Leo Radzihovsky, Spring 2016

PHYS 5260: Quantum Mechanics - II

Homework Set 3

Issued February 8, 2016
Due February 22, 2016

Reading Assignment: Shankar, Ch.18, 21

1. Consider a hydrogen atom in its ground state at $t \to -\infty$, subjected to a uniform electric field $\mathbf{E}(t) = \hat{\mathbf{z}}\mathcal{E}e^{-t^2/\tau^2}$. Show that the probability that at time $t \to \infty$ the atom ends up in any of the $n = 2$ states is, to first order,

$$P(n=2) = \left(\frac{e\mathcal{E}}{\hbar}\right)^2 \left(\frac{2^{15}a_0^2}{3^{10}}\right) \pi\tau^2 e^{-\omega^2\tau^2/2}, \tag{1}$$

   where $\omega = (E_{2\ell m} - E_{100})/\hbar$. Does the answer depend on whether or not the spin is incorporated in the picture?

2. Shifted harmonic oscillator (revisited)

   (a) Consider a charged particle (charge q, mass m) confined to move in 1D in the presence of a harmonic potential $V = \frac{1}{2}m\omega^2x^2$ and subjected to a uniform, constant electric field $\mathcal{E}$. If at time $t = 0^-$ the particle is in its ground state but at $t = 0^+$ the electric field is suddenly shut off, compute the average energy change of the oscillator at subsequent time.

   (b) What is the final energy change after the electric field is instead shut off *adiabatically* and how does it compare to that in part (a)?

   (c) How does your answer compare to the classical result?

   Hint: It is convenient to represent the state of the original displaced oscillator in terms of the displacement operator (by distance α) $D_\alpha = e^{-\alpha\partial_x}$ acting on the state of the undisplaced oscillator. Expressing this displacement operator in terms of creation and annihilation operators and using Baker-Campbell-Hausdorff formula $e^{A+B} = e^Ae^Be^{-\frac{1}{2}[A,B]}$ (valid when $[A,B]$ is a c number) allows a very efficient computation of the probability of finding the system in the n-th quantum state.

3. Perturbed harmonic oscillator transitions

(a) Consider, once again, a charged particle in a ground state of a 1D periodic potential $V(x) = \frac{1}{2}m\omega^2 x^2$, this time perturbed at time $t = 0$ by a weak *oscillating* electric field $\mathcal{E}\cos\omega_e t$. Calculate a transition rate at time t out of the ground state. What is the asymptotic, long-time rate?

(b) Repeat the above analysis if instead of an electric field, the perturbation is a periodically modulated oscillator frequency, i.e., $\omega \to \omega(t) = \omega + \omega_0 \cos\omega_e t$, where the amplitude of frequency modulation is weak, $\omega_0 \ll \omega$.

4. Consider a single spin 1/2 that has been "prepared" (using e.g., a strong polarizing magnetic field pointing along $\hat{\mathbf{z}}$, that now has been shut off) to be in the spin-up ("along" $\hat{\mathbf{z}}$) eigenstate.

   (a) What are the amplitudes $d_\uparrow$, $d_\downarrow$ of finding the spin in the spin-up and spin-down eigenstate, respectively, after an application of a weak (so that 1st order perturbation theory applies) magnetic field pulse (τ is the duration of time over which pulse is applied, taken to be much smaller than any other scale in the problem)

      i. $\mathbf{B} = \hat{\mathbf{z}} B_0 \tau \delta(t)$, i.e., along z, parallel to the initial spin configuration?

      ii. $\mathbf{B} = \hat{\mathbf{y}} B_0 \tau \delta(t)$, i.e., along y, transverse to the initial spin configuration?

   Comment: Take for concreteness the interaction of the spin with a magnetic field to be described by a Hamiltonian $H = -\mu_B \sigma \cdot \mathbf{B}$

   (b) Repeat analysis of all of the above questions for a magnetic field of magnitude B_0 constant during time $0 < t < \tau$ and zero otherwise, i.e., turned on at $t = 0$ and shut off at $t = \tau$.

   Notice that this case should reduce to the previous case for a vanishing "on" duration τ.

   (c) For the last case with B_0 along $\hat{\mathbf{y}}$, deduce the duration τ_* beyond which the perturbation theory breaks down, and interpret this time physically, e.g., by thinking about the full spin dynamics (at least classically first) for this case.

