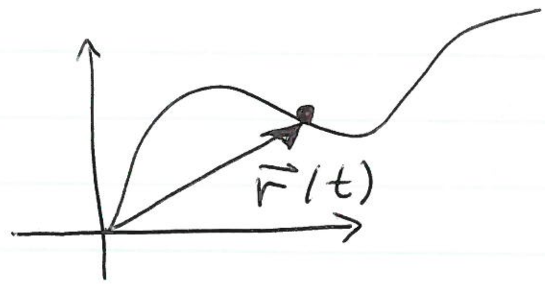
Introduction I.

Lecture 1: Review of Classical Mechanics

- Newton's Law (1660):

$$m\ddot{\vec{r}} = \vec{F}$$



formulated in terms of $\vec{r}(t)$ w/ initial cond. $\vec{r}(0)$, $\dot{\vec{r}}(0)$.

$\Rightarrow$ All of classical mechanics!

- Three equivalent formulations

$\rightarrow$ Lagrangian
$\rightarrow$ Hamiltonian
$\rightarrow$ Hamilton-Jacobi

} based on Least Action Principle.

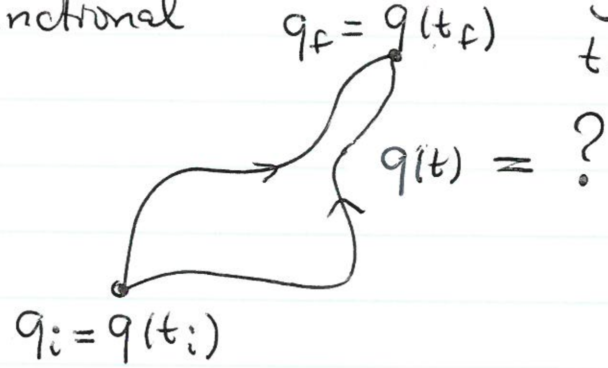
Advantages:
- canonical coord. invariant
- single scalar fnc.
- simpler to incorporate constraints (cf. Lagrange multiplier vs. solving constraint)
- symmetry & conservation laws.

A. Lagrangian formalism

- Dynamical system defined by a Lagrangian $L(\underbrace{q, \dot{q}}, t)$

  stands for many "positions", "velocities".

- Action functional $S[q(t)] = \int_{t_i}^{t_f} dt\, L(q(t), \dot{q}(t), t)$



$q_{cl}(t)$ is that path s.t.

- $$\left.\frac{\delta S[q(t)]}{\delta q(t)}\right|_{q_{cl}(t)} = 0$$

  need calculus of variations to compute variational derivative.

Calculus of variations Interlude

- $S[q(t)]$ – functional i.e. given fnc $q(t)$ returns a number

$$\frac{\delta S}{\delta q(t)} = ?$$

- Look at sums:

$$S(q_1, q_2, \ldots, q_N) \equiv S[\{q_n\}] = \sum_n f(q_n)$$

$$\frac{\partial S}{\partial q_m} = \frac{\partial}{\partial q_m} \sum_n f(q_n) = \sum_n f'(q_n) \underbrace{\frac{\partial q_n}{\partial q_m}}_{\delta_{n,m}}$$

$$\Rightarrow \frac{\partial S}{\partial q_m} = f'(q_m)$$

now take $t \to n\Delta t \equiv t_n \Rightarrow q(t) \to q_n \equiv q(t_n)$

$\Rightarrow$ functional $\Leftrightarrow$ function of $N \to \infty$ variable, one for each value of t_n, i.e. $q(t_1), q(t_2), \ldots$

- In practice: $S[q(t)] = \int dt' \, V(q(t'))$

$$\Rightarrow \frac{\delta S}{\delta q(t)} = \int dt' \, V'(q(t')) \underbrace{\frac{\delta q(t')}{\delta q(t)}}_{\delta(t'-t)} = \frac{\partial V}{\partial q}$$

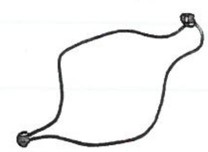
Ex. $S[q(t)] = \frac{1}{2} \int dt' \, (\partial_{t'} q)^2$

$$\frac{\delta S}{\delta q(t)} = \int dt' \, \partial_{t'} q \, \partial_{t'} \frac{\delta q(t')}{\delta q(t)} = \int dt' \, \partial_{t'} q \, \partial_{t'} \delta(t'-t)$$

$$= -\partial_t^2 q + \text{boundary terms.}$$

... back to Least Action principle:

- $\left.\dfrac{\delta S[q(t)]}{\delta q(t)}\right|_{q_{cl}(t)} = 0$, $S = \displaystyle\int_{t_i}^{t_f} dt' \, L(q,\dot{q})$



- $L = ?$ $\qquad \underline{\delta q(t_f) = \delta q(t_i) = 0}$

"Simplest" systems where $T(\dot{q})$, $V(q)$

$\Rightarrow L = T - V$ reproduces Newton's Eqn's.

$$\frac{\delta S[q(t)]}{\delta q(t)} = \int_{t_i}^{t_f} dt' \frac{\delta L}{\delta q}[\dot{q}, q] = 0$$

$$0 = \int_{t_i}^{t_f} dt' \left[\frac{\partial L}{\partial \dot{q}'} \, \partial_{t'}\underbrace{\left(\frac{\delta q'}{\delta q}\right)}_{\delta(t'-t)} + \frac{\partial L}{\partial q'} \underbrace{\frac{\partial q'}{\partial q}}_{\delta(t'-t)} \right]$$

☆ $\boxed{\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0}$ ← Lagrange's Eqn.

identify: $p = \dfrac{\partial L}{\partial \dot{q}}$, $F = \dfrac{\partial L}{\partial q}$

$\Rightarrow \dot{p} = F$ ✓

generalized "momentum" & "force".
(can be angular moment & torque, constraint forces, etc...)

