MULTIPLICATIVE SCHWARZ ALGORITHMS FOR SOME NONSYMMETRIC AND INDEFINITE PROBLEMS

XIAO-CHUAN CAI* AND OLOF B. WIDLUND †

Abstract. The classical Schwarz alternating method has recently been generalized in several directions. This effort has resulted in a number of new powerful domain decomposition methods for elliptic problems, in new insight into multigrid methods and in the development of a very useful framework for the analysis of a variety of iterative methods. Most of this work has focused on positive definite, symmetric problems. In this paper a general framework is developed for multiplicative Schwarz algorithms for nonsymmetric and indefinite problems. Several applications are then discussed including two- and multi-level Schwarz methods and iterative substructuring algorithms. Some new results on additive Schwarz methods are also presented.

Key words. nonsymmetric elliptic problems, preconditioned conjugate gradient type methods, finite elements, multiplicative Schwarz algorithms

AMS(MOS) subject classifications. 65F10, 65N30, 65N55

1. Introduction. The analysis of the classical Schwarz alternating method, discovered more than 120 years ago by Hermann Amandus Schwarz, was originally based on the use of a maximum principle; cf. e.g. [30]. The method can also conveniently be studied using a calculus of variation. This approach is quite attractive because it allows us to include elliptic problems, such as the systems of linear elasticity, which do not satisfy a maximum principle. Such a framework is also as convenient for a finite element discretization as for the original continuous problem.

It is easy to show, see e.g. P.-L. Lions [22], that the fractional steps of the classical Schwarz method, applied to a selfadjoint elliptic problem and two overlapping subregions covering the original region, can be expressed in terms of projections onto subspaces naturally associated with the subregions. Let $a(u, v)$ be the inner product, which is used in the standard weak formulation of the elliptic problem at hand, and let V be the corresponding Hilbert space. The projections, $P_i : V = V_1 + V_2 \rightarrow V_i,\ i = 1, 2$, are defined by

$$a(P_i u, v) = a(u, v), \quad \forall v \in V_i,\ i = 1, 2.$$

For this simple *multiplicative* Schwarz method, the error propagation operator is

$$(I - P_2)(I - P_1);$$

cf. Lions [22], or Dryja and Widlund [12, 14, 16]. The projections P_i are symmetric, with respect to the inner product $a(u, v)$, and they are also positive semidefinite. (In

* Department of Mathematics, University of Kentucky, Lexington, KY 40506. The work of this author was supported in part by the National Science Foundation and the Kentucky EPSCoR Program under grant STI-9108764.

† Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, N.Y. 10012. This work was supported in part by the National Science Foundation under Grant NSF-CCR-8903003 and in part by the U.S. Department of Energy under contract DE-FG02-88ER25053 at the Courant Mathematics and Computing Laboratory.

this paper, symmetry is always with respect to a symmetric, positive definite form $a(u,v)$ and the adjoint S^T of an operator S is given by $a(S^Tu,v) = a(u,Sv)$.)

The classical product form of Schwarz's algorithm can be viewed as a simple iterative method for solving

$$(P_1 + P_2 - P_2P_1)u_h = g_h,$$

with an appropriate right-hand side g_h. The algorithm can be extended immediately to more than two subspaces. Recently, there has also been a lot of interest in an *additive* variant of Schwarz's algorithm in which the equation

$$(P_1 + \cdots + P_N)u_h = g_h$$

is solved by a conjugate gradient algorithm; cf. Dryja and Widlund [14], Matsokin and Nepomnyaschikh [26] and Nepomnyaschikh [27]. It has been discovered that we can view many domain decomposition and iterative refinement methods as Schwarz algorithms and a general theory is being developed; cf. e.g. Bjørstad and Widlund [2], Cai [6], Dryja, Smith and Widlund [13], Dryja and Widlund [16, 17, 15, 19], Mathew [23, 25, 24], Smith [31] and Widlund [32, 33].

As already noted, both the multiplicative and additive Schwarz methods can be extended to the case of more than two subspaces. We can also replace the projections by other operators, $T_i : V \rightarrow V_i$, which approximate them. The analysis of the general multiplicative case introduces additional difficulties. Recently, Bramble, Pasciak, Wang, and Xu [4] and Xu [34] have made substantial progress towards developing a general theory for the symmetric, positive definite case. In this paper, we extend the theory to a class of nonsymmetric and indefinite problems.

In many interesting applications to elliptic equations, one of the subspaces, V_0, plays a special role. It often corresponds to an intentionally coarse mesh, and provides global transportation of information between the different parts of the region in each step of the iteration. If, for a particular application, it is not necessary to include such a space, we can just drop V_0. We note that Bramble et al. [4] considered a somewhat more general situation; however, in the interest of keeping the presentation simple, we limit our discussion to the case of one special subspace.

With $J+1$ subspaces, $V_0, \cdots, V_J$, and $V = V_0 + \cdots + V_J$, the error propagation operator of the multiplicative Schwarz algorithm becomes

$$E_J = (I - T_J) \cdots (I - T_0).$$

Our main task is to estimate the spectral radius $\rho(E_J)$ of this operator.

