Thermodynamics 4230

Aug. 20, 2011

Lecture Set 2

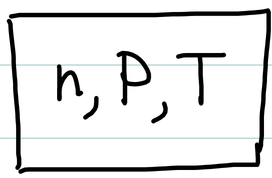
(Leo Radzihovsky)

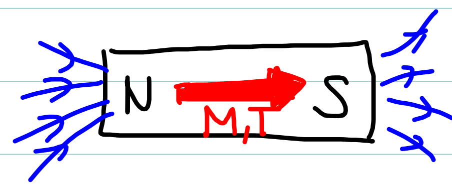
Microcanonical Stat. Mech: 2^{nd} Law of Thermodynamics

States & their counting

Macrostates: gas (n, P, T), magnet (n, M, T), solid (u, σ, T)

- characterize bulk system
- very different probabilities



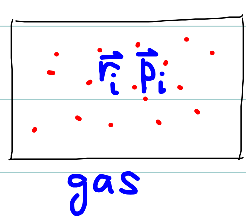


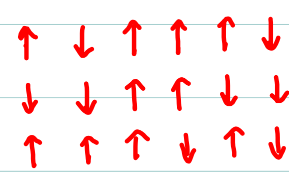
Microstates:

gas: (r_i, p_i), $\vec{k}_n = \frac{2\pi \vec{n}}{L}$ (Q.M) $\psi_{\vec{k}_n}$, ... (position/momentum)

magnet: $\vec{S}_i$ (spin on every ion/site)

- microscopic, unique characterization of system
- equally probably at fixed E (postulate)





Huge # of microstates for each macrostate

$\rightarrow$ bulk properties determined by multiplicity (#) $\Omega(E)$ of μ-states for each macrostate

$\Rightarrow$ **need to count μ-states, $\Omega(E)$**

Counting of μ-states

relation to bulk/thermodynamics:

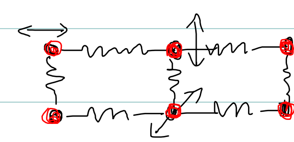
$$\text{Prob. of macrostate } (A) = \frac{\#\,\Omega_A \text{ of } \mu\text{-states}}{\text{total } \# \text{ of } \mu\text{-states}}$$

isolated system w/ fixed E ("μ-canonical ensemble")

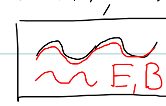
$$\Omega_A(E) = \;???$$

counting depends on system type:

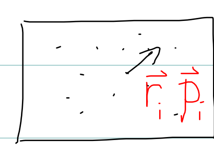
- binary systems:
  - e.g.: • "Ising" magnet w/ spin ↑, ↓
    - • coin up/down
    - • 1d random walk step R, step left
    - • binary #'s 0, 1
- quantum oscillators (bosons):



  - e.g.: • ion vibrations in a crystal (phonons)
    - • EM modes e.g. BB radiation, μ-wave oven, etc



    - • bosonic atoms ($\rightarrow$ BEC)
- Boltzmann (classical) gas:



Binary systems

Mathematics of counting: combinatorics

- permutations of N distinct objects $N! \simeq \left(\frac{N}{e}\right)^N$
- "N choose m" – choose m identical objects out of N

$$\Omega(N,m) = \binom{N}{m} = \frac{N!}{m!\,(N-m)!} = C^N_m$$

e.g.:

→ binomial coeff. in:

$$(x+y)^N = \underbrace{(x+y)(x+y)\dots(x+y)}_{\text{choose } x \text{ or } y \text{ from each of the } N \text{ factors}} = \sum_{m=0}^{N} C^N_m\, x^m\, y^{N-m}$$

↳ Pascal triangle

→ flip N coins:

\# of ways to get m heads $\Omega(m,N) = C^N_m$

\# of total combinations 2^N

$$\Rightarrow P(m,N) = \frac{C^N_m}{2^N} = C^N_m \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{N-m}$$

↳ prob of m heads

↳ prob of $N-m$ tails

note: $\sum_{m=0}^{N} P(m,N) = \left(\frac{1}{2}+\frac{1}{2}\right)^N = 1$ ✓

- $P(N,N) = P(0,N) = \frac{1}{2^N} \to 0$ → very unlikely!
- $P(\frac{N}{2},N) = \frac{N!}{(\frac{N}{2})!\,(\frac{N}{2})!}\,\frac{1}{2^N} \approx 1$ → most likely (check)

binary #'s: 4 bits $\Rightarrow 2^4 = 16$ total

0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

$$\Omega(0,4) = \Omega(4,4) = 1 = \frac{4!}{0!\,4!} = 1 \checkmark$$

$$\Omega(1,4) = \Omega(3,4) = 4 = \frac{4!}{1!\,3!} = 4 \checkmark$$

$$\Omega(2,4) = 6 = \frac{4!}{2!\,2!} = 6 \checkmark$$

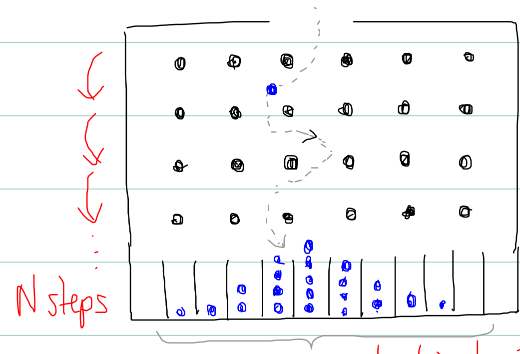
note:

