Lecture 8

N-particles in higher dimensions Q.M.

I. Distinguishable particles

A. One dimension

Look at Hilbert space of 2 particles

- $[X_i, P_i] = i\hbar\,\delta_{ij}$

- states: $|n\rangle \rightarrow |n_1\, n_2\rangle$

Ex: coord. basis:

$\rightarrow$ $\hat{X}_i |x_1\, x_2\rangle = x_i |x_1\, x_2\rangle$

similarly $|p_1\, p_2\rangle$, etc.

$\rightarrow$ $\langle x_1' x_2' | x_1 x_2\rangle = \delta(x_1'-x_1)\,\delta(x_2'-x_2)$

$\rightarrow$ $\hat{P}_i \rightarrow -i\hbar \dfrac{\partial}{\partial x_i}$

$\rightarrow$ wave fnc: $\langle x_1\, x_2 | \psi\rangle = \psi(x_1, x_2)$

$\rightarrow$ $P(x_1, x_2) = |\langle x_1\, x_2|\psi\rangle|^2 = |\psi(x_1, x_2)|^2$

$\rightarrow$ $\langle\psi|\psi\rangle = \int dx_1\, dx_2\, |\psi(x_1, x_2)|^2$

Formally Hilbert space of a 2 particle system is a <u>direct</u> <u>product</u> of two 1 particle Hilbert spaces:

- $\mathbb{H}_{1,2} \equiv \underset{\substack{N_1 N_2 \\ \text{dimensional}}}{\mathbb{H}_{1\otimes 2}} = \underset{N_1}{\mathbb{H}_1} \otimes \underset{N_2}{\mathbb{H}_2}$ - dimensional

Namely:

- $|n_1\, n_2\rangle = |n_1\rangle \otimes |n_2\rangle \rightarrow \begin{cases} \text{particle 1 in state } n_1 \\ \text{particle 2 in state } n_2 \end{cases}$

  <u>different</u> spaces.

  $\neq \langle n_1 | n_2 \rangle$ (inner product)

  $\neq |n_1\rangle\langle n_2|$ (outer product)

  vectors from same space

- Note: direct product is a linear operation

$$(\alpha_1 |n_1\rangle + \alpha_1' |n_1'\rangle) \otimes \beta |n_2\rangle = \alpha_1 \beta |n_1\rangle \otimes |n_2\rangle + \alpha_1' \beta |n_1'\rangle \otimes |n_2\rangle$$

- $\mathbb{H}_{1\otimes 2}$ is spanned by $|n_1\rangle \times |n_2\rangle$ if $|n_1\rangle$ & $|n_2\rangle$ respectively span $\mathbb{H}_1$, $\mathbb{H}_2$.

- Note: $|\psi\rangle = \sum_{n_1, n_2} a_{n_1 n_2} |n_1\rangle \otimes |n_2\rangle$

  <u>not</u> every element of $\mathbb{H}_{1\otimes 2}$ is a direct-product state.

- inner product: $\langle x_1' x_2' | x_1 x_2 \rangle =$

$$= \left(\langle x_1'| \otimes \langle x_2'|\right)\left(|x_1\rangle \otimes |x_2\rangle\right) = \langle x_1'|x_1\rangle \langle x_2'|x_2\rangle$$

$$= \delta(x_1' - x_1)\,\delta(x_2' - x_2)$$

$$\Rightarrow \langle x_1 x_2 | \psi \rangle \equiv \psi(x_1, x_2) = \langle x_1 | \psi \rangle \langle x_2 | \psi \rangle$$

simplest example $|\psi\rangle = |n_1\rangle \otimes |n_2\rangle$

$\Rightarrow \quad \psi_{n_1, n_2}(x_1, x_2) = \psi_{n_1}(x_1)\,\psi_{n_2}(x_2)$

- operators acting in $\mathbb{H}_{1\otimes 2}$:

$\Rightarrow \hat{X}_1 \rightarrow \hat{X}_1^{1\otimes 2} = \hat{X}_1 \otimes \hat{\mathbb{1}}$

such that: $\hat{X}_1^{1\otimes 2} |x_1 x_2\rangle = (\hat{X}_1 \otimes \mathbb{1})(|x_1\rangle \otimes |x_2\rangle)$

$$= \hat{X}_1 |x_1\rangle \otimes \mathbb{1}|x_2\rangle = x_1 |x_1\rangle \otimes |x_2\rangle = x_1 |x_1 x_2\rangle$$

$\Rightarrow \hat{P}_2 \rightarrow \hat{P}_2^{1\otimes 2} = \hat{\mathbb{1}} \otimes \hat{P}_2$

$\Rightarrow$ more generally direct product of Θ_1 & Θ_2 is $\Theta^{1\otimes 2} \equiv \underset{\substack{\uparrow \\ \text{only} \\ \text{acts on } \mathbb{H}_1}}{\Theta_1} \otimes \underset{\substack{\uparrow \\ \text{only acts} \\ \text{on } \mathbb{H}_2}}{\Theta_2}$

$\Rightarrow$ matrix representation: $\langle n_1 n_2 | \Theta^{1\otimes 2} | m_1 m_2 \rangle$

$= \langle n_1 | \Theta^{(1)} | m_1 \rangle \langle n_2 | \Theta^{(2)} | m_2 \rangle$

$\Rightarrow$ distributive, associative, commutative

<u>Example</u>:

$\mathbb{H}_1$ – two dimensional, spanned by $|+\rangle, |-\rangle$

$\mathbb{H}_2$ – three dim., spaned by $|1\rangle, |2\rangle, |3\rangle$

$\Rightarrow \mathbb{H}^{1\otimes 2} \equiv \mathbb{H}_1 \otimes \mathbb{H}_2$ is spanned by

$$\left.\begin{matrix} |+\rangle\otimes|1\rangle, & |+\rangle\otimes|2\rangle, & |+\rangle\otimes|3\rangle, \\ |-\rangle\otimes|1\rangle, & |-\rangle\otimes|2\rangle, & |-\rangle\otimes|3\rangle \end{matrix}\right\} \; 2\times 3 = 6 \; \checkmark$$

$$\langle n_1|\mathcal{O}^{(1)}|m_1\rangle \equiv \mathcal{O}^{(1)}_{n_1 m_1} = \begin{array}{c} \\ |+\rangle \\ |-\rangle \end{array} \begin{array}{c} \begin{array}{cc} |+\rangle & |-\rangle \end{array} \\ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \end{array}$$

with $a_1 = \langle +|\mathcal{O}^{(1)}|+\rangle$, etc.