In Section 2, we develop an abstract theory for the multiplicative Schwarz method just introduced. This work is inspired by the work by Bramble, Pasciak, Wang, and Xu [4] and Xu [34]. Their papers are confined to the positive definite, symmetric case; here we consider problems with nonsymmetric and indefinite iteration operators T_i. In Section 3, we introduce a family of nonsymmetric and indefinite elliptic problems and in the rest of the paper we use our abstract theory to derive a number of results on the convergence rate of several algorithms applied to such elliptic problems. Throughout the paper, we also comment on additive Schwarz methods.

This paper does not include any numerical results. We refer to Cai, Gropp, and Keyes [9] for an extensive experimental study of many methods for nonsymmetric and indefinite problems.

So far, most of the work on Schwarz methods has been restricted to the symmetric case. See Bramble, Leyk, and Pasciak [3], Cai [6, 7, 8], Cai and Widlund [10], Cai and Xu [11], Mathew [23, 25, 24] and Xu [35] for previous work on Schwarz methods for nonsymmetric and indefinite problems.

2. An Abstract Theory for Schwarz Methods.

Our main task is to provide an estimate of the spectral radius of the error propagation operator E_J arising in the multiplicative Schwarz method. We begin by observing that with

$$E_j = (I - T_j)\cdots(I - T_0), \qquad E_{-1} = I,$$

$$\text{and} \qquad R_j = T_j + T_j^T - T_j^T T_j,$$

we have

$$E_j^T E_j - E_{j+1}^T E_{j+1} = E_j^T R_{j+1} E_j.$$

This leads to the identity

$$I - E_J^T E_J = \sum_{j=0}^{J} E_{j-1}^T R_j E_{j-1}. \tag{1}$$

It is easy to see that a satisfactory upper bound for $\rho(E_J)$ can be obtained by showing that the operator on the right hand side of (1) is sufficiently positive definite. It might therefore seem natural to assume that the operators R_i are positive semidefinite. This is so if $T_i^T = T_i \geq 0$ and $\|T_i\|_a \leq 2$ but such an assumption on R_i can often not be established in our applications. In the general case, we therefore make a different

Assumption 1. *There exist a constant $\gamma > 0$ and parameters $\delta_i \geq 0$, such that $\sum \delta_i$ can be made sufficiently small and*

$$R_i = T_i + T_i^T - T_i^T T_i \geq \gamma T_i^T T_i - \delta_i I. \tag{2}$$

We note that if we can bound $T_i + T_i^T + \delta_i I$ from below by a positive multiple of $T_i^T T_i$, then Assumption 1 is satisfied for αT_i for a sufficiently small α. It is well known that such a rescaling (underrelaxation) often is necessary to obtain convergence in nonsymmetric cases.

We now establish some simple consequences of Assumption 1. In the proof, we give a simple argument, which we also use in several other proofs.

Lemma 1. *If Assumption 1 is satisfied, then*

$$\|T_i\|_a \leq \omega_i \equiv (1+\sqrt{1+\delta_i(1+\gamma)})/(1+\gamma) \leq 2/(1+\gamma)+\delta_i/2, \tag{3}$$

and

$$\|I-T_i\|_a \leq 1+\delta_i/2. \tag{4}$$

Proof. It follows from Assumption 1 that

$$a(T_iu, T_iu) \leq 2/(1+\gamma)a(T_iu, u) + \delta_i/(1+\gamma)a(u,u).$$

Therefore,

$$\|T_iu\|_a^2 \leq 2/(1+\gamma)\|T_iu\|_a\|u\|_a + \delta_i/(1+\gamma)\|u\|_a^2.$$

By considering the solutions of the quadratic equation

$$x^2 - 2/(1+\gamma)x - \delta_i/(1+\gamma) = 0,$$

we easily obtain (3). Inequality (4) is obtained by a straightforward computation. □

In the case studied previously, with T_i symmetric, positive semidefinite, $\delta_i = 0$,

$$0 \leq T_i \leq \omega I \equiv 2/(1+\gamma)I, \tag{5}$$

and

$$R_i \geq (2-\omega)T_i \geq 0; \tag{6}$$

cf. Bramble et al. [4]. In the general case, to simplify our calculations and formulas, we set $\omega = \max_i \omega_i$ and always assume that $\omega \geq 1$.

A $J \times J$ matrix $\mathcal{E}$ provides a convenient measure of the extent by which the range of the operators T_i are mutually orthogonal:

Definition 1. *The matrix* $\mathcal{E} = \{\varepsilon_{i,j}\}_{i,j=1}^J$ *is defined by strengthened Cauchy-Schwarz inequalities, i.e.* $\varepsilon_{i,j}$ *are the smallest constants for which*

$$|a(T_iu, T_jv)| \leq \varepsilon_{i,j}\|T_iu\|_a\|T_jv\|_a \ , \forall u,v \in V \tag{7}$$

hold.

Note that $\varepsilon_{i,i} = 1$ and that $0 \leq \varepsilon_{i,j} \leq 1$. In favorable cases, $\rho(\mathcal{E})$ remains uniformly bounded even when J grows. By Gershgorin's theorem, $\rho(\mathcal{E}) \leq J$ always holds.

We next establish an auxiliary result.