If $L = T - V$
$= \frac{1}{2} m \dot{q}^2 - V(q)$

$\Rightarrow p = m\dot{q}$, $F = -\dfrac{\partial V}{\partial q}$ ✓

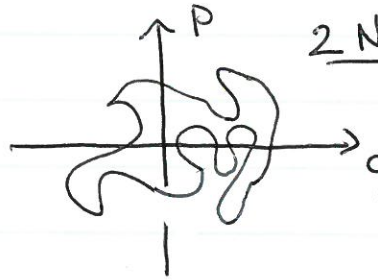
easy & obvious generalization to N variables $q^2 \to |\vec{q}|^2$, etc.

B. Hamiltonian formalism

- Dynamical system defined by a Hamiltonian $H(\underbrace{q, p}, t)$

canonical "position" & momentum, live in phase space.

phase space



2Nd-dimens.

$H(q,p)$ – Legendre transform of $L(q,\dot{q})$

↳ change of vars from $\dot{q} \to p = \dfrac{\partial L}{\partial \dot{q}}$

s.t. $\dfrac{\partial H}{\partial p} = \dot{q}$

$\Rightarrow$ $\underline{H(q,p) = p\dot{q} - L(q,\dot{q})}$

Note: $\dfrac{\partial H}{\partial p} = \dot{q} + \cancel{p \dfrac{\partial \dot{q}}{\partial p}} - \cancel{\dfrac{\partial L}{\partial \dot{q}} \dfrac{\partial \dot{q}}{\partial p}}$ ✓

$$\left.\frac{\partial H}{\partial q}\right|_p = \cancel{p \frac{\partial \dot{q}}{\partial q}} - \frac{\partial L}{\partial q} - \cancel{\frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q}} = -\frac{\partial L}{\partial q} = -\dot{p}$$

$\Rightarrow$ Hamilton's Eqns:

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

$2 \cdot N \cdot d$ 1^{st} order ode.
(vs. Nd 2^{nd} order ode in Lagr. form)

- For $\dot{q}$ independent "potentials"

$$L = \frac{1}{2} m \dot{q}^2 - V$$

$$\Rightarrow H = p\dot{q} - L = m\dot{q}^2 - \frac{1}{2} m \dot{q}^2 + V$$

$$H = T + V$$

- Phase-space evolution: $\dot{q} = \frac{\partial H}{\partial p}$

$\mathcal{O}(p, q, t)$ evolution? $\quad \dot{p} = -\frac{\partial H}{\partial q}$

$$\frac{d}{dt}\mathcal{O} = \partial_t \mathcal{O} + \frac{\partial \mathcal{O}}{\partial p} \underbrace{\frac{\partial p}{\partial t}}_{-\frac{\partial H}{\partial q}} + \frac{\partial \mathcal{O}}{\partial q} \underbrace{\frac{\partial q}{\partial t}}_{\frac{\partial H}{\partial p}}$$

$$\Rightarrow \dot{\mathcal{O}} = \partial_t \mathcal{O} + \{\mathcal{O}, H\}_{p,q}$$

where $\underset{\text{Poisson bracket}}{\{A, B\}_{p,q}} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$

Note: Hamilton's Eqns $\Leftrightarrow$ $\dot{q} = \{q, H\}$

$\{q, q\} = \{p, p\} = 0$; $\{q, p\} = 1$ $\quad \dot{p} = \{p, H\}$

$\{q,p\} \to$ later $\{q, p\} \to \frac{1}{i\hbar}[q, p]$

Poisson bracket props:

$\{A, B\} = -\{B, A\}$, antisymm.

$\{A, B + C\} = \{A, B\} + \{A, C\}$, linearity

$\{A, BC\} = \{A, B\}C + B\{A, C\}$, distributive.

$\{A, B\}_{\bar{q},\bar{p}} = \{A, B\}_{q,p} \{q, p\}_{\bar{q},\bar{p}}$, chain rule

- Symmetry & cyclic coordinates.

→ if q is missing in L or H

$\Rightarrow \dot{p}_q = 0 \rightarrow$ constant of motion.

→ $\dot{\mathcal{O}} = \{\mathcal{O}, H\} = 0 \rightarrow \mathcal{O}$ conserved by H evolution.

- Canonical transformations

→ $q \rightarrow Q(q,p)$, $p \rightarrow P(q,p)$

Q, P are canonical coord. if Hamilton's Eqns preserve their form

<u>require</u>: $\{Q,Q\} = \{P,P\} = 0$, $\{Q,P\} = 1$.

note: $\{A,B\}_{q,p} = \{A,B\}_{Q,P}$ for canon. vars.

→ generator of canonical transformations.

pick Q, P to be "evolving" from q, p via new "Hamiltonian" $\mathcal{G}$: $\partial_\varepsilon q = \frac{\partial \mathcal{G}}{\partial p}$, $\partial_\varepsilon p = -\frac{\partial \mathcal{G}}{\partial q}$

$$Q = q + \varepsilon \frac{\partial \mathcal{G}}{\partial p}, \quad P = p - \varepsilon \frac{\partial \mathcal{G}}{\partial q}$$

guaranteed to be canonical

$\mathcal{G}$ - generator of canonical transformation

$\{Q,Q\} = \{P,P\} = 0$, $\{Q,P\} = 1$ (check)

Symmetry:

$$\delta \mathcal{H} = \varepsilon \{H, g\} = 0$$

$\Rightarrow$ g is a constant of motion $\Rightarrow$ conserved.

Ex. $g = p \Rightarrow \delta x = \varepsilon \frac{\partial p}{\partial p} = \varepsilon$

$\delta p = -\varepsilon \frac{\partial p}{\partial x} = 0$

generates translations

$\{H, p\} = 0 \Rightarrow \dot{p} = 0 \Rightarrow p = \text{const.}$

C. Hamilton-Jacobi formalism

Canonical transfomation $q \to Q$, $p \to P$

$$\Rightarrow \quad \dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

K - Hamiltonian in new coordinates.