$$\langle n_2|\mathcal{O}^{(2)}|m_2\rangle \equiv \mathcal{O}^{(2)}_{n_2 m_2} = \begin{array}{c} \\ |1\rangle \\ |2\rangle \\ |3\rangle \end{array} \begin{array}{c} \begin{array}{ccc} |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} \end{array}$$

with $a_2 = \langle 1|\mathcal{O}^{(2)}|1\rangle$, etc.

$$\hat{\mathcal{O}}^{(1)} \otimes \hat{\mathcal{O}}^{(2)} \equiv \mathcal{O}^{(1\otimes 2)} \Rightarrow \langle n_1 n_2|\mathcal{O}^{(1\otimes 2)}|m_1 m_2\rangle$$

$$= \begin{array}{c} \\ |+1\rangle \\ |+2\rangle \\ |+3\rangle \\ |-1\rangle \\ |-2\rangle \\ |-3\rangle \end{array} \begin{array}{c} \begin{array}{cccccc} |+1\rangle & |+2\rangle & |+3\rangle & |-1\rangle & |-2\rangle & |-3\rangle \end{array} \\ \begin{pmatrix} a_1 a_2 & a_1 b_2 & a_1 c_2 & b_1 a_2 & b_1 b_2 & b_1 c_2 \\ a_1 d_2 & a_1 e_2 & a_1 f_2 & b_1 d_2 & b_1 e_2 & b_1 f_2 \\ a_1 g_2 & a_1 h_2 & a_1 i_2 & b_1 g_2 & b_1 h_2 & b_1 i_2 \\ c_1 a_2 & c_1 b_2 & \cdots & d_1 a_2 & d_1 b_2 & \cdots \\ \vdots & & & \vdots & & \\ & & & & & \end{pmatrix} \end{array}$$

6×6 matrix.

- For simplicity of notation will often suppress $1 \otimes 2$, where obvious.

<u>Ex</u>: $(\hat{p}_1 + \hat{p}_2)^2 = \hat{p}_1^2 + \hat{p}_2^2 + 2\hat{p}_1 \hat{p}_2$

$$\equiv (\hat{p}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{p}_2)^2 = \hat{p}_1^2 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{p}_2^2 + 2\,\hat{p}_1 \otimes \hat{p}_2 . \quad \checkmark$$

- coordinate representation:

states: $|\psi\rangle = \sum_{n_1, n_2} c_{n_1 n_2} |n_1\rangle \otimes |n_2\rangle$

$$\Rightarrow \psi(x_1, x_2) \equiv (\langle x_1| \otimes \langle x_2|)|\psi\rangle = \sum_{n_1, n_2} c_{n_1 n_2} \langle x_1|n_1\rangle \langle x_2|n_2\rangle$$

$$\Rightarrow \psi(x_1, x_2) = \sum_{n_1, n_2} c_{n_1 n_2} \psi_{n_1}(x_1) \psi_{n_2}(x_2)$$

simple product, distinguished by $x_{1,2}$ argument.

operators: <u>ex</u>. $\hat{p}_1^{1\otimes 2} = \hat{p}_1 \otimes \mathbb{1}$

$$\langle x_1 x_2 | \hat{p}_1^{1\otimes 2} | x_1' x_2' \rangle = \underbrace{\langle x_1 | \hat{p}_1 | x_1' \rangle}_{-i\hbar \frac{\partial}{\partial x_1} \delta(x_1 - x_1')} \underbrace{\langle x_2 | \hat{\mathbb{1}} | x_2' \rangle}_{1}$$

$$\Rightarrow \langle x_1 x_2 | \hat{p}_1^{1\otimes 2} | x_1' x_2' \rangle = -i\hbar \frac{\partial}{\partial x_1} \delta(x_1 - x_1')$$

<u>partial</u> derivative ensures that $\hat{p}_1 \otimes \mathbb{1}$ only acts on $\mathbb{H}_1$.

- Evolution in $\mathbb{H}^{1\otimes 2}$

$\boxed{\text{formally}}$

$$\hat{H}(\hat{p}_1, \hat{p}_2, \hat{x}_1, \hat{x}_2)|\psi\rangle = i\hbar\,\partial_t|\psi\rangle$$

Simple if $\hat{H}$ is <u>separable</u>:

$$\hat{H}(p_1, p_2, \hat{x}_1, \hat{x}_2) = \hat{H}_1(\hat{p}_1, \hat{x}_1) + \hat{H}_2(\hat{p}_2, \hat{x}_2)$$

i.e. in formal notation:

$$\hat{H}^{1\otimes 2} = \hat{H}_1^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_2^{(2)}$$

otherwise <u>not</u> separable.

Look at <u>separable</u> case:

$$\hat{H}|\psi(t)\rangle = i\hbar\,\partial_t|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-iEt/\hbar}|E\rangle$$

$$\Rightarrow \hat{H}|E\rangle = E|E\rangle$$

$$(H_1(p_1, x_1) + H_2(p_2, x_2))|E\rangle = E|E\rangle$$

<u>but</u> $[H_1(p_1, x_1), H_2(p_2, x_2)] = 0$, since $[p_i, x_j] = -i\hbar\delta_{ij}$ etc.