Lemma 2. *The following two inequalities hold:*

$$a(\sum_{i=1}^J T_i^T T_i v, v) \leq (2\rho(\mathcal{E})^{1/2}/(1+\gamma) + \sum_{i=1}^J \delta_i/(2\rho(\mathcal{E})^{1/2}))^2 a(v,v),$$

and

$$\|\sum_{i=1}^{J} T_i\|_a \leq 2\rho(\mathcal{E})/(1+\gamma) + \sum_{i=1}^{J} \delta_i/2.$$

Proof. By using the strengthened Cauchy-Schwarz inequalities, we obtain

$$a(\sum_{i=1}^{J} T_i v, \sum_{i=1}^{J} T_i v) \leq \rho(\mathcal{E}) \sum_{i=1}^{J} a(T_i v, T_i v) = \rho(\mathcal{E}) \sum_{i=1}^{J} a(T_i^T T_i v, v). \tag{8}$$

We now use Assumption 1 and the standard Cauchy-Schwarz inequality obtaining

$$\begin{aligned} a(\sum_{i=1}^{J} T_i^T T_i v, v) &\leq (2/(1+\gamma)) a(\sum_{i=1}^{J} T_i v, v) + (\sum_{i=1}^{J} \delta_i/(1+\gamma))\|v\|_a^2 \\ &\leq 2\rho(\mathcal{E})^{1/2}/(1+\gamma) a(\sum_{i=1}^{J} T_i^T T_i v, v)^{1/2}\|v\|_a + (\sum_{i=1}^{J} \delta_i/(1+\gamma))\|v\|_a^2. \end{aligned}$$

The inequalities now follow by using an argument very similar to that in the proof of Lemma 1. □

An upper bound for $\|\sum_{i=0}^{J} T_i\|_a$ is required in the analysis of additive Schwarz methods, see Cai and Widlund [10]; it is often relatively easily obtained by providing an upper bound for $\rho(\mathcal{E})$. A lower bound on the same operator is obtained, in the symmetric, positive definite case, by estimating the parameter C_0 of an inequality similar to that of Assumption 2 introduced below. Note that we now work with the operators $T_i^T T_i$ instead of the T_i that were used in the symmetric, positive definite case; cf. Bramble et al. [4].

Assumption 2. *There exists a constant $C_0 > 0$, such that*

$$\sum_{i=0}^{J} T_i^T T_i \geq C_0^{-2} I.$$

Obtaining a bound for C_0 is often one of the most difficult part of the analysis of Schwarz methods in any specific application.

In the symmetric, positive definite case, an estimate of the condition number of the operator that is relevant for the additive algorithm is obtained straightforwardly:

$$C_0^{-2}\omega^{-1} I \leq \sum_{i=0}^{N} T_i \leq (\rho(\mathcal{E}) + 1)\omega I. \tag{9}$$

The upper bound is an easy consequence of Lemma 2; cf. (5) for the definition of ω. The lower bound follows from Assumption 2 and an elementary inequality; $T_i^2 \leq \omega T_i$ in the symmetric, positive semidefinite case.

A bound on the rate of convergence of the conjugate gradient method follows from (9) in a routine way. Similarly, in the theory developed by Cai [6] and Cai and Widlund [10], a lower bound for $a(\sum_{i=0}^{N} T_i u, u)$ and an upper bound for $\|\sum_{i=0}^{J} T_i\|_a$ are required to obtain an estimate of the rate of convergence of the GMRES and other Krylov space based iterative methods that are used for nonsymmetric problems; cf. Eisenstat, Elman, and Schultz [20].

We can now prove, under Assumptions 1 and 2, that the symmetric part of the operator $\sum_{i=0}^{J} T_i$ is positive definite, provided that $\sum_0^J \delta_i$ is small enough.

Lemma 3. *For any $v \in V$,*

$$a(\sum_{i=0}^{J} T_i v, v) \geq \left(C_0^{-2}\omega^{-1} - \sum_{0}^{J} \delta_i/2 \right) \|v\|_a^2.$$

We note that we recover the lower bound in (9) by setting the $\delta_i = 0$.

Proof. It follows from Assumption 1 that

$$a(T_i v, v) \geq \frac{1+\gamma}{2} a(T_i^T T_i v, v) - \frac{\delta_i}{2} a(v, v).$$

The proof is completed by forming a sum and by using Assumption 2 and the relation between ω and γ. □

The main effort goes into establishing the following Lemma. (Throughout, C and c denote generic positive constants, which are independent of the mesh parameters that will be introduced later.)

Lemma 4. *In the general case, there exists a constant $c > 0$ such that*

$$\sum_{i=0}^{J} E_{i-1}^T T_i^T T_i E_{i-1} \geq \tilde{\gamma} \sum_{i=0}^{J} T_i^T T_i, \ \text{where } \tilde{\gamma} = c(\omega^2 \rho(\mathcal{E})^2 + (\sum_{i=0}^{J} \delta_i)^2 + 1)^{-1}. \tag{10}$$

Proof. We first note that the terms with $i = 0$ can be handled separately and without any difficulty.