$$(x+y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$

Binary systems

- Galton board

$$P(m,N) = \frac{N!}{m!\,(N-m)!}\,\frac{1}{2^N}$$

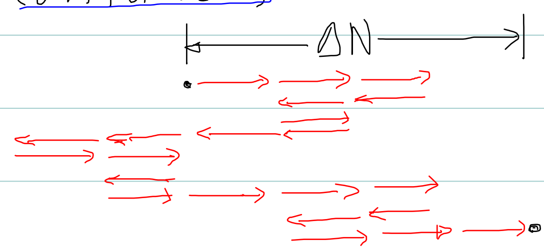


# of bounces left vs right

$\approx$ Normal Gaussian distrib.

binomial distribution

- 1d random walk (diffusion): (cf rubber)



N steps left or right

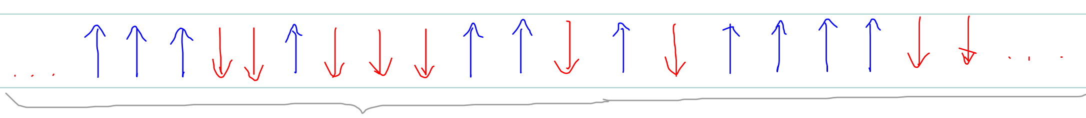
N_L, $N_R = N - N_R$

$$P(N_L, N) = \frac{N!}{N_L!\,(N-N_L)!}\,\frac{1}{2^N} \rightarrow P(\Delta N, N) = \frac{N!\;2^{-N}}{\left(\frac{N-\Delta N}{2}\right)!\left(\frac{N+\Delta N}{2}\right)!}$$

$$\langle \Delta N \rangle = 0 \Leftrightarrow \langle N_L \rangle = \langle N_R = N - N_L \rangle = \frac{N}{2}$$

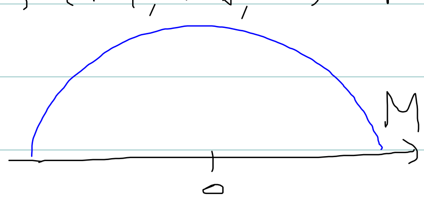
$$(N_R - N_L)_{rms} = \Delta N_{rms} \simeq \sqrt{N}$$ random walk

- "Ising" (up/down) magnet:



N - spins, N_{up} - up, $N_{down} = N - N_{up}$ - down

$$P(N_\uparrow, N_\downarrow; N) = \tilde{P}(M = \Delta N, N) = \frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!}\,\frac{1}{2^N}$$



$M = N_\uparrow - N_\downarrow$

$$\Rightarrow \langle M \rangle = 0 \Leftrightarrow \langle N_\uparrow \rangle = \langle N_\downarrow \rangle = \frac{N}{2}$$

Other fun problems

- Birthday problem / paradox

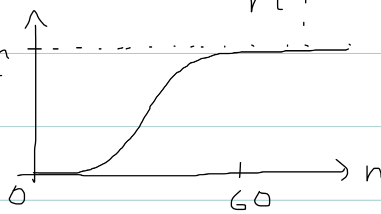
Prob$_N$ of 2 people out of N having same b-date?

$P_{366} = 1, \quad P_{60} \simeq 0.99, \quad P_{23} \simeq \frac{1}{2}, \quad P_2 \simeq \frac{1}{365}$

$$\overline{P}_n = 1 \cdot \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

not having same b-day $= \dfrac{365!}{(365-n)!} \dfrac{1}{365^n}$ $\rightarrow$ no $\dfrac{1}{n!}$ b.c. n are distinct

$$\Rightarrow P_n = 1 - \overline{P}_n \approx 1 - e^{-\frac{n^2}{2 \cdot 365}}$$



- Monty Hall problem / paradox
(Let's make a deal)

- prize behind one of 3 doors
- choose a door $\rightarrow \frac{1}{3}$ prob of winning
- host opens door w/ no prize & asks if you would like to change between your choice & 2nd closed door
- always switch, since $P = \frac{1}{3} \rightarrow P_{switch} = \frac{2}{3}$!

mathematics

- $N! \simeq \underbrace{N^N e^{-N}\sqrt{2\pi N}}_{\text{(Stirling approx)}} \left(\simeq \left(\frac{N}{2}\right)^N\right) = \underbrace{N\cdot(N-1)\cdot(N-2)\ldots 3\cdot 2\cdot 1}_{N}$