$\Rightarrow$ diagonalizable simultaneously.

$$|E\rangle = |E_1\rangle \otimes |E_2\rangle$$

$$(H_1 \times \mathbb{1})|E\rangle + (\mathbb{1} \times H_2)|E\rangle = E|E\rangle$$

$$H_1|E_1\rangle = E_1|E_1\rangle \,, \quad H_2|E_2\rangle = E_2|E_2\rangle$$

with $E_1 + E_2 = E$

$$\Rightarrow |\psi(t)\rangle = |E_1\rangle \otimes |E_2\rangle \, e^{-\frac{i}{\hbar}E_1 t} \, e^{-\frac{i}{\hbar}E_2 t}$$

$\boxed{\text{coordinate representation}}$

project all of above steps onto $|x_1\rangle \otimes |x_2\rangle$

$$\Rightarrow \psi(x_1, x_2; t) = (\langle x_1| \otimes \langle x_2|)|\psi(t)\rangle = \langle x_1 x_2|\psi(t)\rangle$$

$\langle x_1 x_2|\hat{H}|x_1' x_2'\rangle$, etc ...

$$\Rightarrow H\left(-i\hbar\frac{\partial}{\partial x_1}, x_1 ; -i\hbar\frac{\partial}{\partial x_2}, x_2\right)\psi(x_1, x_2, t)$$

$$= i\hbar\,\partial_t\,\psi(x_1, x_2, t)$$

separable: $H = H_1\left(-i\hbar\frac{\partial}{\partial x_1}, x_1\right) + H_2\left(i\hbar\frac{\partial}{\partial x_2}, x_2\right)$

Ex. of particle in a potential:

$$H = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1) - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)$$

$$\psi(x_1, x_2, t) = \psi_E(x_1, x_2)\, e^{-\frac{i}{\hbar}Et}$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + V_1(x_1) - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi_E(x_1 x_2) = E\,\psi_E(x_1 x_2)$$

Assume/Try: $\psi_E(x_1, x_2) = \psi_{E_1}(x_1)\,\psi_{E_2}(x_2)$

$$\Rightarrow \underbrace{\frac{1}{\psi_{E_1}(x_1)}\left[-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1)\right]\psi_{E_1}(x_1)}_{\text{fnc of } x_1 \text{ only} \Rightarrow \text{const } E_1} +$$

$$+ \underbrace{\frac{1}{\psi_{E_2}(x_2)}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi_E(x_2)}_{\text{fnc of } x_2 \text{, only} \Rightarrow \text{const } E_2} = E$$

const. x_1, x_2 independent.

$$\Rightarrow \left[-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + V_1(x_1)\right]\psi_{E_1}(x_1) = E_1\,\psi_{E_1}(x_1)$$

$$\left[-\frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V_2(x_2)\right]\psi_{E_2}(x_2) = E_2\,\psi_E(x_2)$$

& $\quad E = E_1 + E_2$ ✓

$$\Rightarrow \Psi_E(x_1, x_2, t) = \psi_{E_1}(x_1)\,\psi_{E_2}(x_2)\,e^{-\frac{i}{\hbar}E_1 t}\,e^{-\frac{i}{\hbar}E_2 t}$$

Note: even if $\hat{H}$ not separable in one set of vars may become separable in another set of coordinates $\Rightarrow x_1, x_2 \rightarrow$ normal modes of e.g. harmonic oscill.

(then usually still call separable)

<u>Important Example</u>: 2 interacting particles
$\Rightarrow V(x_1, x_2) = V(x_1 - x_2)$

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(x_1, x_2)$$

considerable simplification even if not separable in x_1, x_2, but only due to mutual interaction $\Rightarrow V(x_1, x_2) = V(x_1 - x_2)$

$\Rightarrow$ guaranteed new coordinates in which separable:

<u>transform to</u>: center of mass & relative coordinates:

$$X_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad X = x_1 - x_2$$

("normal modes" of this problem)

$$\rightarrow x_1 = X_{cm} + \frac{m_2}{m_1 + m_2} X, \quad x_2 = X_{cm} - \frac{m_1}{m_1 + m_2} X$$

$$\Rightarrow p_1 = m_1 \dot{x}_1 = m_1 \dot{X}_{cm} + \frac{m_1 m_2}{m_1 + m_2} \dot{X}$$

$$p_2 = m_2 \dot{x}_2 = m_2 \dot{X}_{cm} - \underbrace{\frac{m_1 m_2}{m_1 + m_2}}_{\equiv \mu \text{ — reduced mass}} \dot{X}$$

$m_1 + m_2 \equiv M$

$$\Rightarrow p_i = \frac{m_i}{M} P_{cm} \pm p$$

where $P_{cm} \equiv M \dot{X}_{cm}$, $\mu \dot{X} \equiv p$

verified a posteriori from new $\hat{H}$.

$$\boxed{P_{cm} = p_1 + p_2}, \quad \boxed{p = \frac{m_2}{M} p_1 - \frac{m_1}{M} p_2}$$

$$\Rightarrow H_{tot} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1 - x_2)$$

$$H_{tot} = \frac{P_{cm}^2}{2M} + \frac{p^2}{2\mu} + V(x)$$

$$\Rightarrow H_{tot} = H_{cm} + H_{relative} \leftarrow \underline{\text{now separable}}$$

<u>Note</u>: $H_{cm} = \frac{P_{cm}^2}{2M} \Leftrightarrow$ <u>free</u> $(V_{cm}(X_{cm}) = 0)$

$H_{rel} = \frac{p^2}{2\mu} + V(x) \Leftrightarrow$ <u>a</u> particle of mass μ moving in potential $V(x)$

→ Many examples:

- Earth around sun $\Rightarrow M = m_E + m_S$, $\mu = \frac{m_E m_S}{m_E + m_S}$
- e in a Hydrogen atom $\Rightarrow$
  $M = m_e + m_p$, $\mu = \frac{m_e m_p}{m_e + m_p}$

etc.

→ $\dot{X}_{cm} = \{X_{cm}, H\} = \frac{P_{cm}}{M}$ ✓

→ $\dot{x} = \{x, H\} = \frac{p}{\mu}$ ✓

$\Rightarrow$ Canonical quantization:

$$[X_{cm}, P_{cm}] = i\hbar, \quad [x, p] = i\hbar$$

$$P_{tot} V = -i\hbar \sum_{i=1}^{N} \partial_i \left(V(x_1 - x_2,\ x_1 - x_3,\ \ldots\ x_1 - x_N,\ x_2 - x_3, \ldots \right)$$

clearly vanishes since $(\partial_1 + \partial_2) V(x_1 - x_2) = 0$

$$\Rightarrow [P_{tot}, V] = 0 \quad \Rightarrow [P_{tot}, H] = 0$$

$$\Rightarrow \quad \psi_E(X_{cm}, x) = \frac{e^{i p_{cm} X_{cm}/\hbar}}{(2\pi\hbar)^{1/2}} \quad \psi_{E_{rel}}(x)$$

$$E = \frac{P_{cm}^2}{2M} + E_{rel}.$$

CM drifts like free particle → convenient to study problem in co-moving CDM frame
$\Rightarrow p_{cm} = 0 \Rightarrow$ reduces to relative coord. prob.