A direct consequence of the definition of the operator E_j is that

$$I = E_{i-1} + \sum_{j=0}^{i-1} T_j E_{j-1} = E_{i-1} + T_0 + \sum_{j=1}^{i-1} T_j E_{j-1}. \tag{11}$$

For $i > 0$, we therefore obtain

$$a(T_i^T T_i v, v) = a(T_i v, T_i E_{i-1} v) + a(T_i v, T_i T_0 v) + a(T_i v, T_i \sum_{j=1}^{i-1} T_j E_{j-1} v).$$

Let $d_j = \|T_j E_{j-1} v\|_a$, $1 \leq i \leq J$, be the components of a vector d. We find that

$$\|T_i v\|_a^2 \leq \|T_i v\|_a d_i + \|T_i v\|_a \|T_i T_0 v\|_a + \|T_i v\|_a \|T_i \sum_{j=1}^{i-1} T_j E_{j-1} v\|_a.$$

Cancelling the common factor and squaring, we see that we need to estimate $\sum_{i=1}^{J} a(T_i^T T_i T_0 v, T_0 v)$ and $\sum_{i=1}^{J} a(T_i^T T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v)$ appropriately in order to complete the proof of the Lemma.

We can use Lemma 2 directly to estimate the first expression. We use Assumption 1 to estimate the second expression and obtain,

$$a(T_i^T T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v) \leq (2/(1+\gamma) a(T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v)$$
$$+\delta_i/(1+\gamma) a(\sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v)).$$

After forming the sum over i, the second term on the right hand side can be estimated straightforwardly from above by

$$\rho(\mathcal{E})/(1+\gamma) \sum_{i=1}^{J} \delta_i |d|_2^2.$$

What remains, primarily, is to estimate

$$\sum_{i=1}^{J} a(T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v).$$

By using the strengthened Cauchy-Schwarz inequalities, we obtain

$$a(T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v) \leq \|T_i \sum_{j=1}^{i-1} T_j E_{j-1} v\|_a \sum_{j=1}^{i-1} \varepsilon_{i,j} d_j.$$

Therefore, by using the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{J} a(T_i \sum_{j=1}^{i-1} T_j E_{j-1} v, \sum_{j=1}^{i-1} T_j E_{j-1} v) \leq (\sum_{i=1}^{J} \|T_i \sum_{j=1}^{i-1} T_j E_{j-1} v\|_a^2)^{1/2} (\sum_{i=1}^{J} (\sum_{j=1}^{i-1} \varepsilon_{i,j} d_j)^2)^{1/2}.$$

We now use the fact that all $\varepsilon_{i,j}$ and d_j are nonnegative and obtain

$$\sum_{i=1}^{J} (\sum_{j=1}^{i-1} \varepsilon_{i,j} d_j)^2 \leq \rho(\mathcal{E})^2 |d|_{l_2}^2.$$

The proof is now completed by an argument similar to that in the proof of Lemma 1. $\square$

We next note that it follows from Assumption 1 that

$$E_{j-1}^T R_j E_{j-1} \geq \gamma E_{j-1}^T T_j^T T_j E_{j-1} - \delta_j E_{j-1}^T E_{j-1}.$$

We use Lemma 4 to estimate the first term from below and the bound (4) to show that the second term is bounded by

$$\delta_j exp(\sum_{i=0}^{j-1} \delta_i) I \ .$$

We can now put it all together and obtain

THEOREM 1. *In the general case, the multiplicative algorithm is convergent if*

$$\frac{\gamma}{(\omega^2\rho(\mathcal{E})^2 + (\sum \delta_i)^2 + 1)C_0^2}$$

dominates

$$\sum_{j=0}^{J} \delta_j exp(\sum_{i=0}^{j-1} \delta_i)$$

by a sufficiently large constant factor. Under this assumption, there exists a constant $c > 0$ *such that*

$$\rho(E_J) \leq \|E_J\|_a \leq \sqrt{1 - \frac{c}{(\omega^2\rho(\mathcal{E})^2 + (\sum \delta_i)^2 + 1)C_0^2}} \ .$$

We note that in the positive definite, symmetric case, the $\delta_i = 0$. By using very similar arguments, we can show that

$$\rho(E_J) \leq \|E_J\|_a \leq \sqrt{1 - \frac{(2-\omega)}{(2\omega^2\rho(\mathcal{E})^2 + 1)C_0^2}} \ .$$

The multiplicative Schwarz algorithm can also be accelerated by using the GMRES algorithm; cf. subsection 4.2. In the analysis of the resulting method, we need to establish that the symmetric part of the operator $I - E_J$ is positive definite; see Eisenstat, Elman and Schultz [20] for the underlying theory. To obtain a result, we note that since

$$a(E_J v, v) \leq \|E_J\|_a \|v\|_a^2 = -(1 - \|E_J\|_a)\|v\|_a^2 + a(v,v),$$

we obtain

$$a((I - E_J)v, v) \geq (1 - \|E_J\|_a)\|v\|_a^2, \quad \forall v \in V. \tag{12}$$

This bound can now be combined with that of Theorem 1.