- $P_{Bin}(m,N) = \frac{N!}{m!(N-m)!}\, p^m (1-p)^{N-m}$

$$\approx \begin{cases} P_{Poisson}(m) \simeq \frac{\bar{m}^m}{m!} e^{-\bar{m}}, & \text{for } p\to 0,\ N\to\infty \text{ w/ } pN = \text{const} \\ (\bar{m} = Np,\ \overline{\delta m^2} = \bar{m}) & \\ P_{Gauss\,(Normal)} \simeq \frac{1}{\sqrt{2\pi N}} e^{-\frac{(m-\bar{m})^2}{2N}}, & \text{for } p \text{ fixed, } N\to\infty \\ & \text{e.g. } p = \frac{1}{2} \Rightarrow \bar{m} = \frac{N}{2} \end{cases}$$

$$\left(\begin{array}{l} \overline{\delta m^2} = \overline{(m-\bar{m})^2} = N \\ \Rightarrow \delta m_{rms} \simeq \sqrt{N} \ll \bar{m} = \frac{N}{2} \end{array}\right)$$

details:

$$P(m) = \frac{N(N-1)\ldots(N-m+1)}{m!} \frac{p^m (1-p)^N}{(1-p)^m}$$

Poisson limit: $N \gg m \gg 1$, $pN \to$ fixed

$$\approx \frac{(pN)^m}{m!} e^{-pN} \Rightarrow P_{Poisson}(m) = \frac{\bar{m}^m}{m!} e^{-\bar{m}}, \text{ with } \bar{m} = pN$$

moments of $P_{Poisson}(m)$:

- $\sum_{m=0}^{\infty} P_p(m) = 1$
- $\langle m \rangle = \sum_{m=0}^{\infty} m P_p(m) = \sum_{m=1} \frac{\bar{m}^m}{\cancel{m!}_{(m-1)!}} e^{-\bar{m}} \cancel{m} = \bar{m} \underbrace{\sum_{m=1} \frac{\bar{m}^{m-1}}{(m-1)!} e^{-\bar{m}}}_{1}$

$\langle m \rangle = \bar{m} = pN$ ✓

- $\langle m^2 \rangle = \sum_{m=0}^{\infty} m^2 P_p(m) \underset{???}{=} \bar{m} + \bar{m}^2$

$$\langle m^2\rangle = \sum_{m=0}^{\infty} m^2 \frac{\bar{m}^m}{m!} e^{-\bar{m}} = \bar{m} \sum_{m=1} (m-1+1) \frac{\bar{m}^{m-1}}{(m-1)!} e^{-\bar{m}}$$

$$= \bar{m}^2 \sum_{m=2} \frac{\bar{m}^{m-2}}{(m-2)!} e^{-\bar{m}} + \bar{m} \sum_{m=1} \frac{\bar{m}^{m-1}}{(m-1)!} e^{-\bar{m}}$$

$$= (\bar{m})^2 + \bar{m} \Rightarrow \underline{\langle m^2\rangle - \langle m\rangle^2 = \bar{m}} = \langle (m - \langle m\rangle)^2 \rangle = m_{rms}^2$$

$$\Rightarrow \underline{m_{rms}^{P} = \sqrt{\bar{m}}}$$

Binomial distribution moments:

- $$\bar{m} = \sum_{m=1}^{N} P(m)\, m = \sum_{m=1}^{N} \frac{m\, N!}{m!\,(N-m)!} p^m (1-p)^{N-m}$$

$$= pN \underbrace{\sum_{m=1}^{N} \frac{(N-1)!\; p^{m-1} (1-p)^{N-1-(m-1)}}{(m-1)!\,(N-1-(m-1))!}}_{1} = \underline{pN = \bar{m}}$$

- $$\overline{m^2} = \sum_{m=1}^{N} \frac{m^2\, N!}{m!\,(N-m)!} p^m (1-p)^{N-m} = p^2 N \sum_{m=1}^{N} \frac{(N-1)!\,(m-1+1)}{(m-1)!\,(N-m)!} p^{m-2} (1-p)^{N-m}$$

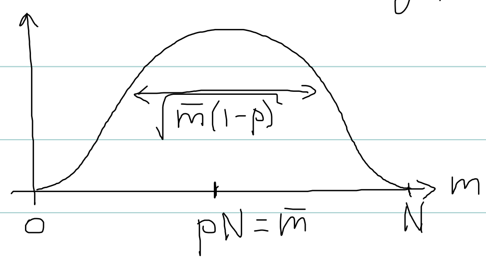
$$= p^2 N(N-1) \underbrace{\sum_{m=2}^{N} \frac{(N-2)!\; p^{m-2} (1-p)^{N-m}}{(m-2)!\,(N-2-(m-2))!}}_{=1} + pN \underbrace{\sum_{m=1}^{N} \frac{(N-1)!\; p^{m-1} (1-p)^{N-m}}{(m-1)!\,(N-m)!}}_{=1}$$

$$\Rightarrow \overline{m^2} = \bar{m}^2 + \bar{m} - p^2 N = \bar{m}^2 + \bar{m} \overbrace{(1-p)}^{\to 1 \text{ in Poisson } p\to 0 \text{ limit}}$$

$$\Rightarrow \overline{m^2} - \bar{m}^2 = m_{rms}^2 = \bar{m}(1-p) \Rightarrow m_{rms} = \sqrt{\bar{m}\,(1-p)}$$

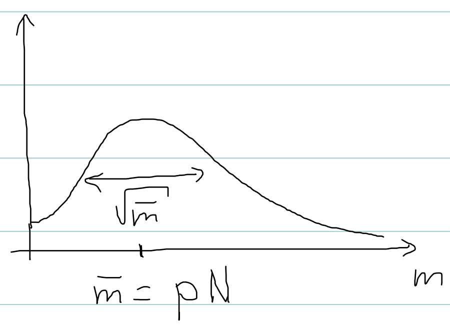
Binomial, Poisson, Gaussian Distributions