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2}\psi_{E_{rel}}(x) + V(x)\,\psi_{E_{rel}}(x) = E_{rel}\,\psi_{E_{rel}}(x)$$

Straightforward generalization to N particles:

$$H_{(N)} = \sum_i \frac{p_i^2}{2m_i} + V(\{x_i\})$$

but even if just particle interaction, even if just 2-body (2 at a time)

$$H^{(N)}_{2body} = \sum_i \frac{p_i^2}{2m_i} + \sum_{i \neq j} V(x_i - x_j)$$

still cannot reduce to decoupled H's, even though can factor out free $X_{cm}(t)$ motion.
$\Rightarrow P_{total} = \sum_i p_i = -i\hbar \sum_i \vec{\nabla}_i$ is conserved!
Prominent exception, where via normal modes can decouple $H^{(N)}$ into N $H^{(1)}$'s is N-coupled harmonic oscillators; N atoms in a crystal →
→ N independent normal modes of vibration → phonons
must make sure that new vars are canonical i.e. satisfy canonical comm. relations (Unitary transf.)

B. Generalization to N particles in d-dimensions

$$\underbrace{X, P}_{2 \text{ vars.}} \rightarrow \underbrace{X_i, P_i}_{2N \text{ vars}} \rightarrow \underbrace{\vec{X}_i, \vec{P}_i}_{2Nd \text{ vars.}}$$

1 particle $|x\rangle \rightarrow |\vec{r}_i\rangle$ with $\langle \vec{r}_i' | \vec{r}_j \rangle = \delta_{ij} \underbrace{\delta^d(\vec{r}' - \vec{r})}$

$\delta(x'-x)$
$\delta(y'-y)$
$\delta(z'-z)$

N particle state in d-dim:

$$|\vec{r}_1 \vec{r}_2 \vec{r}_3 \ldots \vec{r}_N\rangle \in \mathbb{H}^{(dN)}$$

dN-dimensional Hilbert space.

Ex. 3d infinite square box:

$$V(\vec{r}) = \begin{cases} 0, & 0<x<L,\ 0<y<L,\ 0<z<L \\ \infty, & \text{otherwise} \end{cases}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi_E(\vec{r}) + V(\vec{r}) \Psi_E(\vec{r}) = E \Psi_E(\vec{r})$$

separable

$$\Rightarrow \quad \Psi_E = \left(\frac{2}{L}\right)^{1/2} \sin(k_{n_x} x) \left(\frac{2}{L}\right)^{1/2} \sin(k_{n_y} y) \left(\frac{2}{L}\right)^{1/2} \sin(k_{n_z} z)$$

$$E_{n_x n_y n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \frac{\hbar^2 \pi^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2)$$

$$n_i \in \mathbb{Z} \geq 0$$

More generally if $H = H_x + H_y + H_z \rightarrow$ separable
$\Rightarrow$ reducible to d 1d probs. $\Rightarrow$ easy!

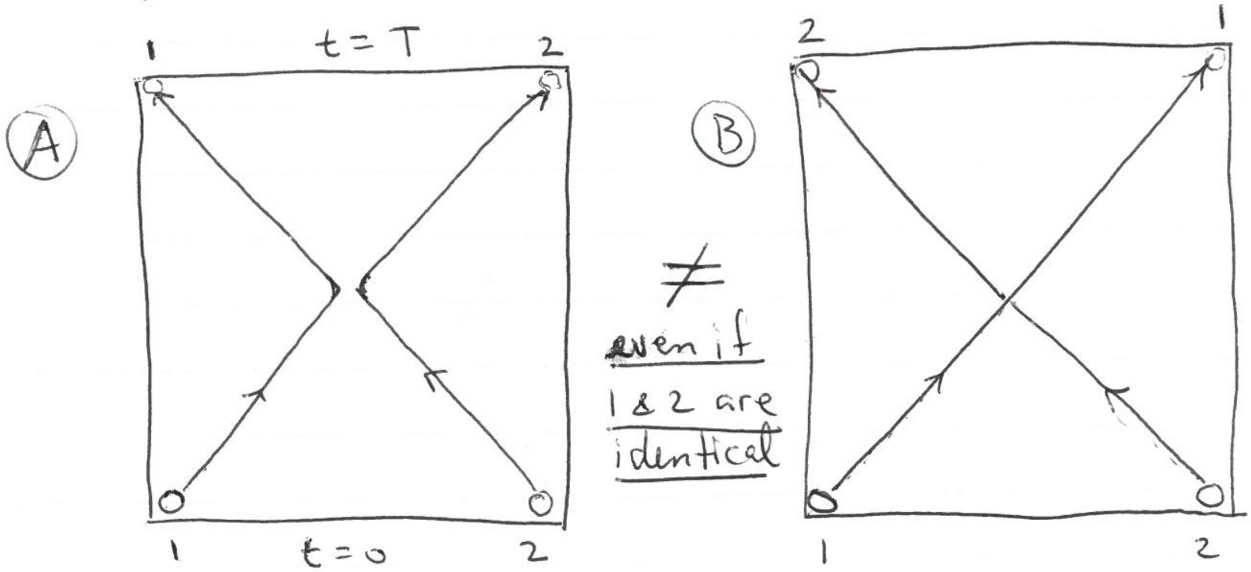
II. Indistinguishable particles

One - Dimension

Very different meaning classically & quantum mechanically! → Q.M. much stronger.

Why?

Classically can follow trajectories of two identical particles $X_1(t)$ & $X_2(t)$ without disturbing them ⟹ identical classical particles are distinguishable via distinct trajectories:



⟹ $|1,2)$ state at $t = T$ is distinct from $|2,1)$ state

i.e. two distinct events A & B!

Quantum mechanically A & B do <u>not</u> correspond to distinct states since <u>no</u> concept of trajectories

$\Rightarrow$ A $\Leftrightarrow$ B ! i.e. $|1,2\rangle \Leftrightarrow |2,1\rangle$ same Q.M. state.