... relation to H ?

require: $\delta \int_{t_i}^{t_f} [P\dot{Q} - K(Q,P,t)]\,dt = 0$

& $\delta \int_{t_i}^{t_f} [p\dot{q} - H[q,p]]\,dt = 0$

$$\Rightarrow \quad \lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{dF}{dt}$$

($\lambda \neq 1$ scale transformations, simple $\Rightarrow$ take $\lambda = 1$)

various types of F fncs: (i) $F_1(q, Q, t)$, (ii) $F_2(q, P, t)$, (iii) $F_3(p, Q, t)$, (iv) $F_4(p, P, t)$ } Legendre transforms

(i) $p = \frac{\partial F_1}{\partial q}$, $P = -\frac{\partial F_1}{\partial Q}$

(ii) $p = \frac{\partial F_2}{\partial q}$, $Q = \frac{\partial F_2}{\partial P}$

...

$F_2(q, P) = F(q, Q) + QP$

& $\boxed{K = H + \frac{\partial F}{\partial t}}$

special choice of Q, P constant, guaranteed by $K = 0$ $\Rightarrow$

Hamilton's principle fnc.

$$\underline{H + \frac{\partial F}{\partial t} = 0}$$

canonical transf. to initial conditions.

take $F_i = F_2(q, P)$ with $p = \frac{\partial F_2}{\partial q}$

$\Rightarrow$ Hamilton-Jacobi Eqn: $\quad F_2 \equiv S(q, P)$

$$H\left(q, \frac{\partial S}{\partial q}; t\right) + \frac{\partial S}{\partial t} = 0$$

Note: $$\frac{dS}{dt} = \underbrace{\frac{\partial S}{\partial q}}_{p} \dot{q} + \frac{\partial S}{\partial P} \cancel{\dot{P}}_{0} + \underbrace{\frac{\partial S}{\partial t}}_{-H}$$

$$\Rightarrow \frac{dS}{dt} = p\dot{q} - H = L \; !$$

$\Rightarrow$ S - action. (up to const).

Hamilton's principle fn executes canonical transf. $q, p \rightarrow Q = q(0), P = p(0)$.

Note: for $H = \frac{p^2}{2m} + V(q)$

$$\Rightarrow \boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = -\frac{\partial S}{\partial t}}$$

Find $S(q, P, t)$
$\Rightarrow$ solve problem

- Geometric optics limit of a wave eqn.

Consider case of conserved energy

$$\Rightarrow S(q, P, t) = W(q, P) - Et$$

Plane wave: $\psi \sim e^{i\vec{k}\cdot\vec{r} - i\omega t}$

more generally $\psi \sim e^{iS/\hbar}$

with $S = W - Et$, $W(q,P)$ smoothly varying phase front

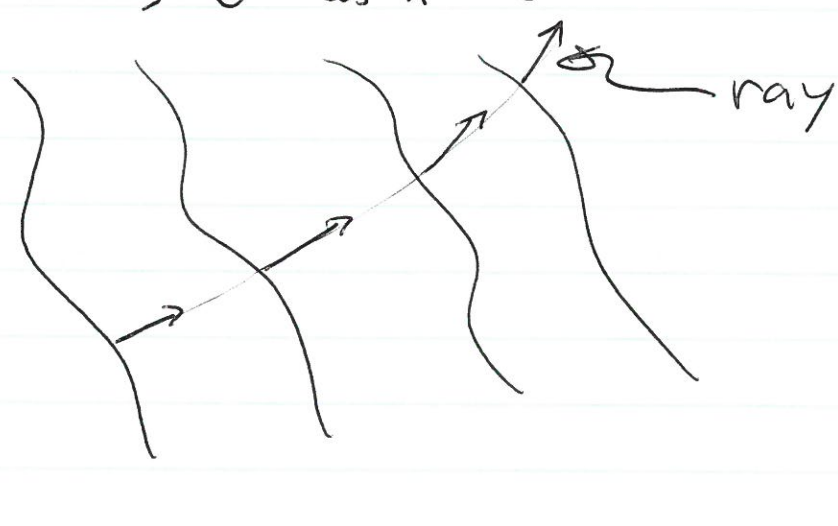
$$W \simeq \frac{P\cdot q}{\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\psi + V(q)\psi = i\hbar\,\partial_t\psi$$

$$\psi \sim e^{iS/\hbar}$$

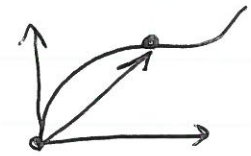
$$\Rightarrow -\frac{\hbar^2}{2m}\frac{i}{\hbar}\frac{\partial^2}{\partial q^2}S + \frac{\hbar^2}{2m}\frac{1}{\hbar^2}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = -\partial_t S$$

$$\underbrace{-i\hbar\frac{1}{2m}\frac{\partial^2}{\partial q^2}S}_{\substack{\text{wavie (quantum)}\\ \text{corrections}\\ \to 0 \text{ as } \hbar\to 0}} + \underbrace{\boxed{\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = -\partial_t S}}_{\substack{\text{Hamilton-}\\ \text{Jacobi eqn}}}$$



Summary:

- Newton's Eqns for $\vec{r}(t)$:



$$m\ddot{\vec{r}} = \vec{F}(\vec{r}, t)$$

Least Action principle.

- Lagrangian formulation: $L(q, \dot{q}; t)$

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Hamiltonian formulation: $H(q, p; t) = p\dot{q} - L$

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\dot{q} = \{q, H\}, \quad \dot{p} = \{p, H\}$$

More generally $\dot{O} = \{O, H\}$

- Hamilton-Jacobi theory:

Find Hamilton's principle fnc $S(\overset{q_f}{q}, \overset{p(0)}{P}; t)$
for canonical transf. to Q, P init. cond.
with $\underset{p_f}{p} = \dfrac{\partial S}{\partial q}, \quad \underset{q(0)}{Q} = \dfrac{\partial S}{\partial P} \Big\} \Rightarrow q = q(Q, P; t)$

with S satisfying: $\left(K' = 0 \Rightarrow H + \dfrac{\partial S}{\partial t} = 0\right)$

$$H(q, \partial_q S; t) \doteq -\frac{\partial S}{\partial t}$$

$$\frac{1}{2m}(\partial_q S)^2 + V(q) = -\partial_t S \iff \text{ray limit of}$$

Schrödinger's Eqn. $\frac{\hbar^2}{2m}\partial_q^2\psi + V\psi = i\hbar\partial_t\psi$.