3. Nonsymmetric and Indefinite Elliptic Problems. We consider a linear, second order elliptic equation with Dirichlet boundary condition, defined on a polygonal region $\Omega \subset R^d$, $d = 2, 3$,

$$\begin{cases} \mathcal{L}u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

The elliptic operator $\mathcal{L}$ is of the form

$$\mathcal{L}u(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}) + 2\sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x),$$

where the quadratic form defined by $\{a_{ij}(x)\}$ is symmetric, uniformly positive definite and all the coefficients are sufficiently smooth. We use a weak formulation of this problem:

Find $u \in H_0^1(\Omega)$, such that,

$$b(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega). \tag{14}$$

Here $(\cdot,\cdot)$ is the usual L^2 inner product, $f \in L^2(\Omega)$ and the bilinear form $b(\cdot,\cdot)$ is given by

$$b(u,v) \;=\; a(u,v) + s(u,v) + c(u,v),$$

where

$$a(u,v) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx,$$

$$s(u,v) = \sum_{i=1}^{d} \int_{\Omega} (b_i \frac{\partial u}{\partial x_i} v + \frac{\partial (b_i u)}{\partial x_i} v) dx,$$

$$c(u,v) = (\tilde{c}u, v),$$

with $\tilde{c}(x) = c(x) - \sum_{i=1}^{d} \partial b_i / \partial x_i$.

We note that the lower order terms of the operator $\mathcal{L}$ are relatively compact perturbations of the principal, second order term. We also assume that (14) has a unique solution in $H_0^1(\Omega)$.

For this problem, it is appropriate to use $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$, a norm equivalent to the $H_0^1(\Omega)$ norm. Integration by parts shows that $s(\cdot,\cdot)$ satisfies $s(u,v) \;=\; -s(v,u)$, for all $u, v \in H_0^1(\Omega)$. Using elementary, standard tools, it is easy to establish the following inequalities:

(i) There exists a constant C, such that $\mid b(u,v) \mid \leq C\|u\|_a \|v\|_a, \quad \forall u, v \in H_0^1(\Omega)$, i.e. $b(u,v)$ is a continuous, bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$.

(ii) Gårding's inequality: There exists a constant C, such that

$$\|u\|_a^2 - C\|u\|_{L^2(\Omega)}^2 \leq b(u,u), \quad \forall u \in H_0^1(\Omega).$$

(iii) There exists a constant C, such that

$$\mid s(u,v) \mid \leq C\|u\|_a \|v\|_{L^2(\Omega)}, \quad \forall u, v \in H_0^1(\Omega).$$

We note that the bound for $s(\cdot,\cdot)$ is different from that of $b(\cdot,\cdot)$; each term in $s(\cdot,\cdot)$ contains a factor of zero order. This enables us to control the skew-symmetric term and makes our analysis possible. This is not the only interesting case; see Bramble, Leyk, and Pasciak [3] in which several interesting algorithms are considered for equations where the skew-symmetric term is not a compact perturbation relative to the leading symmetric term.

We also use the following regularity result; cf. Grisvard [21] or Nečas [28].

(iv) The solution w of the adjoint equation

$$b(\phi, w) \;=\; g(\phi), \;\; \forall \phi \in H_0^1(\Omega)$$

satisfies

$$\|w\|_{H^{1+\sigma}(\Omega)} \leq C\|g\|_{L^2(\Omega)}.$$

Here σ depends on the interior angles of $\partial\Omega$, is independent of g and is at least $1/2$.

Let V^h be a finite dimensional subspace of $H_0^1(\Omega)$ with a mesh parameter h; details are provided in the next section. The Galerkin approximation of equation (13) is:

Find $u_h^\star \in V^h$ such that

$$b(u_h^\star, v_h) \;=\; (f, v_h), \;\; \forall v_h \in V^h. \tag{15}$$

We will now discuss several iterative methods for solving equation (15).

4. Two-level Schwarz Type Methods. In this section, we consider two classes of two-level methods that use overlapping and non-overlapping subregions, respectively. We begin by describing the two-level overlapping decomposition of Ω, which was introduced by Dryja and Widlund in [14]; see also Dryja and Widlund [16] for a fuller discussion.

4.1. A Two-level Subspace Decomposition with Coloring. Let $\Omega \subset R^d$ be a given a polygonal region and let $\{\Omega_i\}_{i=1}^N$ be a shape regular, coarse finite element triangulation of Ω. Here the Ω_i are non-overlapping d-dimensional simplices, i.e. triangles if $d = 2$ and tetrahedra if $d = 3$, with diameters on the order of H. The Ω_i are also called substructures and $\{\Omega_i\}$ the coarse mesh or H-level subdivision of Ω. In a second step, each substructure Ω_i, and the entire domain, are further divided into elements with diameters on the order of h. These smaller simplices also form a shape regular finite element subdivision of Ω. This is the fine mesh or h-level subdivision.

The finite element spaces of continuous, piecewise linear function on these triangulations are denoted by V^H and V^h, respectively. All elements of these spaces vanish on $\partial\Omega$ and V^H and V^h are therefore subspaces of $H_0^1(\Omega)$.

We introduce overlap between the subregions by extending each subregion Ω_i to a larger region Ω_i^{ext}; $\Omega_i \subset \Omega_i^{ext}$ with distance$(\partial\Omega_i^{ext} \cap \Omega, \partial\Omega_i \cap \Omega) \geq \;\alpha H, \;\; \forall i$. Here α is a positive constant. The same construction is used for the subregions that meet the boundary except that we cut off the parts that are outside Ω. In this construction, we also make sure that $\partial\Omega_i^{ext}$ does not cut through any h-level elements.

We note that recent results by Dryja and Widlund [33] provide bounds on the rate of convergence as a function of α and that, in our experience, the performance is often quite satisfactory even when the overlap is on the order h; cf. Cai, Gropp, and Keyes [9].

