi(\vec{r})} =$

$$= e^{-i\phi}[\hat{p}, e^{i\phi}] + \hat{p} = \hat{p} + \hbar\nabla\phi$$

$$\Rightarrow e^{-i\phi} f(\hat{p})\, e^{i\phi} = f(\hat{p} + \hbar\nabla\phi)$$

$$\Rightarrow \boxed{\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}'\right)|\psi'\rangle = (i\hbar\partial_t - V')|\psi'\rangle}$$

(Y. Aharonov, D. Bohm, P.R. **115**, 485 (1959))

## IV. The Aharonov-Bohm Effect (AB)

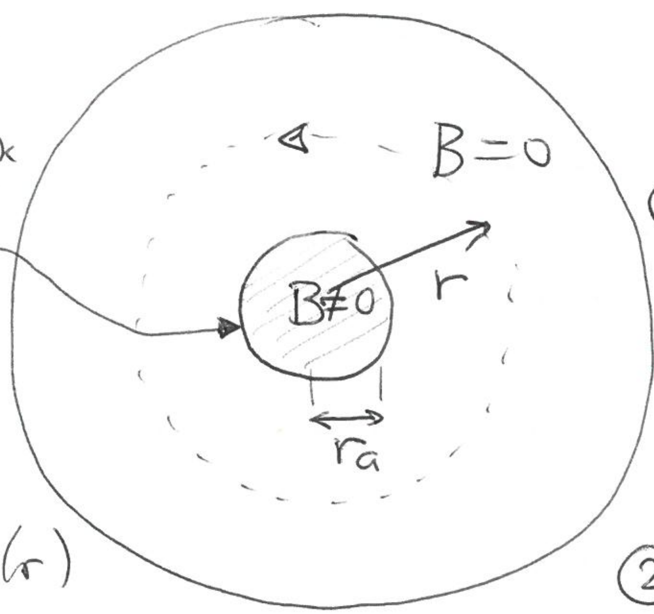
In CM (E&M) only $\vec{E}$ & $\vec{B}$ are physical

In QM $\vec{A}$, V are more fundamental
$\Rightarrow$ no way to formulate QM in terms of $\vec{E}, \vec{B}$!

Key point: $\vec{E}$ & $\vec{B}$ are not the only gauge invnt quantities! Also "Wilson's loops"

$$\oint \vec{A}\cdot d\vec{r} = W$$

"Strange" effects due to non-locality of Q.M. $\Rightarrow$ AB effect




can even shrink $r_a \to 0$, point flux w/ $\vec{A} = \hat{\varphi}\frac{\Phi}{2\pi r}$ $= \frac{\Phi}{2\pi r^2}(-y, x)$

$\vec{A}(\vec{r}) = ?$

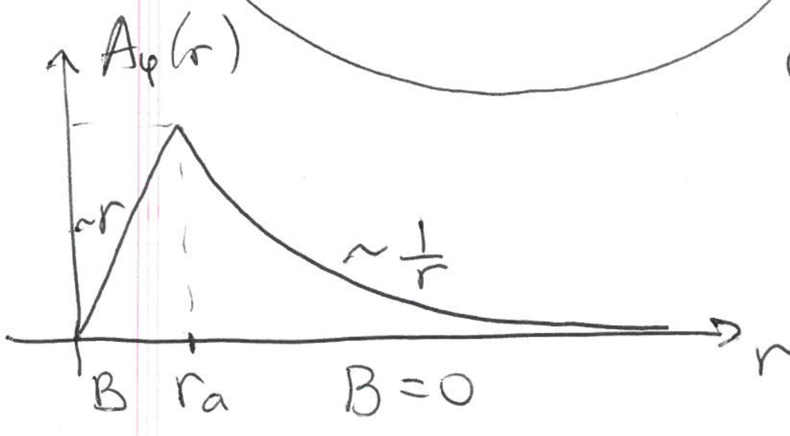
① $\oint_{r<r_a} \vec{A}\cdot d\vec{r} = B\pi r^2$