Lecture 2:
Conflicts with Experiments

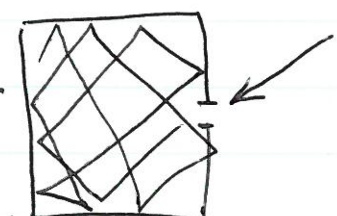
Early 1900's experiments on light & matter at small scales ~ Å

⇒ crisis in classical physics

⇒ new concepts:
- particle nature of light → photons
- wave nature of matter
- quantization of physical quantities.

1 Black-body radiation (Planck 1900):

BB does not emit, only absobs.

T 

Experimentally well-studied spectral density:

$\rho(\nu, T)$ – energy density

Classical equipartition of energy among $2 \times 2 = 4$ transverse E&M modes
$\vec{E}, \vec{B}$

⇒ Rayleigh-Jean Law ⇒ $\rho(\nu, T) \simeq \frac{8\pi\nu^2}{c^3} k_B T$

$$dE_k = \rho(\nu,T)\,d\nu = \frac{d^3k}{(2\pi)^3}\, 2\cdot 2\cdot k_BT/2$$

$$E = 2\sum_k k_BT = V\int_0^\infty d\nu\,\rho(\nu,T) \propto \int_0^\infty d\nu\,\nu^2$$

$\rightarrow \infty$ catastrophy!

Planck (1900) : emission/absorption in discrete quanta of energy: $\Delta E = h\nu$

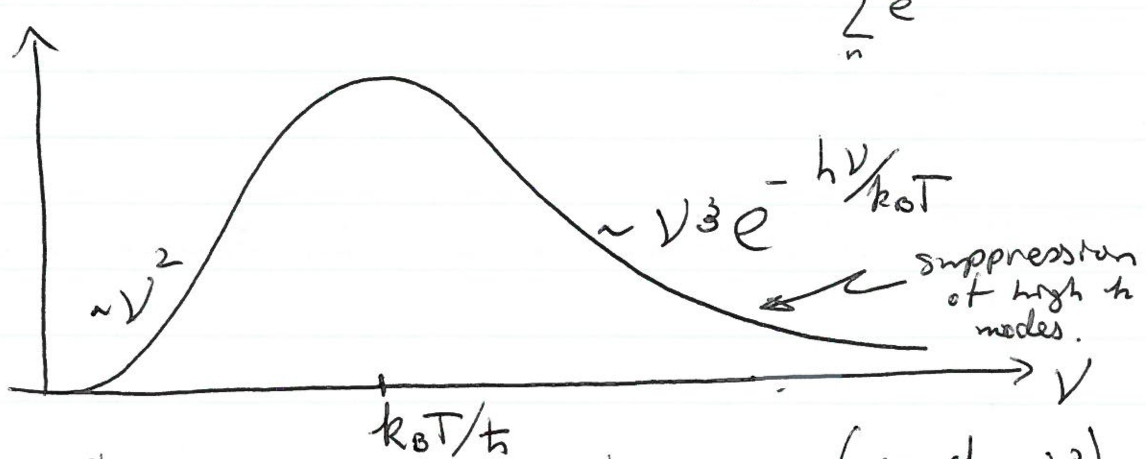
$$E = 2\sum_k h\nu_k \langle n_k\rangle = 2\int \frac{d^3k}{(2\pi)^3}\frac{h\nu_k}{e^{\frac{h\nu_k}{k_BT}}-1}\,, \quad \nu_k = ck$$

$$E/Vol \equiv \int_0^\infty d\nu\,\rho_{\text{Planck}}(\nu)$$

$$\Rightarrow \rho_{\text{Planck}}(\nu) = \frac{8\pi\nu^2}{c^3}\left(\frac{h\nu}{e^{h\nu/k_BT}-1}\right) \sim \begin{cases}\rho_{RJ}\,, & h\nu \ll k_BT \\ \nu^3 e^{-h\nu/k_BT}\,, & h\nu \gg k_BT\end{cases}$$

$\left(\dfrac{h\nu}{e^{h\nu/k_BT}-1}\right.$ not k_BT per E&M mode$\Big)$

$$h - 6.63\times 10^{-34}\ \text{J-s}. \qquad \langle n\rangle = \frac{\sum_n n e^{-nh\nu/k_BT}}{\sum_n e^{-nh\nu/k_BT}}$$



$$E(T) = \int_0^\infty \rho_P(\nu)\,d\nu = \sigma_{SB}T^4 = k_BT\left(8\pi\left(\frac{k_BT}{hc}\right)^3\right)$$

finite!

cf Dulong - Petit law of $C_v = \frac{\partial E}{\partial T} << N k_B$ of solids at low T.

$\Rightarrow$ quantal nature of radiation

$\Delta E = h\nu = \hbar\omega \not\Rightarrow$ equipartition.

2. Photoelectric effect

(Discovered by Hertz 1887, explained by Einstein 1905)

electron emission from metal surface irradiated by light.