We associate an undirected graph with the decomposition $\{\Omega_i^{ext}\}$. Each node of the graph represents an extended subdomain and each edge the intersection of two such subdomains. This graph can be colored, using colors $1, \cdots, J$, in such a way that no nodes of the same color are connected by an edge of the graph. We merge all subdomains of the same color and denote the resulting sets by $\Omega_1', \cdots, \Omega_J'$. Let $V_i^h = V^h \cap H_0^1(\Omega_i')$. (By extending all functions of V_i^h by zero outside Ω_i', we see that $V_i^h \subset V^h$.) For convenience, we set $\Omega_0' = \Omega$ and associate it with color 0. We also use the subspace $V_0^h = V^H$ in our algorithm.

It is easy to see that V^h is the sum of the $J+1$ subspaces;

$$V^h \; = \; V_0^h + V_1^h + \cdots + V_J^h. \tag{16}$$

All the results given in the next subsection are valid for this decomposition, but the algorithms can equally well be used for other choices of the subspaces.

4.2. Algorithms and Convergence Rates Estimates. We begin by introducing oblique projections $P_i : V \longrightarrow V_i$, by

$$b(P_i u_h, v_h) = b(u_h, v_h), \;\; \forall u_h \in V^h, \;\; v_h \in V_i, \; 0 \leq i \leq J.$$

It is often more economical to use approximate rather than exact solvers of the problems on the subspaces. The approximate solvers are introduced in terms of bilinear forms $b_i(u,v)$, defined on $V_i^h \times V_i^h$, such that

$$a(u,u) \leq \omega_b b_i(u,u) \quad \text{and} \quad b_i(u,v) \leq C\|u\|_a \|v\|_a, \; \forall u,v \in V_i^h. \tag{17}$$

Here ω_b is a constant in $(0,2)$. A possible choice is $b_i(u,v) = a(u,v)$ or the bilinear form corresponding to the Laplace operator or to an inexact solver for one of the corresponding finite element problems.

The operators $T_i : V \longrightarrow V_i$, are defined by these bilinear forms:

$$b_i(T_i u_h, v_h) = b(u_h, v_h), \;\; \forall v_h \in V_i.$$

For $i = 0$, we must always, in order to obtain our theoretical results, use an exact solver. Thus, we choose $T_0 = P_0$.

We note that $P_i u_h^\star$ and $T_i u_h^\star$ can be computed, without explicit knowledge of $u_h^\star$, by solving a problem in the subspace V_i.

$$b(P_i u_h^\star, v_h) = b(u_h^\star, v_h) = (f, v_h), \;\; \forall v_h \in V_i^h,$$

or

$$b_i(T_i u_h^\star, v_h) = b(u_h^\star, v_h) = (f, v_h), \;\; \forall v_h \in V_i^h.$$

We now describe the classical Schwarz alternating algorithm in terms of the mappings T_i; we can consider the case of exact oblique projections as a special case.

Algorithm 1 (The classical Schwarz algorithm).

i) Compute $g_i = T_i u_h^\star$, *for* $i = 0, \cdots J$;

ii) Iterate until convergence: Obtain u_h^{n+1}, *the* $(n+1)th$ *approximate solution, from* u_h^n *using* $J+1$ *fractional steps*

$$u_h^{n+\frac{i+1}{J+1}} = u_h^{n+\frac{i}{J+1}} + \left(g_i - T_i u_h^{n+\frac{i}{J+1}}\right), \quad i = 0, \cdots, J.$$

We can regard this algorithm as a Richardson iterative method; cf. discussion in Section 1. More powerful iterative methods can also be used to accelerate the convergence. We recall that the multiplicative Schwarz operator is defined by the operator

$$E_J = (I - T_J)(I - T_{J-1}) \cdots (I - T_1)(I - P_0).$$

Since the polynomial $I - E_J$ does not contain any constant terms, we can compute

$$g_h = (I - E_J)u_h^\star, \tag{18}$$

without knowing the solution $u_h^\star$. We obtain,

Algorithm 2 (The accelerated multiplicative Schwarz algorithm).

i) Compute $g_h = (I - E_J)u_h^\star$;

ii) Solve the operator equation

$$(I - E_J)u_h = g_h \tag{19}$$

by a conjugate gradient-type iterative method, such as GMRES.

We remark that if the T_i are symmetric, positive semi-definite, then the operator E_J can be symmetrized by doubling the number of fractional steps, reversing the order of the subspaces. We can then use the standard conjugate gradient method, in the inner product $a(\cdot, \cdot)$, to solve a linear system with the operator $I - E_J^T E_J$.

The additive variant of the two-level multiplicative Schwarz algorithm, considered here, is given in terms of the operator $T = P_0 + T_1 + \cdots + T_J$.

Algorithm 3 (The additive Schwarz algorithm).

i) Compute $g_h = T u_h^\star$;

ii) Solve the operator equation

$$T u_h = g_h \tag{20}$$

by a conjugate gradient-type iterative method, e.g. the conjugate gradient method if T *is symmetric, positive definite and the GMRES method otherwise.*

To prove the convergence of these algorithms for our class of nonsymmetric and indefinite elliptic problems, we use a lemma that shows that the contribution from the skew-symmetric and zero order terms are of a lower order in H.