<u>Note:</u> of course for nonidentical particles, e.g. e & p

$|a\,b\rangle \neq |ba\rangle$

particle 1 in state a
particle 2 in state b

—॥— state b
—॥— state a

So for Q.M. identical particles 1, 2 measured to be at $X_1 = a$ & $X_2 = b$ the state is neither $|a,b\rangle$ nor $|b,a\rangle$. They are in a state $|\psi(a,b)\rangle$ s.t. $\psi(a,b) = \alpha\,\psi(b,a)$

- measure a, b coordinates $\Rightarrow$ particle at $x=a$ & particle at $x=b$.
  $\not\Rightarrow X_1 = a,\ X_2 = b$
- but <u>does</u> give $\hat{X}_1 + \hat{X}_2$ has eigenvalue of $a+b$ <u>definitely</u>
  similarly $V(\hat{x}_1 + \hat{x}_2) \rightarrow V(a+b)$, etc.
- $\psi(a,b) = \beta|ab\rangle + \gamma|ba\rangle$ require $= \alpha[\beta|ba\rangle + \gamma|ab\rangle]$
  $\Rightarrow \beta = \alpha\gamma,\ \gamma = \alpha\beta \Rightarrow \alpha = \pm 1$

$$\beta = \alpha\gamma,\ \gamma = \alpha\beta \Rightarrow \alpha^2 = 1 \Rightarrow \alpha = \pm 1$$
$$\Rightarrow \underline{\beta = \pm\gamma}$$

$\Rightarrow |\psi_S(a,b)\rangle \equiv |ab, S\rangle = |ab\rangle + |ba\rangle$ ← symmetric

$|\psi_A(a,b)\rangle \equiv |ab, A\rangle = |ab\rangle - |ba\rangle$ ← antisymmetric

$$P|\psi(a,b)\rangle \equiv |\psi(b,a)\rangle = \alpha|\psi(a,b)\rangle$$

↑ exchange operator ↑ required by identical particles

$$P^2|\psi(a,b)\rangle = \alpha^2|\psi(a,b)\rangle = |\psi(a,b)\rangle$$

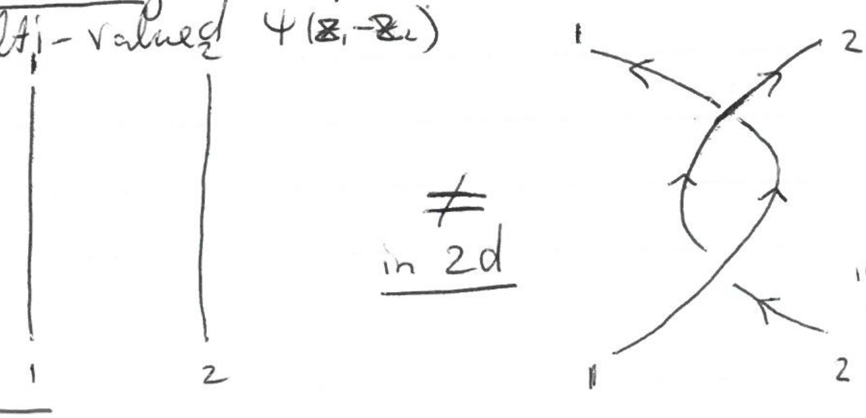
↑ since expect double exchange $\Leftrightarrow \mathbb{1}$

In 3d $\alpha^2 = 1 \Rightarrow \alpha = \pm 1$

However, in 2d $\alpha^2 = e^{i\phi}$, i.e. inner-products returned to same values (e.g. probab.) but wavefnc aquired a phase ϕ after 2π rotation, double exch.

Why? b.c. in 2d can exchange without bringing close together & distinguishing two states by <u>braiding</u> of particle world-lines

$\Rightarrow$ multi-valued $\psi(z_1 - z_2)$



Leinaas & Myrheim (1977) $\Rightarrow$ $\boxed{\text{anyons}}$

see e.g. "Fractional Statistics & anyon S.C." by F. Wilczek.

Note: $PHP = H$, i.e. $[P, H] = 0$

$\Rightarrow$ any linear combination of

$$|n_1 n_2\rangle = \left(\alpha |n_1\rangle |n_2\rangle + \beta |n_2\rangle |n_1\rangle\right)$$

but the only ones occurring in nature are bosons ($\beta = +\alpha$) & fermions ($\beta = -\alpha$) ! exp. fact

$\Rightarrow$ Two types 2-particle wave fnc's

symmetric & antisymmetric

Experimental fact: every particle in nature falls into one of these two classes & does not change this property

no mixed symmetry particles!

$\boxed{\text{Bosons}}$ (e.g. photons, phonons, pions, He^4, graviton) have many-particle wavefnc, that is symmetric under interchange of any two.

Spin-statistic theorem: Bosons $\Leftrightarrow$ integer spin $(0, \hbar, 2\hbar, 3\hbar, \ldots)$

$\boxed{\text{Fermions}}$ (e.g. electron, proton, quark, He^3)

have many-particle wavefnc. that is antisymmetric under interchange of any two

(Finkelstein & Rubenstein '68)

Spin-statistics thm: Fermions $\Leftrightarrow$ half-integer spin. $(\frac{\hbar}{2}, \frac{3}{2}\hbar, \frac{5}{2}\hbar, \ldots)$

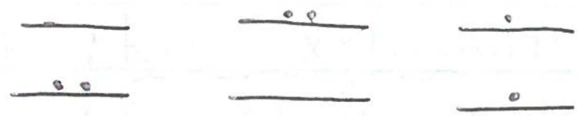
i.e. $|\psi\rangle = \alpha|n_1, n_2; S\rangle + \beta|n_1' n_2'; A\rangle$

not observed in nature!