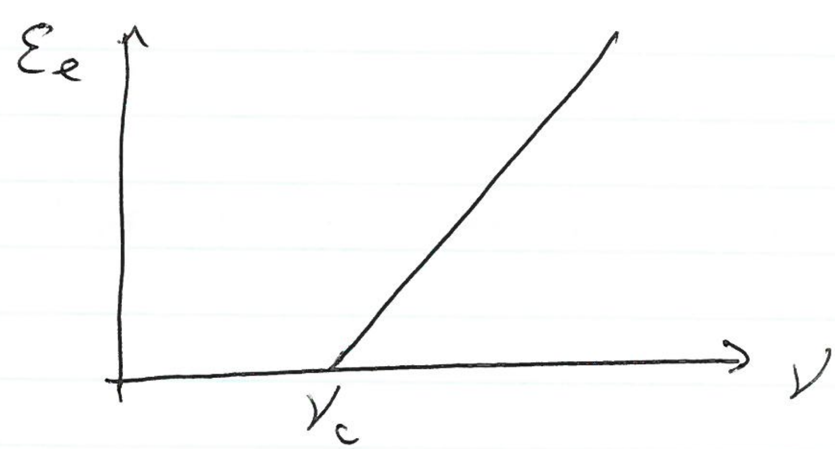
- frequency threshold for emission

  i.e. $\nu < \nu_c$ no e's even at high I.
  ($i_e \propto I$, but $\mathcal{E}_e \not\propto I$)

  conflict since classically
  $\mathcal{E}_e \propto \mathcal{E}_\nu \propto I$

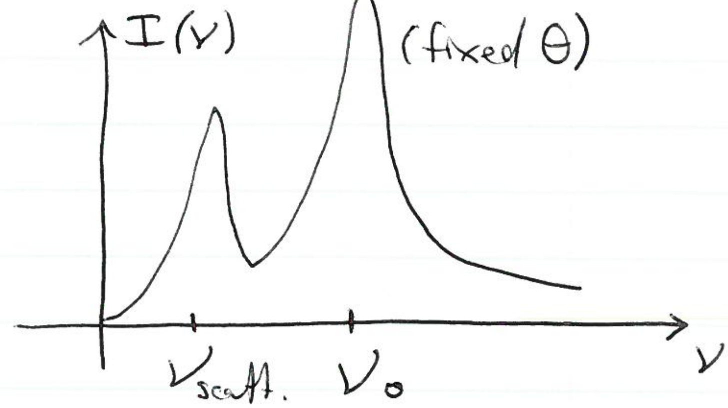
- Einstein $\rightarrow$ photons of energy $h\nu$
  $h\nu_c$ - work fnc of metal

$$\mathcal{E}_e = \frac{1}{2} m v_e^2 = h\nu - W$$



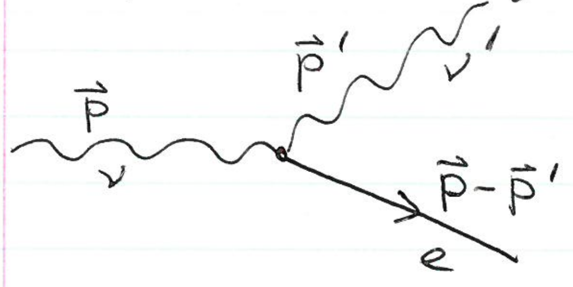
3. Compton effect

- X-rays through metal scattered through $\Theta(\nu)$.



- in conflict with classical idea of absorption of radiation & dipole continuous reradiation $P(\theta) \sim 1 + \cos^2\theta$

- Compton: light – beam of photons.



$E_\nu = h\nu$

$p_\nu = h\nu/c = \hbar k$

(relativity)

$$E_e + h\nu' = h\nu + mc^2$$

$$E_e^2 = h^2(\nu - \nu')^2 + m^2c^4 + 2h(\nu-\nu')mc^2$$

$$c^2(\vec{p}-\vec{p}\,')^2 + m^2\cancel{c^4} = \cancel{h^2\nu^2} + \cancel{h^2\nu'^2} - 2h^2\nu\nu' + 2h(\nu-\nu')mc^2 \cancel{+m^2c^4}$$

$$\cancel{c^2p^2} + \cancel{c^2p'^2} - 2c^2pp'\cos\theta + m^2c^4 = \ldots$$

$$\nu\nu'(1-\cos\theta) = \frac{mc^2}{h}(\nu - \nu')$$

$$\Rightarrow \boxed{\frac{h}{mc}(1-\cos\theta) = \lambda' - \lambda}\,, \quad \frac{h}{mc} \simeq 2.4 \times 10^{-12}\,m \text{ – e's Compton wavelength}$$

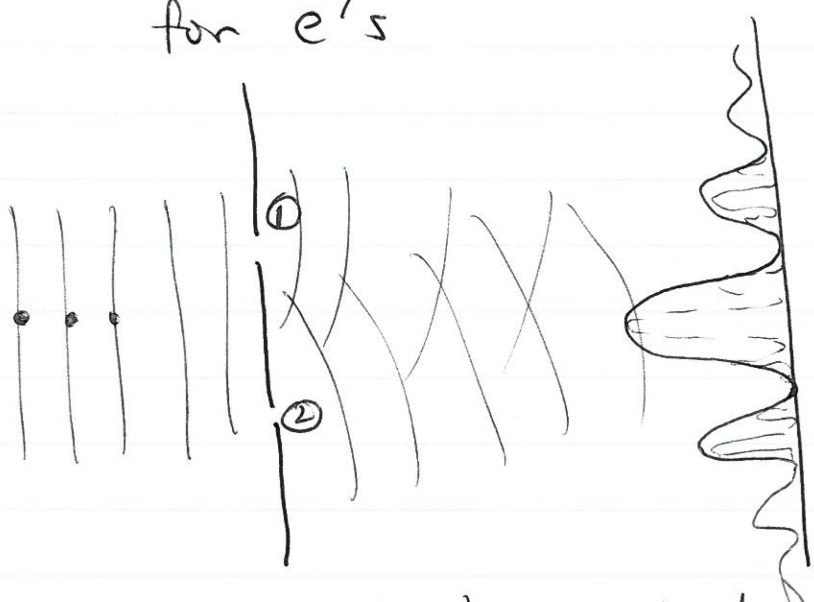
4. Electron diffraction

de Broglie proposed 1923

Davisson - Germer verified through e scattering by a crystal surface e - Bragg scattering.