- Binomial: arbitrary, m, N, p



$$P_B(m) = \frac{N!}{m!\,(N-m)!}\, p^m (1-p)^{N-m}$$

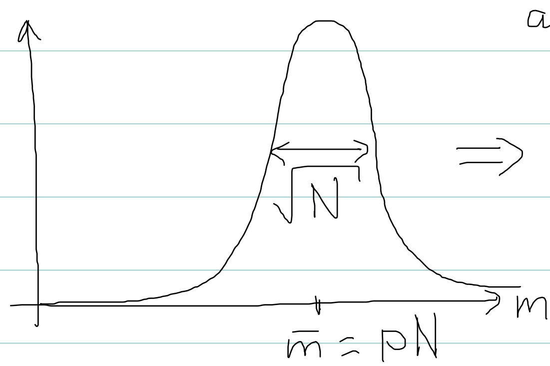
- Poisson: $N \gg m \gg 1$, $p \to 0$ w/ $pN = $ fixed



$$P_P(m) = \frac{\bar{m}^m}{m!} e^{-\bar{m}}$$

- Gaussian (Normal): $N \gg 1$, $m \gg 1$, p - fixed

Taylor-expand $\ln P \simeq -\frac{(m-\bar{m})^2}{\bar{m}}$ about $m = \bar{m}$



$$\Rightarrow \quad P_G(m) \simeq \frac{1}{\sqrt{2\pi \bar{m}}}\, e^{-\frac{(m-\bar{m})^2}{2\bar{m}}}$$

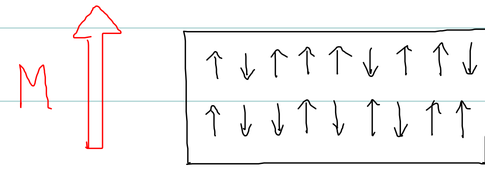
$$m_{rms} \simeq \sqrt{N}$$

narrows (in relative terms) $\Rightarrow$ $\underbrace{\frac{m_{rms}}{\bar{m}} \sim \frac{1}{\sqrt{N}}}$ $\to 0$

e.g. sum random ± 1 $\Rightarrow$ $S = \text{sum}_N$ obeys Gaussian distribution.

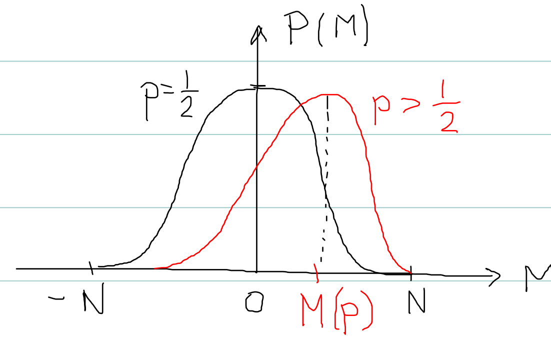
"Ising" magnet

$$P(n_\uparrow, n_\downarrow) = \frac{N!}{n_\uparrow!\, n_\downarrow!}\, p^{n_\uparrow}(1-p)^{n_\downarrow}$$



$n_\uparrow + n_\downarrow = N \quad , \quad n_\uparrow - n_\downarrow = M$

$$P(M) = \underbrace{\frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!}}_{=\,\Omega(M)} \frac{1}{2^N} \quad , \left(p = \tfrac{1}{2} \Rightarrow \uparrow, \downarrow \text{ are equivalent}\right)$$



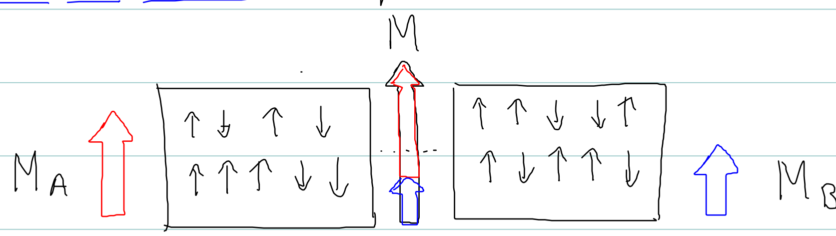
$p \gtrless \frac{1}{2}$ biases for $M \neq 0$

physically via magnetic field $E = -H \cdot M$

$$p = \frac{e^h}{e^h + e^{-h}} \quad , \quad q = 1 - p = \frac{e^{-h}}{e^h + e^{-h}} \quad , \quad h \equiv \frac{H \cdot M}{k_B T}$$