$2\pi r A_\varphi = B\pi r^2$

$\Rightarrow \vec{A}(\vec{r} < r_a\hat{r}) = \frac{Br}{2}\hat{\varphi}$

② $\oint \vec{A}\cdot d\vec{r} = B\pi r_a^2$

$\Rightarrow \vec{A}(r>r_a) = \frac{B r_a^2}{2r}\hat{\varphi}$




check: $\vec{B} = \vec{\nabla}\times\vec{A} = \hat{z}\frac{1}{r}\partial_r(rA_\varphi)$

$$= \begin{cases} B, & r<r_a \\ 0, & r>r_a \end{cases}$$

## A. Time-independent Sch. Eqn.

$$\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 \Psi = E\Psi$$ with $\infty$ potential at $r_a$ & $r_b$

$$\Rightarrow \quad \vec{A} = \hat{\varphi}\,\frac{B r_a^2}{2r}$$

cylindrical coor. 2D: $p^2 = -\hbar^2 \nabla^2 = -\hbar^2\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\right)$

$$\Rightarrow \left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 = \left[-\hbar^2\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \left(\frac{-i\hbar}{r}\partial_\varphi - \frac{q}{c}\frac{1}{r}\frac{B r_a^2}{2}\right)^2\right]$$

$$= -\hbar^2\left(\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{1}{r^2}\left(\partial_\varphi - \underbrace{\left(\frac{iq}{\hbar c}\right)}_{i\frac{2\pi}{\Phi_0}}\underbrace{\frac{B r_a^2}{2}}_{\Phi/2\pi}\right)^2\right)$$

$\Phi_0 \underset{q=e}{=} 4.1\times10^{-7}\,G\,cm^2$ ← flux quantum

$$\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 \Psi = E\Psi$$

$$-\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\left(\partial_\varphi - i\frac{\Phi}{\Phi_0}\right)^2\right]\Psi(r,\varphi) = E\,\Psi(r,\varphi)$$

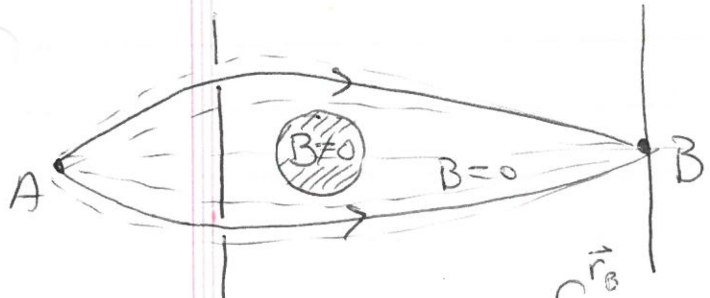
quantize $L_z$ in units of $\hbar$ $\Rightarrow$ integer $l$
for single valued $\psi(\varphi)$

$$\Rightarrow \frac{-\hbar^2}{2mr^2}\left(\partial_\varphi - i\frac{\Phi}{\Phi_0}\right)^2 e^{il\varphi} = \frac{\hbar^2}{2mr^2}\underbrace{\left(l - \frac{\Phi}{\Phi_0}\right)^2}$$

affects the spectrum when $\Phi/\Phi_0 \notin \mathbb{Z}$.

$$\left[-\frac{\hbar^2}{2m}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) + \frac{\hbar^2}{2m}\left(l - \frac{\Phi}{\Phi_0}\right)^2\right]\psi_l(r) = E\,\psi_l(r)$$

# B. Via P.I. Scattering




$$U(\vec r_B,\vec r_A;t)=\int_{\vec r_A}^{\vec r_B}\mathcal{D}\vec r\; e^{\frac{i}{\hbar}\int_0^t dt'\,\underbrace{L(\vec r(t'))}_{\frac{m}{2}\left(\frac{d\vec r}{dt}\right)^2+\frac{q}{c}\frac{d\vec r}{dt}\cdot\vec A}}$$

<u>"Rough" analysis</u>:

$$U(\vec r_B,\vec r_A;t)=\int\mathcal{D}r\; e^{\frac{i}{\hbar}\int_0^t dt'\left[\frac{m}{2}\left(\frac{dr}{dt}\right)^2\right]+\frac{i}{\hbar}\underbrace{\int_0^t dt'\,\frac{q}{c}\frac{d\vec r}{dt}\cdot\vec A}_{\text{above}\ \underbrace{\frac{q}{c}\oint_{r_A}^{r_B}d\vec r\cdot\vec A}_{\text{path indepnt}}}}$$

$$+\int_{\text{Below}}\mathcal{D}r\; e^{\frac{i}{\hbar}\int_0^t dt\left[\frac{m}{2}\left(\frac{dr}{dt}\right)^2\right]+i\frac{2\pi}{\phi_0}\underbrace{\oint_{r_A}^{r_B}d\vec r\cdot\vec A}_{\text{path indep.}}}$$