$\Leftrightarrow$ Young's - like double slit exp. for e's

$I_{12} \neq I_1 + I_2$



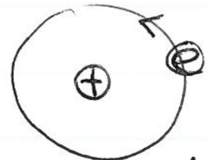
$\Rightarrow$ e's exhibit particle-wave duality

$$\lambda = \frac{h}{p}$$ (by analogy with photons)

The key idea! interferes with itself (wave) e (like E&M wave/photon) "goes through" both slits to create interference pattern, unless look precisely through to determine which one. → no trajectory independent of apparatus.

5 Atomic spectra – discrete.

- Rutherford model of atom



(not Thomson "pudding" model)

based on α particle large angle scattering on thin foil

... but in conflict with

(a) atomic spectra <u>discrete</u>

$$\frac{1}{\lambda_{n_1 n_2}} = \text{Const.}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad n_{1,2} \in \mathbb{Z}$$

(b) stability of atom (matter), absence of radiation of γ due to accelerating e's.

- Niels Bohr (1913) quantization/model:

$\rightarrow L = n\hbar \Rightarrow E_n$

$\rightarrow \Delta h\nu = E_{n_2} - E_{n_1}$

- Sommerfeld, Wilson

$$nh = \int_{\text{closed path}} \vec{p}\cdot d\vec{r}$$

- $n\hbar = L_n$ observed by Stern-Gerlach (1922)

$$E_n = \frac{1}{2} m v_n^2 - \frac{Ze^2}{r_n}$$

$$\frac{m v_n^2}{r_n} = \frac{Ze^2}{r_n^2}$$

$$\Rightarrow r_n = \frac{n^2\hbar^2}{mZe^2} \propto n^2 \lambda/\alpha$$

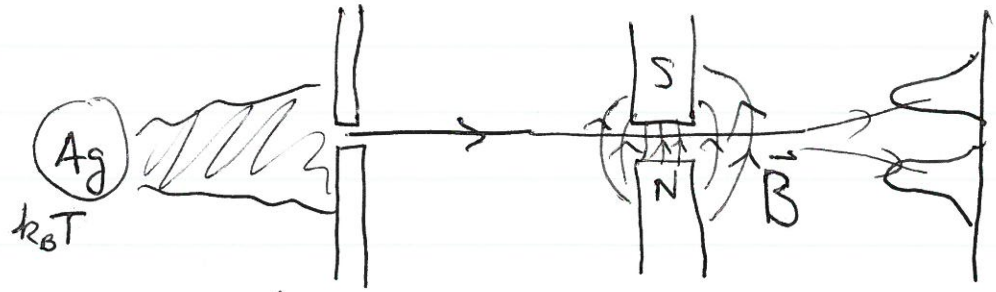
$$\lambda = \frac{h}{mc}, \quad \alpha = \frac{e^2}{\hbar c}$$

$$v_n = \frac{Ze^2}{n\hbar}$$

$$\Rightarrow E_n = -\frac{e^4 Z^2 m}{2\hbar^2}\frac{1}{n^2} = -\frac{mc^2}{2}\frac{(Z\alpha)^2}{n^2}$$

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}\cdots$$

Stern-Gerlach Experiment (1921)



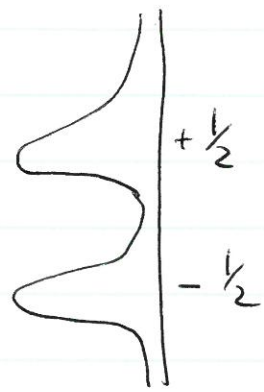
one $5s_1$ valence electron $\Rightarrow$ $s = 1/2$

$$\vec{\mu} \propto \vec{J} = \vec{S} \Rightarrow E = -\vec{\mu} \cdot \vec{B}$$

$$F_z = -\partial_z E = \mu_z \partial_z B_z$$

random thermal distribution

classically continuously distributed $-\frac{1}{2} < \mu_z < \frac{1}{2}$



Q.M. only two states $\mu_z = \pm 1/2$ nothing in between

key experiment to demonstrate nature of q.m. state $|\psi\rangle$ is a vector in Hilbert space; we'll return to this.

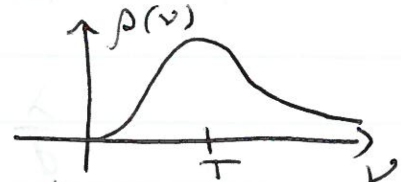
- quantization of observables
- wave nature of matter - interference

Key experiments:

photons

1. Black-body radiation (Planck 1900)

$$E = n\mathcal{E} = h\nu n \,, \quad n \in \mathbb{Z}$$



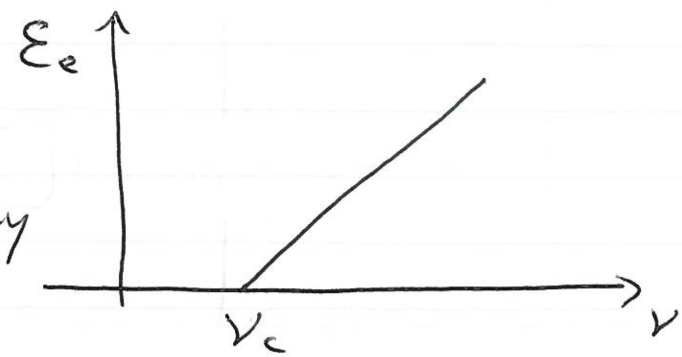
$\Rightarrow$ light quanta $\rightarrow$ photons