Two "Ising" magnets in equilibrium

$\boxed{\text{two}}$ magnets in thermal equilibrium: $M_A + M_B = M$

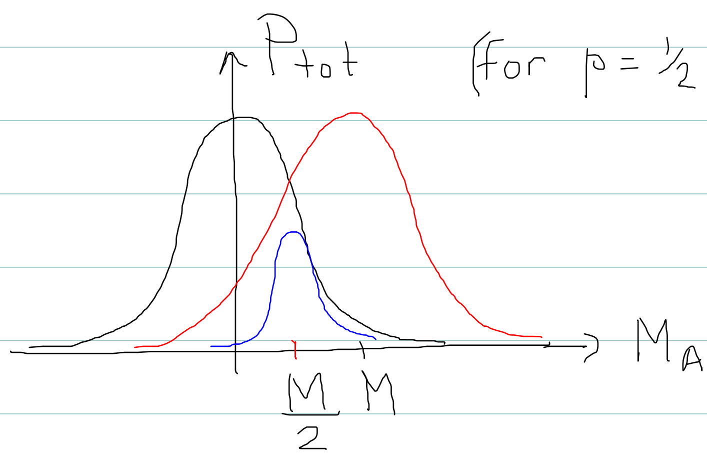


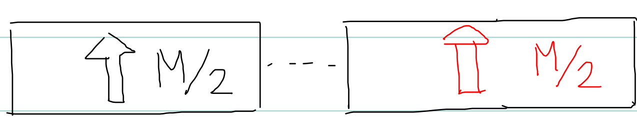
$$\Omega_{tot}(M_A) = \Omega(M_A)\,\Omega(M - M_A) \;;\quad \Omega_{grand} = \sum_{M_A=-N}^{N} \Omega_{tot}(M_A)$$

$$P_{tot}(M_A, M_B) = \frac{\Omega(M_A)\,\Omega(M_B)}{\Omega_{grand}(M)}$$

$$\boxed{\text{maximize } P_{tot} \text{ by } M_A = M_B = \frac{M}{2}}$$

(for $p = \frac{1}{2}$)



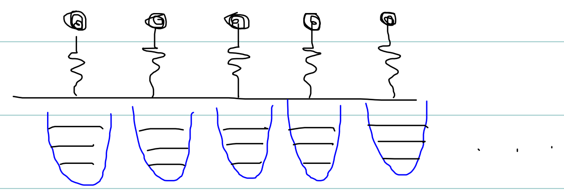


most probable macrostate for $p = \frac{1}{2}$

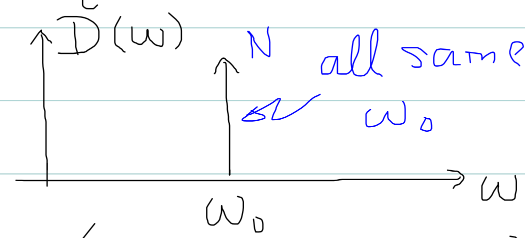
Quantum oscillators (bosons)

e.g. phonons, EM modes, ...

- simple example: Einstein model of phonons (lattice vibrations)



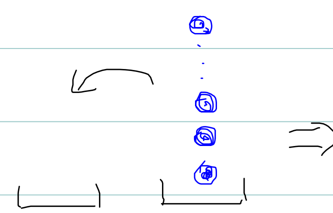
$$E_i = \hbar\omega_0 n_i$$



$$E^{tot}_{n_1 n_2 \dots n_N} \equiv E_{\{n_i\}} = \hbar\omega_0 \underbrace{(n_1 + n_2 + \dots + n_N)}_{n}$$

(in reality 3N coupled oscillators → 3N normal modes with ω_k freq.'s.)

$$\Omega(N, n) = ?$$

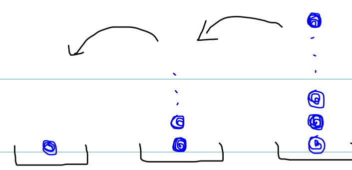


→ $N = 2$, n:

0	n
1	$n-1$
2	$n-2$
3	$n-3$
⋮	
n	0

$$\boxed{\Omega(2, n) = n + 1} = \sum_{n_1=0}^{n} 1 = \sum_{n_1, n_2 = 0}^{n} \delta_{n_1+n_2,\, n}$$

$\to N = 3,\ n$:



0	0	4	
0	1	3	
0	2	2	$\Omega(2,4) = 5$
0	3	1	
0	4	0	
1	0	3	
1	1	2	$\Omega(2,3) = 4$
1	2	1	
1	3	0	
2	0	2	
2	1	1	$\Omega(2,2) = 3$
2	2	0	
3	0	1	$\Omega(2,1) = 2$
3	1	0	
4	0	0	$\Omega(2,0) = 1$

$N = \#\text{atoms} \simeq 10^{23}$

$\Rightarrow$ incomprehensibly huge

$= \Omega(3,4) = 5+4+3+2+1 = \underline{15}$

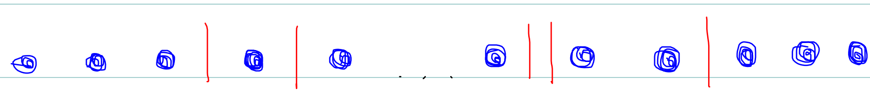
$$\Omega(3,n) = \sum_{n_1 n_2 n_3 = 0}^{N} \delta_{n_1+n_2+n_3,\, n}$$

$$= \frac{1}{2}(n+2)(n+1)$$

$$\left(\underset{n=4}{=} \frac{1}{2} \cdot 6 \cdot 5 = \underline{15} \right)$$

in general: $\Omega(N,n) = \dfrac{(N+n-1)!}{n!\,(N-1)!}$ $\to$ # of ways to distrib. n balls among N boxes

$= \dfrac{\text{total perm. of } n \text{ balls} + N-1 \text{ partitions}}{(\text{identical balls})(\text{ident. partitions})}$