Ex. of 2 particles in 2 states

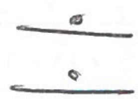
Bosons : 3 states

(BEC)



$|2,0\rangle$ $|0\,2\rangle$ $|1\,1\rangle$

Fermions : 1 state

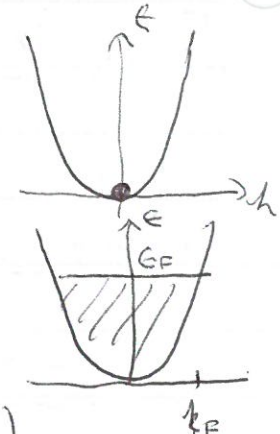


Fermi Sea

classical : 4 states

$|n_1\rangle|n_2\rangle$

BEC (laser), Fermi Sea $\quad \epsilon_k = \frac{\hbar^2 k^2}{2m}$



$|BEC\rangle = |k=0\rangle|k=0\rangle|k=0\rangle \ldots |k=0\rangle$

$|FS\rangle = |k=0\rangle|k=\frac{2\pi}{L}\rangle|k=-\frac{2\pi}{L}\rangle \ldots |k_F\rangle|-k_F\rangle$

$E_{gs} = 0$ (Bosons) , $E = \frac{3}{5}\epsilon_F N$ (Fermions)

SF

Fermi - Liquid

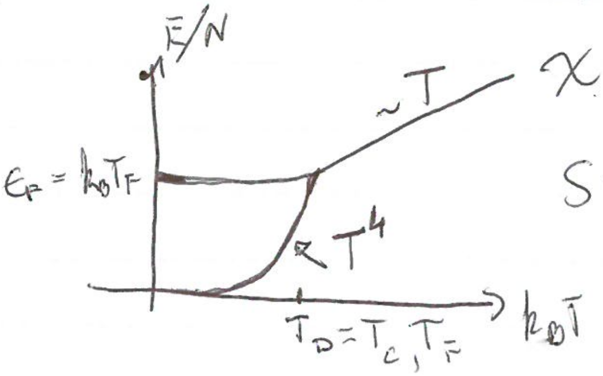
$C_V \sim T^d$

$C_V \sim \frac{T}{\epsilon_F} k_B$

$S = Z$

$\chi \sim \frac{\mu_B}{T_F}$

$S = \frac{1}{2} Z$



Hence: measure location of two identical particles to be a & b, then if:

Bosons: $\Rightarrow |\psi_B\rangle = |ab\rangle + |ba\rangle \equiv |ab;S\rangle$

Fermions: $\Rightarrow |\psi_F\rangle = |ab\rangle - |ba\rangle \equiv |ab;A\rangle$

$\Rightarrow$ Pauli Exclusion Principle (PEP):

Fermions in state:

$$|n_1, n_2; A\rangle = |n_1 n_2\rangle - |n_2 n_1\rangle$$

$$\Rightarrow |n, n; A\rangle = |nn\rangle - |nn\rangle = 0$$

$\Rightarrow$ two fermions cannot be in the same quantum state!

$\Leftrightarrow$ occupation number of any q. state by fermions can only be 0 or 1.

PEP – ~~at the~~ heart of quantum stat. mech., (stability of matter) makes metals what they are, nuclear physics, astrophysics (neutron stars, etc) ...

Note:
- any # of bosons $\rightarrow$ composite boson (like even #'s)
- odd # of fermions $\rightarrow$ composite fermion (like odd #'s)

  ex. $He^3 = \underbrace{2p + 1n + 2e}_{5 \text{ fermions}}$
- even # of fermions $\rightarrow$ composite boson

  ex. $He^4 = \underbrace{2p + 2n + 2e}_{6 \text{ fermions}}$

- Two particle Hilbert space breaks up into bosonic & fermionic $\mathbb{H}$'s.

$$\mathbb{H}_{1\otimes 2} = \mathbb{H}_S \oplus \mathbb{H}_A$$

- Normalization:

$\mathbb{H}^{(2)}_{1\otimes 2}$: $|n_1, n_2; S/A\rangle = N(|n_1 n_2\rangle \pm |n_2 n_1\rangle)$

$$\Rightarrow 1 = \langle n_1 n_2; S/A | n_1 n_2; S/A\rangle = |N|^2 \Big(\underbrace{\langle n_1 n_2 | n_1 n_2\rangle}_{1} + \underbrace{\langle n_2 n_1 | n_2 n_1\rangle}_{1} \pm \underbrace{\langle n_1 n_2 | n_2 n_1\rangle}_{0} \pm \underbrace{\langle n_2 n_1 | n_1 n_2\rangle}_{0} \Big)$$

if $n_1 \neq n_2$ orthogonal

$$\Rightarrow N = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |n_1 n_2 S/A\rangle = \frac{1}{\sqrt{2}}(|n_1 n_2\rangle \pm |n_2 n_1\rangle)$$

$$P(n_1 n_2; S/A) = |\langle n_1 n_2; S/A | \psi_{S/A}\rangle|^2$$

$$1 = \sum_{n_1 \geq n_2}\sum P(n_1 n_2; S/A) = \frac{1}{2}\sum_{n_1}\sum_{n_2} P(n_1 n_2; S/A)$$

e.g. $1 = \int \frac{dx_1 dx_2}{2} P_{S/A}(x_1 x_2)$ ← to compensate double counting $x_1 x_2$ & $x_2 x_1$

Sometimes convenient to absorb $\frac{1}{2}$ into $P(n_1 n_2)$ & $\frac{1}{\sqrt{2}}$ in $\psi_{S/A}(n_1 n_2)$ i.e.

$$\Rightarrow \boxed{\psi_{S/A}(x_1, x_2) = \frac{1}{\sqrt{2}}\langle x_1 x_2; S/A | \psi_S\rangle} = \frac{1}{2}\left[\langle x_1 x_2 | \psi_{S/A}\rangle \pm \langle x_2 x_1 | \psi_{S/A}\rangle\right]$$

since $|x_1 x_2\rangle$ symm/antisymm $= \langle x_1 x_2 | \psi_{S/A}\rangle$

~~2D~~

particles are representations of Braid group.

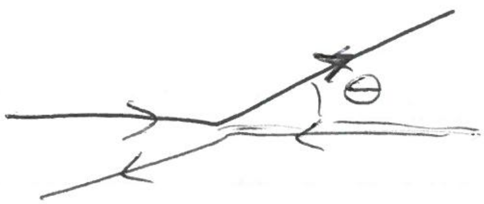
In 2D wind one around the other $= P^2 \neq 1$

1d representation $\Rightarrow$ mult. by $e^{i2\theta}$

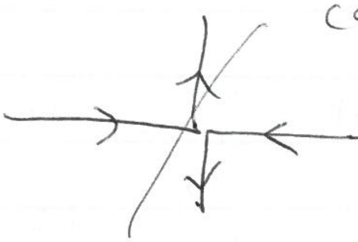
θ depends particle statistics.