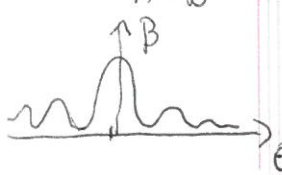
Above

path indep. since $B=0$

If A & B are on axis then $U_{above}=U_{below}$

$$U_{A\to B}=U^{(B=0)}_{above/below}\; e^{i\frac{2\pi}{\phi_0}\oint_{r_A\,\text{above}}^{r_B}dr}\left[1+e^{i\frac{2\pi}{\phi_0}\underbrace{\oint d\vec r\cdot\vec A}_{\phi}}\right]$$

$$P_{A\to B}=|U_{A\to B}|^2=\underbrace{\frac{1}{\cancel{4}}\left|U^{(B=0)}_{A\to B}\right|^2}_{\text{constructive interference}\Rightarrow\text{add}}\;\underbrace{\cancel{4}\cos^2\left(\pi\frac{\phi}{\phi_0}\right)}_{\frac{1}{2}\left(1+\cos\left(2\pi\frac{\phi}{\phi_0}\right)\right)}$$




can turn max at $\theta=0$ to minimum for $\frac{\phi}{\phi_0}=\frac{1}{2}$

but B changes relative phase ⇒ shifts max depending on $\phi/\phi_0$: <u>no shift for $\phi/\phi_0=n$</u>

<u>shift to min</u> for $\phi/\phi_0=\frac{1}{2}(2n+1)$.

## Physical consequences

even though particle is only in $B=0$ region! i.e., absolutely <u>no</u> effect in E.O.M. (Newton) ⇒ nonlocality of QM, particle "senses" $\oint\vec A\cdot d\vec l$ <u>not just</u> $\vec\nabla\times\vec A=\vec B$.

# B. details of AB via PI

Evolution in the presense of pnt flux.

- Free $B=0$ evolution

$$U(\vec{r}\,',\vec{r};t) = A_t\, e^{\frac{im}{\hbar}\frac{(\vec{r}\,'-\vec{r})^2}{2t}}$$

$$= A_t e^{\frac{im}{2\hbar t}(r'^2+r^2)}\, e^{-\frac{im}{\hbar t} r' r \cos\underbrace{(\theta'-\theta)}}$$

limited to $[0, 2\pi)$

$$= A_t e^{\frac{im}{2\hbar t}(r'^2+r^2)} \sum_{\ell=-\infty}^{\infty} e^{i\ell(\theta'-\theta)}\, I_\ell\left(-\frac{imr'r}{\hbar t}\right)$$

decompose into angular harmonics.

$$\left(\text{used } e^{a\cos\theta} = \sum_\ell e^{i\ell\theta} I_\ell(a)\right)$$

Fourier series coefficients

$$= \sum_\ell e^{i\ell(\theta'-\theta)}\, U_\ell(r',r;t)$$

$$= \int_{-\infty}^{\infty} d\lambda \sum_{n=-\infty}^{\infty} e^{i\lambda(\theta'-\theta+2\pi n)}\, U_\lambda(r',r;t)$$

(using Poisson sum: $\sum_n e^{i2\pi\lambda n} = \sum_\ell \delta(\lambda-\ell)$)

$$= \int_{-\infty}^{\infty} d\phi\, U_\phi(r',r,\theta',\theta;t)$$

$$= \int_{-\infty}^{\infty} d\phi \sum_{n=-\infty}^{\infty} \delta(\theta'\underbrace{-\theta+2\pi n-\phi}) \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda\phi}\, U_\lambda(r',r;t)$$

fixes to angle $\phi$ paths.
$\Leftrightarrow$ winding # $n$.