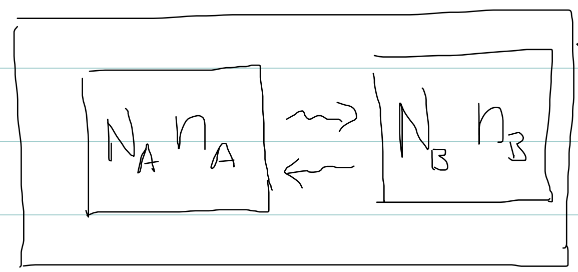
contrast: $C^N_n\, x^{n}\, y^{N-n}$ $\to$ 2 "boxes" (x,y) $\Rightarrow 2^N$ terms

here: "C^N_n" $\to 1 \Rightarrow \underline{N+1 \text{ terms}} \ll 2^N$

check: $\Omega(N,n) = \dfrac{(N+n-1)!}{n!\,(N-1)!}$

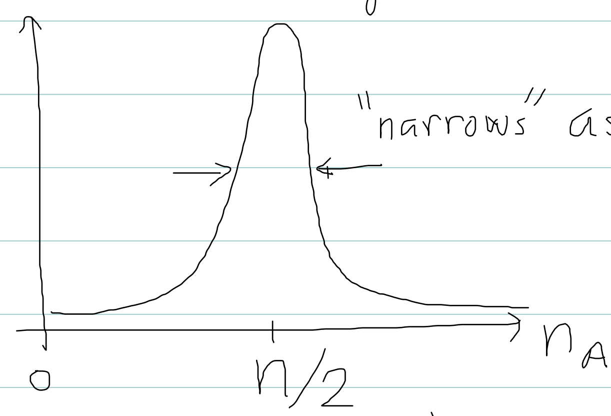
$$\Omega(2,n) = \frac{(n+1)!}{n!\,1!} = n+1 \;\checkmark; \qquad \Omega(3,n) = \frac{(n+2)!}{n!\,2!} = \frac{1}{2}(n+1)(n+2) \;\checkmark$$

Two (Einstein) solids in contact $\to$ thermal equilibrium:
(A, B)



$N = N_A + N_B$
$n = n_A + n_B$

$$P(n_A) = \frac{\Omega_A(n_A)\,\Omega_B(n-n_A)}{\Omega_{grand}(n)}$$



"narrows" as $\sqrt{n} \approx \sqrt{10^{23}} \ll 10^{23}$

(if $N_A = N_B$)

equilibrates s.t. $n_A \to n/2$

$$\Omega(N,n) = \frac{(n+N-1)!}{n!\,(N-1)!} \quad \text{(exact)}$$

cf 2^N

(classical limit) $\approx \left(\dfrac{e\,n}{N}\right)^N \to$ high occup. $\simeq \left(\dfrac{\#\text{ excitations}}{\text{oscill}}\right)^N$

2^{nd} Law of thermodynamics

$n_{rms} \simeq \sqrt{n} << n \Rightarrow$ essentially single macrostate in "thermodynamic limit", i.e. $N \to \infty$; all others $P \to 0$!

Entropy $= S = k_B \ln \Omega(E)$

2^{nd} Law: large system in equilibrium will be in a macrostate with maximum entropy S.

S increases for bulk (large) system

S is additive:

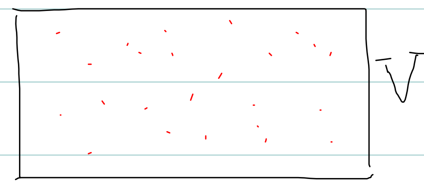
$S_{tot} = S_A + S_B$

$k_B \ln \Omega = k \ln \Omega_A \Omega_B = k_B \ln \Omega_A + k_B \ln \Omega_B$

note: $\Omega_{tot} \simeq \Omega_{grand} = \sum_{n_A} \Omega_{tot}(n_A)$

Boltzmann gas

$$\Omega(E,V) = ?$$



determined by phase-space volume of shell w/ $R = \sqrt{E}$

states determined by $(\vec{r}_i, \vec{p}_i) \leftarrow$ 6N-dimension.

$$\Omega(E,V) = \sum_{\{\vec{r}_i, \vec{p}_i\}}{}' 1, \quad \text{w/ energy } E \pm \Delta E$$

$$= \frac{1}{N!} \int \frac{d^{3N}r \, d^{3N}p}{(2\pi\hbar)^{3N}} \delta\left(E - \sum_{i=1}^{N} \frac{p_i^2}{2m}\right) \Delta E$$

$$\simeq \frac{V^N}{N!} \frac{2\pi^{3N/2}}{\left(\frac{3N}{2}-1\right)!} \frac{(2mE)^{\frac{3N-1}{2}}}{h^{3N}} \frac{2m}{2\sqrt{2mE}} \Delta E$$

$$\simeq \frac{1}{N!} \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} \frac{(2m)^{3N/2}}{h^{3N}} V^N E^{3N/2}$$

$$\Rightarrow S = k_B \ln \Omega(E) = k_B N\left[\frac{5}{2} + \ln\left(\frac{V}{N}\left(\frac{4\pi m E}{3N h^2}\right)^{3/2}\right)\right]$$