   (d) Solve the problem (b) *exactly* by explicitly finding the expression for the spinor $d_\sigma(t)$ at time t, and compare the lowest order Taylor expansion with the perturbative solutions above. Also for case (ii) calculate the magnetic field pulse duration τ_π for which the spin is in the spin-down eigenstate with 100% certainty, and compare it to τ_*

5. Berry's phase

   As we discussed in class, for an extremely slowly (adiabatically) changing Hamiltonian, $H(\alpha(t))$ (where $\alpha(t)$ is some parameter that changes slowly in time) we expect that a system that starts out in an eigenstate $|n, \alpha(0)\rangle$ separated from other states by a large gap will remain in an *instantaneous* eigenstate of the time-dependent Hamiltonian, i.e., will satisfy $H(\alpha)|n,\alpha\rangle = E_n(\alpha)|n,\alpha\rangle$. Naturally, the state will evolve in time due to t-dependence of $\alpha(t)$, as well as the usual phase factor $e^{-i/\hbar\int_0^t E(\alpha(t'))dt'}$.

However, as we learned in class, the state will acquire an additional (nonobvious) phase factor, $e^{i\gamma(t)}$. The neat thing about this phase γ is that it can be nontrivial even if the Hamiltonian and all adiabatic parameters, e.g., $\alpha(t)$ have returned back to their original values (i.e. the adiabatic parameters have executed a cyclic variation). This phase (discovered by Michael Berry, 1984) has many amazing consequences in physics (see Shankar, Ch. 21). It is given by

$$\begin{aligned}\gamma_n &= i\int_0^t dt'\langle\psi_n(\alpha(t'))|\partial_{t'}|\psi_n(\alpha(t'))\rangle, && (2)\\ &= i\int_{\vec{\alpha}_0}^{\vec{\alpha}_1} d\vec{\alpha}\cdot\langle\psi_n(\alpha)|\vec{\nabla}_\alpha|\psi_n(\alpha)\rangle, && (3)\\ &\equiv \hbar^{-1}\int d\vec{\alpha}\cdot\vec{A}_B, && (4)\end{aligned}$$

where $\vec{A}_B$ is Berry's vector potential corresponding to the phase γ. Notice that $\vec{A}_B$ can have a Berry's magnetic field associated with it, given by $\vec{\nabla}\times\vec{A}_B = \vec{B}_B$. This field is not real, but its effects are quite similar to those of a real magnetic field!

Consider a spin 1/2 spinor that is an "up" eigenstate of $H = -\mathbf{B}(t)\cdot\sigma$, i.e., $|\uparrow,\theta,\phi\rangle = (\cos\theta/2, e^{i\phi}\sin\theta/2)$, where θ and ϕ are angles defining the direction of $\mathbf{B}$.

(a) Compute Berry's phase $\gamma_\uparrow$ associated with $\mathbf{B}$ going slowly once around:

i. a great (vertical) circle lying in the x-z plane,

ii. the equator, lying in the x-y plane,

iii. a (horizontal) circle at a latitude θ, going once around.

Hint: The adiabatic parameter here are the polar and azimuthal angles describing the orientation of $\mathbf{B}$; notice that your answers differ in cases (i) and (ii), even though physically they are equivalent operations on $\mathbf{B}$.

(b) What is the change in the state for cases (i) and (ii) (after one complete revolution of $\mathbf{B}$, *without* the Berry's phase, i.e., just due to θ,ϕ dependence explicit in $|\uparrow,\theta,\phi\rangle$. Note that in the case (i) (but *not* in (ii)) the spinor is not single-valued function of orientation of $\mathbf{B}$, and gets multiplied by -1 after a complete revolution of $\mathbf{B}$ along a great circle.

The resolution, is that indeed the Berry's phase makes up the difference between (i) and (ii).

(c) Show that if the state is multiplied by an arbitrary phase factor $e^{i\chi}$, Berry's vector potential $\vec{A}_B$ transforms under a gauge transformation $\vec{A}_B \to \vec{A}_B - \hbar\vec{\nabla}_\alpha\chi$, and that this shifts Berry's phase exactly by $-\Delta\chi$ so as to exactly cancel $\Delta\chi$ due to changes from the phase factor χ.