Lemma 5. *There exists a constant C, independent of H and h, such that, for all $u_h \in V^h$,*

(i) $|s(u_h, P_i u_h)| \le CH(a(u_h, u_h) + a(P_i u_h, P_i u_h))$ *for* $i > 0$*;*

(ii) $|s(u_h - P_i u_h, P_i u_h)| \le CH(a(u_h, u_h) + a(P_i u_h, P_i u_h))$ *for* $i > 0$*.*

For $i = 0$, (ii) holds with H replaced by H^σ. The same estimates hold if we replace the bilinear form $s(\cdot,\cdot)$ by $c(\cdot,\cdot)$ and/or P_i by T_i.

The proof for the exact oblique projections follows directly from Section 4 of Cai and Widlund [10]. For general T_i the result follows from a minor modification of these arguments.

We can now prove that Assumption 1 is satisfied for the mappings P_i and T_i.

Lemma 6. *For $i > 0$, there exists a constant $H_0 > 0$, such that for $H \le H_0$ Assumption 1 is satisfied with $\delta_i = 4CH$ and*

$$\gamma = \begin{cases} 1 - 2CH > 0 & \text{for } P_i \\ 2/\omega_b - 1 - CH > 0 & \text{for } T_i. \end{cases}$$

Here H is the coarse mesh size and C the constant in Lemma 5. For $i = 0$, the same estimates hold with H replaced by H^σ.

Proof. We give a proof of the lemma only for the P_i; the proof for the T_i can be obtained similarly with the aid of inequality (i) of Lemma 5.

We must establish that

$$a(P_i u_h, u_h) \ge \frac{1+\gamma}{2} a(P_i u_h, P_i u_h) - \frac{\delta_i}{2} a(u_h, u_h), \quad \forall u_h \in V^h.$$

From the definition of P_i, it easily follows that

$$a(P_i u_h, u_h) = a(P_i u_h, P_i u_h) - s(u_h - P_i u_h, P_i u_h) - c(u_h - P_i u_h, P_i u_h). \tag{21}$$

The proof is concluded by bounding the second and third terms using (ii) of Lemma 5. $\square$

We refer to Section 4 of Cai and Widlund [10], or Lemma 8 of this paper, for a proof of Assumption 2, i.e. that

$$\sum_{i=0}^{J} P_i^T P_i \ge C_0^{-2} I.$$

Here C_0 is independent of the mesh parameters h and H. A minor modification of the proof in [10] shows that this bound also holds if P_i is replaced by T_i.

In the study of Schwarz methods with this subspace decomposition and the coloring introduced earlier in this section, a bound for $\rho(\mathcal{E})$ is very easy to obtain; we only need the elementary inequality $\rho(\mathcal{E}) \le J$.

We can now summarize our results for this two-level decomposition. We note that the constant c generally depends on ω_b.

Theorem 2. *There exist constants $H_0 > 0$ and $c(H_0) > 0$, such that if $H \le H_0$, then*

$$\|E_J u_h\|_a^2 \le \left(1 - \frac{c}{(J^2 + (\sum \delta_i)^2 + 1) C_0^2}\right) \|u_h\|_a^2, \quad \forall u_h \in V^h. \tag{22}$$

Here J is the number of colors used for the set of extended subregions.

The proof follows directly from the abstract theory and the previous results of this section.

For the additive Schwarz algorithm, we similarly obtain

Theorem 3. *There exist constants $H_0 > 0$ and $c(H_0) > 0$, such that if $H \le H_0$, then*

$$\|Tu_h\|_a \le C(J + \sum \delta_i + 1)\|u_h\|_a, \quad \forall u_h \in V^h, \tag{23}$$

and

$$a(Tu_h, u_h) \ge cC_0^{-2}\|u_h\|_a^2, \quad \forall u_h \in V^h. \tag{24}$$

We note that a proof of Theorem 3 is given already in Cai and Widlund [10].

4.3. Iterative Substructuring Algorithms in Two Dimensions. We now consider an iterative substructuring method for problems in two dimensions. In defining the partition of the finite element space into subspaces, we use the same coarse space V^H as in Section 4.1. In addition, we use local subspaces corresponding to the subregions $\Omega_{ij} = \Omega_i \bigcup \Gamma_{ij} \bigcup \Omega_j$, which play the same role as the Ω_i^{ext} in Section 4.2. Here Ω_i and Ω_j are adjacent substructures with a common edge Γ_{ij}. We note that an interior substructure is the intersection of three such regions. By coloring the subdomains, as in the previous subsection, we obtain $\Omega_0' = \Omega, \Omega_1', \cdots, \Omega_J'$ where each $\Omega_i', i > 0$, is a union of nonoverlapping subregions that share the same color. As before, the local subspaces are defined by $V_i^h = H_0^1(\Omega_i') \cap V^h$. It is easy to show that

$$V^h = V_0^h + V_1^h + \cdots + V_J^h.$$

We can now introduce additive and multiplicative Schwarz algorithms based on this decomposition. For this decomposition, the constant C_0^2 can be estimated by

$$C_0^2 = C\left(1 + \log(H/h)\right)^2,$$

where C is independent of H and h; cf. Dryja and Widlund [17]. Theorems 2 and 3 hold with this C_0^2. The estimates of the other parameters, such as $\rho(\mathcal{E}) \le J$, can easily be found using the techniques as before. We note that a proof of the result for the additive case is given already in Cai and Widlund [10]. The corresponding problems for three dimensions appears to be open.