$$= \sum_{n=-\infty}^{\infty} \underbrace{U_n(\theta',\theta;t)}_{\text{amplitude at fixed winding } n.}$$

$$U_n = \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda(\theta'-\theta+2\pi n)}\, A_t e^{\frac{im}{2\hbar t}(r'^2+r^2)}\, I_\lambda\left(-\frac{imr'r}{\hbar t}\right)$$

$$U = \sum_n U_n$$

- $B \neq 0$ inside a infinitesimal tube w/ $\Phi$.

$$\Rightarrow U_{B\neq 0} = \sum_n U_n \, e^{i 2\pi n \Phi/\phi_0}$$

extra phase from $U_\lambda^{B\neq 0}$

$$= \sum_n \int_{-\infty}^{\infty} d\lambda \, e^{i\lambda(\theta'-\theta+2\pi n) + i(\theta'-\theta+2\pi n)\left(\frac{\phi}{\phi_0}\right)} U_\lambda^{B=0}$$

$$= \int_{-\infty}^{\infty} d\lambda \, e^{i\left(\lambda+\frac{\phi}{\phi_0}\right)(\theta'-\theta)} \underbrace{\sum_n e^{i\left(\lambda+\frac{\phi}{\phi_0}\right)2\pi n}}_{\sum_l \delta\left(\lambda+\frac{\phi}{\phi_0}+l\right)} U_\lambda^{B=0}$$

$$U_{B\neq 0} = \sum_l e^{il(\theta'-\theta)} U^{B=0}_{\left|l+\frac{\phi}{\phi_0}\right|}$$

$$U_{B\neq 0}(\vec{r}', \vec{r}, t) = \sum_l e^{il(\theta'-\theta)} A_t \, e^{\frac{im}{2\hbar t}(r'^2+r^2)} I_{\left|l+\frac{\phi}{\phi_0}\right|}\left(\frac{-imr'r}{\hbar t}\right)$$

modified wavefnc quantization of $\lambda$, changes b.c.

Expand $I_{|l+\phi/\phi_0|}$ for large $r, r'$ to get AB oscillations w/ $\phi/\phi_0$ (see F. Wilczek's book)

$$\Rightarrow \frac{P(\vec{r}')}{\text{Vol.}} = \left|U_{B\neq 0} - U_{B=0}\right|^2 \propto \boxed{\frac{d\sigma}{d\varphi} = \frac{1}{2\pi k} \frac{\sin^2 \pi \frac{\phi}{\phi_0}}{\sin^2 \frac{\varphi}{2}}}$$

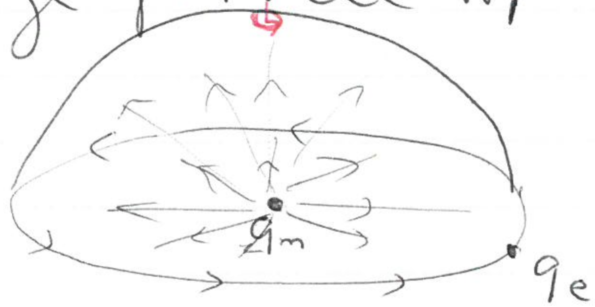
# V. Magnetic monopole & $q_m$ quantization

(P.A.M. Dirac) 1931

suppose magnetic monopole of m.charge $q_m$ exists $\Rightarrow$ $\vec{\nabla}\cdot\vec{B} = 4\pi q_m \delta^3(r) \neq 0$

$\Rightarrow$ $\vec{B} = q_m \dfrac{\hat{r}}{r^2}$

What happens to wavefnc of electrical charge particle w/ charge $q_e$?




$$\Psi(\vec{r}) \xrightarrow[2\pi \text{ encircling}]{} \Psi(\vec{r})\, e^{i\frac{2\pi}{\phi_0}\underbrace{\oint \vec{A}\cdot d\vec{r}}}$$

$$\phi = \frac{4\pi r^2 q_m}{r^2} = 4\pi q_m$$

$$\to \Psi(\vec{r})\, e^{i 2\pi \frac{\phi}{\phi_0}}$$

in 3d require single valued $\Psi(\vec{r})$

$\Rightarrow$ $\underline{\phi = n\,\phi_0}$ $\Rightarrow$ $\boxed{q_m = n\dfrac{\phi_0}{4\pi} = n\dfrac{hc}{4\pi q_e}}$

Dirac quantization condition