In $D>2$ (3D, ...) $B_n = P_n$ with $P^2 = 1$

$\Rightarrow e^{i2\theta} = e^{i2\pi, 0} = 1 \Rightarrow \underline{\theta = 0, \pi}$



$\pm$



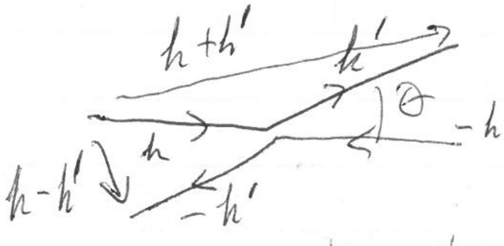
cancel for fermions $(-)$

for $\theta = \frac{\pi}{2}$

$\cos\theta - \cos(\pi-\theta)$

$$f_{B/F}(\vec{k}, \vec{k}') \propto U(|k-k'|) \pm U(|k+k'|)$$

$k^2 + k'^2 - 2kk'\cos\theta \qquad k^2 + k'^2 + 2kk'\cos\theta$



$\cos\theta = -\cos\theta$

$\Rightarrow \theta = \pi/2$

$|k-k'| = |k+k'|$

Identical particles in Q.M.

→ unitary interchange operator P_{12}

$$P_{12}|ab\rangle = |ba\rangle = e^{i\theta}|ab\rangle$$

$$[P_{12}, H] = 0 \iff P_{12}^{+} H P_{12} = H \quad \text{unchanged.}$$

$$\Rightarrow H|ba\rangle = E|ba\rangle, \quad H|ab\rangle = E|ab\rangle$$

$$|\psi_{ab}\rangle = \alpha|ab\rangle + \beta|ba\rangle$$

$$P_{12}^2 = 1 \Rightarrow P_{12}|ab\rangle = e^{i\theta}|ab\rangle$$

P_n group. representations are particles $\pm 1 \Rightarrow$ B's & F's. $\theta = 0, \pi$

→ $P_{ij}|N \text{ identical bosons}\rangle = +|N \text{ i.b.'s}\rangle \to \mathbb{Z}$ spin

$P_{ij}|N \text{ i.f.'s}\rangle = -|N \text{ i.f.'s}\rangle \to \frac{1}{2}\mathbb{Z}$ spin.

⇒ no mixed symmetry particles; exp. fact.
(could in principle if internal q.#'s, e.g. spin.)

→ spin - statistics theorem law of nature.
Relat. QFT

→ props. PEP & SF.

Finkelstein & Rubenstein
Leinaas & Myrheim.

all of chemistry { atomic & molec. physics.

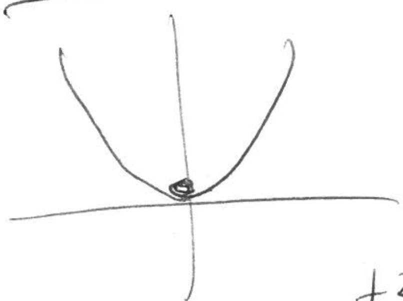
metals

insulators

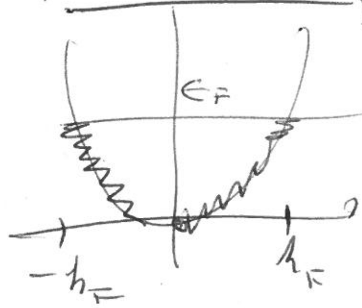
scattering.

20 B_n Braid group. σ_i $\vec{A}\cdot\vec{v} = \frac{q}{\pi}\Phi\,\partial_t\varphi$.

- single particle $H = \frac{p^2}{2m}$
- states $|k\rangle \Rightarrow \langle x|k\rangle = e^{i\vec{k}\cdot\vec{x}}$
- Bosons



Fermions



$$T_d = T_{BEC} = \frac{\hbar^2 n^{2/3}}{2m} \qquad T_{Fermi} = \frac{\hbar^2 n^{2/3}}{2m}$$

$$N_e = \frac{\Phi}{\phi_0} = \frac{BA}{\frac{hc}{e}} =$$

$$\frac{N_e}{A} = \frac{B}{\frac{hc}{e}} = \frac{1}{2\pi l^2}$$

$$\Psi^{(m)} = \prod (z_i - z_j)^m e^{-\frac{1}{4}|z|^2}$$

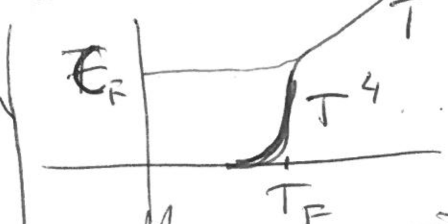
$$\prod_i (z_i - z_0)\, \Psi^{(m)}$$

$e + m\phi_0 = \bar{e}$

$\frac{1}{m}e + \phi_0 = q$

$\theta = \frac{\pi}{m}$

~~Rb87~~
$K\,41, 39$

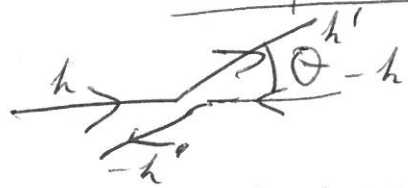


$K40$ $\boxed{F}$ n 21 } 40, p 19, e 19

$\boxed{B}$ K^{39} n 20 } 39, p 19, e 19

$\boxed{B}$ K41 n 22 } 41, p 19, e 19

- scattering $\longrightarrow$



- non interacting

$$\Psi_{\{n\}}(x_1 \ldots x_N) = \psi_{n_1}(x_1)\,\psi_{n_2}(x_2) \ldots ?$$

Slater det / Perm.

- $P(x_1, x_2)$ example.

B.

F.

- Anyons

Symmetries :

$\boxed{3d}$

$$\boxed{s = \frac{\theta}{2\pi}}$$

$(z_i - z_j)^m \leftarrow$ odd.

Only bosons & fermions.

$q = \frac{1}{m}$

$$P_{12}\psi(x_1, x_2) = \psi(x_2, x_1) = e^{i\theta}\psi(x_1, x_2)$$

$$\boxed{\phi = \frac{2\pi}{m}}$$

$\theta = 0, \pi$

why?