2. Photoelectric effect (Hertz 1887, Einstein 1905)

$\mathcal{E}_e \propto \nu - \nu_c$

$\neq$ Intensity

$(i_e \propto I)$

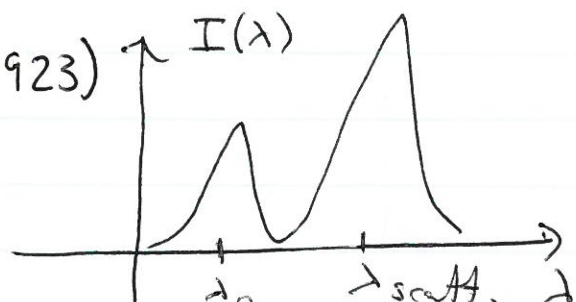


$\Rightarrow$ energy transfered to e in discrete quanta $h\nu$

$$\mathcal{E}_e = \frac{1}{2} m v_e^2 = h\nu - W_{metal}$$

$\Longrightarrow$ photons

3. Compton effect (1923)



X-ray scattering in metal

$\Rightarrow$ photon particle with $E = h\nu$

$p = \dfrac{h\nu}{c}$

$$\Rightarrow \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

4. e - diffraction (de Broglie 1923, Davisson - Germer)

$\Rightarrow$ electron wave with "intensity" $\propto$ probability of finding e at position r.

matter wave

$\lambda = \frac{h}{p}$ $\Longleftrightarrow$ cf Compton for photon

$p = \hbar k = \frac{h\nu}{c} = \frac{h}{\lambda}$

$I_{12} \neq I_1 + I_2$

Wave - particle duality

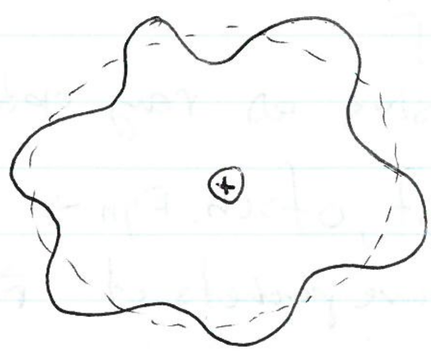
Intensity $\propto$ probability of observing

- double slit exp. (Bragg scatt.)
- polarizer. $I = I_0 \cos^2\theta$.

5. Atomic spectra - discrete (Niels Bohr 1913, Stern-Gerlach 1922)

$\rightarrow L_n = m v_n r_n = n\hbar$

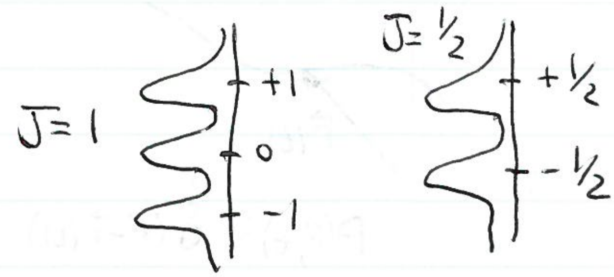
$\rightarrow E_n$ $\Longleftrightarrow$ $\frac{2\pi r}{\lambda} = n \in \mathbb{Z}$ Wilson 1915, Sommerfeld 1913.



with $\lambda = \frac{h}{p}$

confirmed by Stern - Gerlach (1921)

quantization of $\vec{J}$



Lecture 3:
Key ideas in Quantum mechanics

- wave nature of matter with

state of system $\rightarrow \Psi(\vec{r}, t)$ — amplitude (complex in general)

Born interprett. $\rightarrow |\Psi(\vec{r}, t)|^2 = \text{Intensity} \propto \text{Prob}(\vec{r}, t)$ density.

$\rightarrow$ modified by measurement

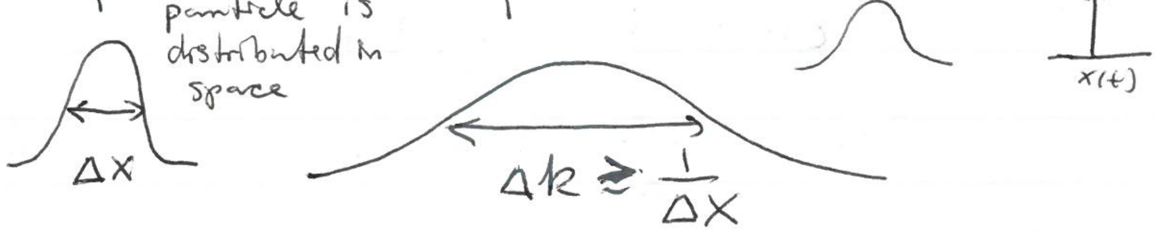
dispersion $\rightarrow E_p = \frac{p^2}{2m}, \quad p = \hbar k = \frac{h}{\lambda}$

$$\Rightarrow \Psi(x) = \int dk\, \tilde{\Psi}_k\, e^{ikx}$$

"particle" is distributed in space

$P(x,t) \neq \delta(x - x(t))$

all wave like phenomena follow!



$$\Delta x\, \Delta k \gtrsim 1 \Leftrightarrow \Delta x\, \Delta p \gtrsim \hbar$$

$\Rightarrow$ Heisenberg (1927) uncertainty principle & Bohr's complementery principle for mutually exclusive (canonically conjugate) observables.

$$\Delta x\, \Delta p \gtrsim \hbar, \quad \Delta\varphi\, \Delta L \gtrsim \hbar, \quad \Delta t\, \Delta E \gtrsim \hbar$$

$\rightarrow$ interference & superposition

$$\Psi_{12} = \Psi_1 + \Psi_2 \Rightarrow |\Psi_{12}|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2\mathrm{Re}\,\Psi_1^* \Psi_2$$

$\Rightarrow P_{12} \gtrless P_1 + P_2$

(double slit exp)

E&M field

$$\nabla\times H = \frac{1}{c}\partial_t E, \quad \nabla\times E = -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \nabla\cdot E = \nabla\cdot B = 0$$

$$F = E + iH, \quad \bar{F} = E - iH.$$

$$\Rightarrow \nabla\times F = \frac{i}{c}\partial_t F, \quad \nabla\times\bar{F} = -\frac{i}{c}\partial_t \bar{F}$$

$$\nabla\cdot F = \nabla\cdot\bar{F} = 0$$

$$\nabla\times F = -i\,\partial_\beta(-i\,\epsilon_{\beta\alpha\gamma})F_\gamma$$

$$\Rightarrow \quad p_\beta = -i\hbar\partial_\beta$$

$$\Rightarrow (p\cdot S)F = i\hbar\partial_t F$$

where $S^\beta_{\alpha\gamma} = -i\,\epsilon_{\beta\alpha\gamma}$ (3x3 matrix).

spin 1 operator.