(Sackur-Tetrode eqn)

$$\left(A_d = \frac{2\pi^{d/2}}{\left(\frac{d}{2}-1\right)!} r^{d-1}\right)$$

"quick & dirty":

$$\Omega(E) = \frac{1}{N!}\int \frac{d^{3N}r\, d^{3N}p}{(2\pi\hbar)^{3N}} \underset{\text{box quantiz.}}{=} \frac{L^{3N}}{N!\,(2\pi\hbar)^{3N}} \underbrace{\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}-1\right)!}\, P^{3N-1}}_{\text{Area}_{3N}(P)} \underbrace{\Delta P}_{} \approx P^{3N}$$

$$\Omega(E) \simeq \underbrace{\frac{V^N}{N!}\,\frac{1}{(2\pi)^{\frac{3N}{2}}}\cdot\frac{1}{\frac{3N}{2}!}}_{(LK)^{3N}\text{ - dimensionless}} \left(\frac{2mE}{\hbar^2}\right)^{\frac{3N}{2}}$$

$$\boxed{\ln\Omega = N\left[\frac{5}{2} + \ln\left(\frac{V}{N}\cdot\left(\frac{4\pi m E}{3h^2 N}\right)^{3/2}\right)\right]}$$

$$S(E) \simeq N\, f(\underbrace{V E^{3/2}}_{})$$

extensive!

const for isentropic/adiabatic

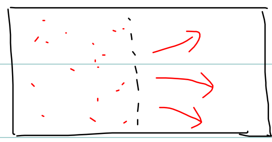
$$\Rightarrow V T^{3/2} \sim V T^{f/2} \quad (f=3)$$

Gibb's $\frac{1}{N!}$ factor

Ideal gas: $\Omega(N,U,V) = f(N)\, V^N U^{3N/2}$

<u>V increasing processes:</u>

- Free expansion: $\underset{\text{fast}}{Q=0}$, $\underset{P=0}{W=0} \Rightarrow \Delta U = 0$



$V_1 \to V_2 > V_1$

$Q = 0$, but $\Delta S = S_f - S_i = N k_B \ln \frac{V_f}{V_i} > 0$

<u>created (increased) entropy in irreversible process!</u>

- Quasi-static, <u>isothermal</u> expansion: $T = \text{const} \Rightarrow \Delta U = 0$
  $\Rightarrow W = -Q$

$\Delta S = N k_B \ln \frac{V_f}{V_i} = \frac{Q}{T}$ ✓

transfer of S from environment:

$S_{universe} = S_{system} + S_{environment} = \text{const}$

$\Rightarrow$ reversible: $TdS = Q$

in general: $TdS \geq Q$

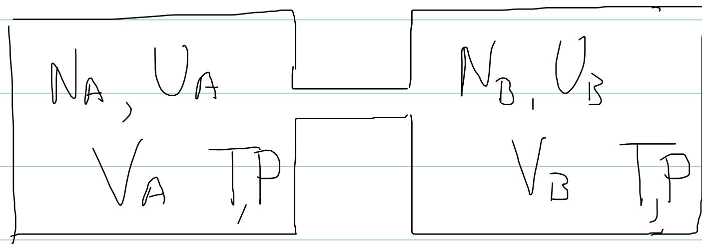
- quasi-static, <u>adiabatic</u> expansion: $\Delta U = W < 0$
  $U \searrow$, $V \nearrow$

<u>$Q = 0$</u>: $V U^{3/2} = \text{const}$

$\Rightarrow S_{system} = \text{const} \Rightarrow \Delta S_{universe} = 0$, for reversible adiabatic process.

Entropy of mixing

A, B gases $\rightarrow$ free expansions:



$$\Rightarrow V_f = V_A + V_B$$

$$S_{tot} = S_A + S_B, \quad \Delta S = \Delta S_A + \Delta S_B$$

$$\Delta S_{tot} = N_A k_B \ln\left(\frac{V_f}{V_A}\right) + N_B k_B \ln \frac{V_f}{V_B} = \text{entropy of mixing}$$

$$\Delta S_{tot} = -N k_B \left[x \ln x + (1-x) \ln (1-x) \right]$$

Ising magnet distribution

- **binomial distribution of m up spins (probability p) out of N spins**

In[88]:=
```
P[m_, N_, p_] := N! / m! / (N - m) ! p^m (1 - p) ^ (N - m)
```

In[89]:=
```
P[m, N, p]
```

Out[89]= $\frac{N!\, p^m (1-p)^{N-m}}{m!\,(N-m)!}$

Normal (Gaussian) distribution limit:

In[91]:=
```
Log[P[m, N, p]]
```

In[94]:= $\text{Series}\left[\log\left(\frac{N!\, p^m (1-p)^{N-m}}{m!\,(N-m)!}\right) /.\ p \to 1/2, \{m,\ N/2, 2\}\right]$

Out[94]= $\log\left(\frac{2^{-N} N!}{\Gamma\left(\frac{N}{2}+1\right)^2}\right) - \left(m - \frac{N}{2}\right)^2 \psi^{(1)}\left(\frac{N}{2}+1\right) + O\left(\left(m-\frac{N}{2}\right)^3\right)$