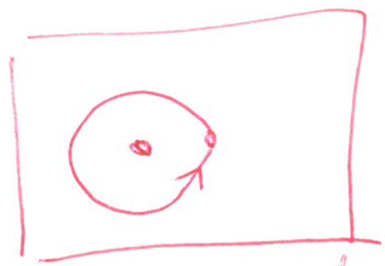
Permutation group of N particles

~~assum~~ in 3d took $P_{12}^2 = \mathbb{1} = e^{i2\theta} = 1$

i.e. double interchange $P_{12}^2\psi = \psi$

↑ identity element

<u>But</u> in 2D



topologically distinct configuration

ψ is not physical on $\langle \psi_1 | \psi_2 \rangle$

$2\theta \neq 0, 2\pi \leftarrow \psi$ is not physical on $\langle \psi_1 | \psi_2 \rangle$

Braid group B_N

(no distinction in 3d).

$e^{i\phi_{12}} \leftarrow$ cannot be defined <u>unambiguously</u>

Look at F P.I.

Classical $e^{iS/\hbar}$ weight, but only infinit.

close traject., but not topological classes.

Topological classes tracked by P_n in $d>2$ & by B_n in $d=2$.

$$\Rightarrow \underline{1} = \langle \psi_{S/A} | \psi_{S/A} \rangle = \iint dx_1 dx_2 \, |\psi_{S/A}(x_1, x_2)|^2$$ 8.19

• Note: <u>Slater determinant / Permanent</u>

$$\psi^{(A)}_{n_1 n_2}(x_1, x_2) = \langle x_1 x_2 | \psi_A \rangle = \frac{1}{\sqrt{2}} \left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2) \right]$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) \end{vmatrix} = \frac{1}{\sqrt{2}} \det\left(\psi_{n_i}(x_j)\right)$$

$$\psi^{(S)}_{n_1 n_2}(x_1, x_2) = \frac{1}{\sqrt{2}} \underset{\substack{\uparrow \\ \text{det but with} \\ \text{no } (-1)\text{'s}}}{\text{Permanent}} \left(\psi_{n_i}(x_j)\right)$$

• ψ_A, ψ_S, $\psi_{\text{Distiguish.}}$ have very diff. $P(x_1, x_2)$

$$P_{S/A}(x_1, x_2) = \frac{1}{2}\left[|\psi_{n_1}(x_1)\psi_{n_2}(x_2) \pm \psi_{n_2}(x_1)\psi_{n_1}(x_2)|^2 \right]$$

$$= \frac{1}{2}\Big[\underbrace{|\psi_{n_1}(x_1)|^2 |\psi_{n_2}(x_2)|^2 + |\psi_{n_2}(x_1)|^2 |\psi_{n_1}(x_2)|^2}_{2P_D(x_1, x_2) \ \leftarrow \text{ if were just classical indisting.}}$$

$$\underbrace{\pm \left(\psi^*_{n_1}(x_1)\psi_{n_2}(x_1)\psi^*_{n_2}(x_2)\psi_{n_1}(x_2) + \psi^*_{n_2}(x_1)\psi_{n_1}(x_1)\psi^*_{n_1}(x_2)\psi_{n_2}(x_2) \right)}_{\text{interference of exchange!}} \Big]$$

→ Analog of interference between amplitudes of diff. way of going through two slits.
i.e. no assignment of labels to particles 1, 2
just like cannot say "went through slit 1 or 2"

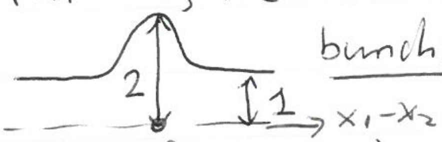
<u>Also vanishes as $\hbar \to 0$</u>

→ Key in ~~a~~ scattering problems; eg. 2 identical fermions → <u>vanishing</u>
... Prob. to scatter to angle $\theta = \pi/2$

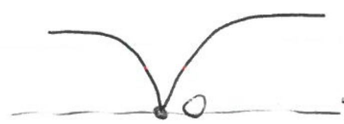
Very different P_S & P_A $\Rightarrow$ distinguish bosons from fermions.

$$P_S(x_1 \to x, x_2 \to x) = 2\left[|\psi_{n_1}(x)|^2 |\psi_{n_2}(x)|^2\right].$$

bunching — twice as big as $P_D(x,x)$.



$$P_A(x_1 \to x, x_2 \to x) = 0 \leftarrow \text{PEP}$$



Also, P_A vanishes for all x_1, x_2 when $n_1 = n_2$

Bosons (Fermions) — obey Bose-Einstein (Fermi-Dirac) states.
e.g. metals, s.c.'s.

Note: If particles are confined by a potential to be far apart e.g. Earth & Moon or even two atoms in a crystal state

the interference term is negligible since exp. small overlap $\Rightarrow$ $\simeq$ no need to symmetrize/antisymm.

- N fermions & N boson states:

$$\mathbb{H}^{(N)} = \mathbb{H}_S^{(N)} \oplus \mathbb{H}_A^{(N)}$$

(no particles "in-between" observed)

fully symmetric on all 2 particle exchanges

fully antisymmetric

only if $n_1 \neq n_2 \neq n_3 \neq \dots \neq n_N$
otherwise $\frac{1}{\sqrt{N!}} \to (C_N^{N_1 N_2 \dots N})^{-1/2} = \sqrt{\frac{N_1! \cdots N_N!}{N!}}$

$$|n_1, n_2, \dots n_N; S/A\rangle = \frac{1}{\sqrt{N!}}\underbrace{\left[|n_1 n_2 n_3 \dots n_N\rangle \pm |n_2 n_1 n_3 n_4 \dots n_N\rangle \pm \dots\right]}_{N!\ \text{terms}}$$

$$\boxed{\psi^{S/A}_{\{n_i\}}(\{x_i\}) = \frac{1}{\sqrt{N!}} \text{Perm}/\det\left(\psi_{n_i}(x_j)\right)}$$

Note: det vanishes if $n_i = n_j$ for any i, j.

... N_1 N_2 N_3 ... in state 1, 2, 3 ...