(3.1a)

All these experiments/"paradoxes" can be understood with one leap

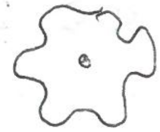
particle is a wave with:

- amplitude: $\psi(r,t)$ ← <u>field</u>
- dispersion: $\omega_k = \frac{\hbar k^2}{2m}$
- intensity $|\psi|^2$ = prob. density of finding particle at point (r,t)

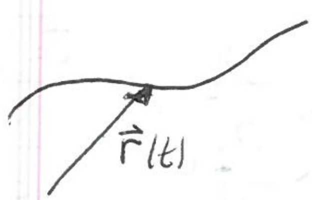
Everything follows:

→ interference/diffraction $\psi_{12} = \psi_1 + \psi_2 \Rightarrow P_{12} \neq P_1 + P_2$

→ spectrum quantization like E&M modes in a cavity, notes on a guitar string

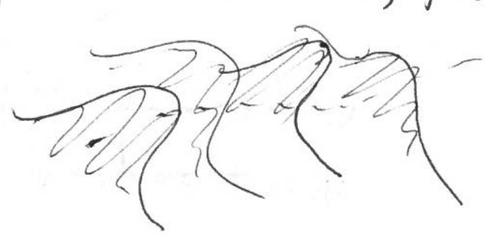


→ classical physics as ray optics, $\lambda/d \to 0$ limit, Sch. Eqn → H-J Eqn.
particles are wave packets at $\vec{r}(t)$, $\vec{p}(t) = -i\hbar\vec{\nabla}\psi$



$P(\vec{r},t) = \delta^3(\vec{r} - \vec{r}(t))$

vs.



$P(\vec{r},t)$ arbitrary fn of $\vec{r}, t$.

• Wave equation for matter

[Schrödinger, Ann. Physik **79**, 361, 489 (1926)
81, 109 (1926)]

"Photon wave" $\vec{E}, (\vec{B})$ obeys Maxwell eqns

i.e. $\partial_t^2 \vec{E} - c^2 \nabla^2 \vec{E} = 0$

massless limit of $E_p = \sqrt{p^2c^2 + m^2c^4}$

$E \sim e^{i\vec{k}\cdot\vec{r} - i\omega t} \iff \omega_k^2 = c^2 k^2 \iff E_p = cp$

$(\hbar\omega_k) = c(\hbar k)$

wave eqn for matter wave $\Psi(\vec{r}, t)$?

nonrel. limit of $E_p = mc^2\left(1 + \frac{p^2c^2}{m^2c^4}\right)^{1/2} \simeq$

look at dispersion $E_p = \frac{p^2}{2m}$ (non relativistic)

$\iff \hbar\omega_k = \frac{\hbar^2 k^2}{2m}$

$\Rightarrow$ Schrödinger's Eqn for $\Psi(\vec{r}, t) \sim e^{i\vec{k}\cdot\vec{r} - i\omega t}$

free particle

$$\boxed{i\hbar\,\partial_t \Psi = -\frac{\hbar^2 \nabla^2}{2m} \Psi}$$

in a potential:

$$\boxed{i\hbar\,\partial_t \Psi = \underbrace{\left(-\frac{\hbar^2 \nabla^2}{2m} + V(r)\right)}_{\frac{\hat{p}^2}{2m} + V(\hat{r}) \;=\; \hat{H}(\hat{p}, \hat{r})} \Psi}$$

$$\underbrace{i\hbar\,\partial_t \Psi = \hat{H}}_{\text{must be linear}} \Psi,$$

must be linear, Hermitian $H^\dagger = H$

$\Psi \sim e^{iS/\hbar} \Rightarrow -\partial_t S = H$ H-J eqn

Schrödinger's Eqn plausability

(ultimate justification: experiments)

- correct classical limit (correspondence principle)

▲ $\hbar\omega_k = \frac{\hbar^2 k^2}{2m} \Rightarrow E_p = \frac{p^2}{2m}$

▲ $\Psi \sim e^{i\vec{k}\cdot\vec{r} - i\omega t} \Rightarrow \Psi \sim e^{iS/\hbar}$ ← rays $\frac{\nabla k}{k} \ll 1$

$\left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\Psi = i\hbar\partial_t\Psi$ with $\omega \simeq -\frac{1}{\hbar}\partial_t S$

$\vec{k} \simeq \frac{1}{\hbar}\vec{\nabla} S$

$-\frac{i\hbar}{2m}\nabla^2 S + \boxed{\frac{(\vec{\nabla} S)^2}{2m} + V = -\partial_t S}$ ← $\frac{\hbar\nabla^2 S}{(\nabla S)^2} = \frac{\nabla k}{k}$

$\hbar \to 0 \Rightarrow$ Hamilton-Jacobi Eqn
with S - Hamiltons characteristic fnc
= classical action
$\Longleftrightarrow$ Newton's Eqns.

- interference of matter waves (e-diffraction)
Davisson-Germer.

- Heisenberg uncertainty

... more later via connection to Heisenberg & Feynman's formulation with their simple classical limits.

Important points:

- in measurement of observerable $\hat{O}$ only one of eigenvalues of $\hat{O}$, o_n can be found !!! → very strange if $\hat{O}$ has discrete eigenvalues. (cf Sound with only discrete set of frequencies / colors)
- overall constant factor in Ψ has no physical content, including overall phase factor $e^{i\varphi}$.