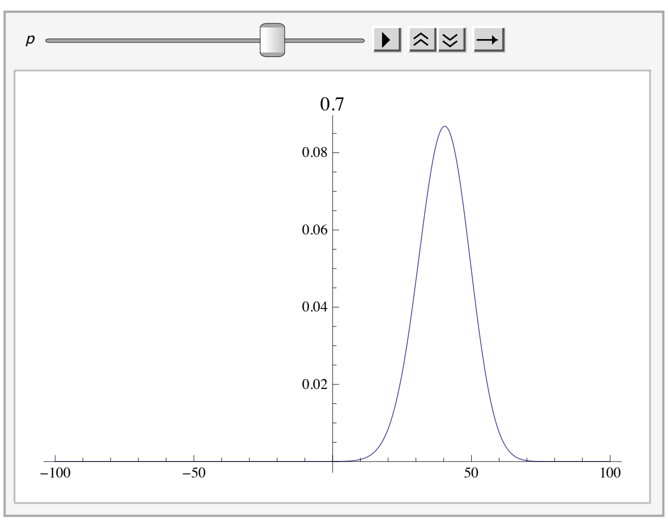
In[102]:=
```
Series[
 N Log[N] - N - m Log[m] + m - (N - m) Log[N - m] + N - m - N Log[2], {m, N / 2, 2}]
```

Out[102]= $-\frac{2\left(m-\frac{N}{2}\right)^2}{N} + O\left(\left(m-\frac{N}{2}\right)^3\right)$

- **Plots as function of magnetization ma for system A, B for different biases $p = \frac{e^h}{e^h + e^{-h}}$ of up spin (corresponds to external field h) and magnetization M:**

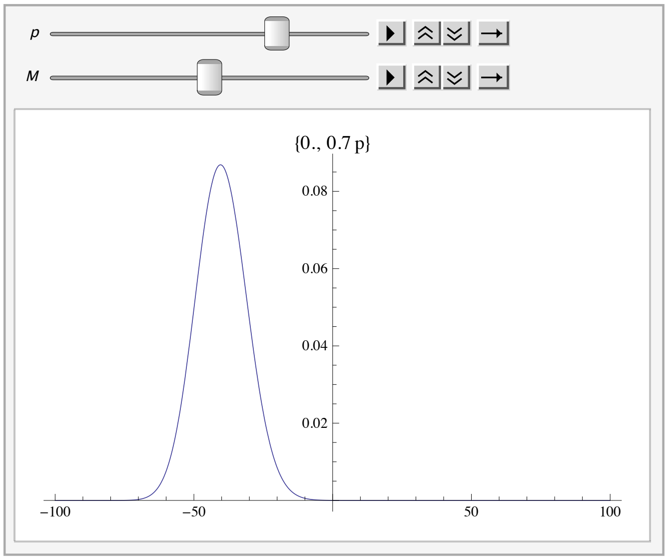
In[114]:=
```
Animate[Plot[P[(100 + ma) / 2, 100, p], {ma, -100, 100}, PlotLabel → p,
  PlotRange → All], {p, 0.1, 0.9}, AnimationRunning → False]
```

Out[114]=



In[122]:=
```
Animate[Plot[P[(100 + M - ma) / 2, 100, p],
   {ma, -100, 100}, PlotLabel → {"M" M, "p" p}, PlotRange → All],
  {p, 0.1, 0.9}, {M, -90, 90}, AnimationRunning → False]
```

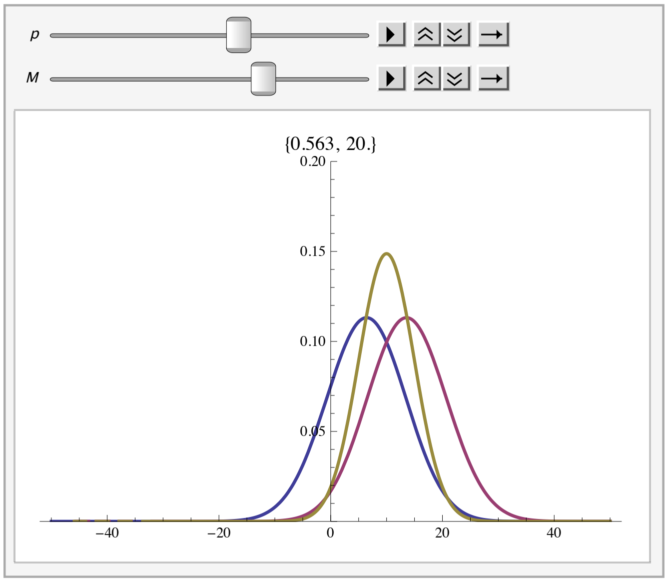
Out[122]=



- **Plots of P for a combined A-B system (with ma + mb = M) as function of ma for different biases p = $\frac{e^h}{e^h + e^{-h}}$ of up spin (corresponds to external field h) and magnetization M:**

In[153]:=
```
Animate[Plot[{P[(N + ma) / 2, N, p] /. N → 50, P[(N + M - ma) / 2, N, p] /. N → 50,
    15 P[(N + ma) / 2, N, p] P[(N + M - ma) / 2, N, p] /. N → 50}, {ma, -50, 50},
   PlotStyle -> Thick, PlotLabel → {p, M}, PlotRange → {0, 0.20}],
  {p, 0.2, 0.8}, {M, -50, 50}, AnimationRunning → False]
```

Out[153]=



In[149]:=
```
Export["magnetAB.pdf", %147]
```

Out[149]= magnetAB.pdf

std = $< m^2 > - < m >^2$ = <m>(1-p) for binomial distribution:

In[15]:=
```
Sum[P[m, N, p] (m^2 - N^2 p^2), {m, 0, N}]
```

Out[15]= $-N\left(p^2 - p\right)$