5. Multilevel Schwarz Type Methods. In this section, we consider a class of additive and multiplicative Schwarz methods using $(L+1)$ rather than two levels.

Following Dryja and Widlund [18] and Zhang [37], we introduce a sequence of nested quasi-uniform finite element triangulations $\{\mathcal{T}^l\}_{l=0}^L$. Here $\mathcal{T}^0 = \{\tau_i^0\}_{i=1}^{N_0}$ is the coarsest triangulation and τ_i^0 represents a substructure. The successively finer triangulations $\mathcal{T}^l = \{\tau_i^l\}_{i=1}^{N_l}$, $l = 1, \cdots, L$, are obtained by dividing each element of the triangulation

$\mathcal{T}^{l-1}$ into several elements. Let $h_i^l = \mathrm{diam}(\tau_i^l)$, $h_l = \max_i\{h_i^l\}$, $H = \max_i h_i^0$ and $h = h_L$. We also assume that there exists a constant $0 < r < 1$ such that $h_l = O(r^l H)$. Let V^l be the finite element space of continuous, piecewise linear functions associated with $\mathcal{T}^l$.

On each level, except the coarsest, we introduce and color an overlapping subdomain decomposition $\Omega = \cup_{i=1}^{J_l} \hat{\Omega}_i^l$. Here J_l is the number of subdomains on level l, each corresponding to a color. We note that there is a fixed upper bound J for J_l. Such a construction has already been introduced in Dryja and Widlund [18, 33] and it is quite similar to that described in Subsection 4.1 for two levels. We assume, as before, that the overlap is relative generous and uniform as measured by the parameter α.

Let $V_i^l = V^l \cap H_0^1(\hat{\Omega}_i^l)$, $i = 1, \cdots, J_l$, $l = 1, \cdots, L$, be subspaces of V^h and set $J_0 = 1$ and $V^0 = V_1^0 = V^H$. The finite element space $V^h = V^L$ can be represented as

$$V^L = \sum_{l=0}^{L} V^l = \sum_{l=0}^{L}\sum_{i=1}^{J_l} V_i^l. \tag{25}$$

Xuejun Zhang [37] has shown that the decomposition (25) is uniformly bounded in the sense of the following Lemma

Lemma 7. *For any $u \in V^h$, there exist $u_i^l \in V_i^l$, such that*

$$u \ = \ \sum_{l=0}^{L}\sum_{i=1}^{J_l} u_i^l.$$

Moreover, there exists a constant C_0, which is independent of the parameters h, H and L, such that

$$\sum_{l=0}^{L}\sum_{i=1}^{J_l} \|u_i^l\|_a^2 \le C_0^2 \|u\|_a^2, \ \ \forall u \in V^h.$$

This result is first established, in Zhang [37], under the assumption of H^2- regularity, e.g. in the case of convex regions, and then a proof, based on a recent result by Oswald [29], is given in the general regularity-free case.

For $0 \le l \le L$, $1 \le i \le J_l$, we define the mapping $P_i^l : V^h \to V_i^l$, by

$$b(P_i^l u, \phi) = b(u, \phi), \ \ \forall \phi \in V_i^l,$$

and similarly, for $1 \le l \le L$ and $1 \le i \le J_l$, we define $T_i^l : V^h \to V_i^l$, by

$$b_i(T_i^l u, \phi) = b(u, \phi), \ \ \forall \phi \in V_i^l.$$

As before, we choose $T_1^0 = T_0 = P_1^0 = P_0$.

The techniques of the proof of Lemma 6 can be applied directly to show that Assumption 1 holds for the mappings P_i^l and T_i^l. The estimates for δ_i can be obtained in the same way as in Lemma 6 with H replaced by h_l for the mappings defined for the level l subspaces.

We can now turn to Assumption 2.

LEMMA 8. *Assumption 2 holds, i.e. there exist positive constants H_0 and $C_0(H_0)$, such that if $H \leq H_0$, then*

$$\sum_{l=0}^{L} \sum_{i=1}^{J_l} (P_i^l)^T P_i^l \geq C_0^{-2} I \ .$$

The estimate also holds if the P_i^l are replaced by the T_i^l.

Proof. Our point of departure is an inequality established in Lemma 5 of Cai and Widlund [10]:

$$(1 - CH^{2\sigma}) a(u,u) \leq b(u,u) + C \|P_0 u\|_a \|u\|_a, \quad \forall u \in V^h. \tag{26}$$

From the definition of the operators P_i^l and Lemma 7, we find that

$$b(u,u) = \sum_{l=0}^{L} \sum_{i=1}^{J_l} b(u, u_i^l) = \sum_{l=0}^{L} \sum_{i=1}^{J_l} b(P_i^l u, u_i^l), \quad \forall u \in V^h.$$

From the continuity of $b(\cdot,\cdot)$ follows that

$$\sum_{l=0}^{L} \sum_{i=1}^{J_l} b(P_i^l u, u_i^l) \leq C \sum_{l=0}^{L} \sum_{i=1}^{J_l} \|P_i^l u\|_a \|u_i^l\|_a.$$

By Lemma 7 and the Cauchy-Schwarz inequality, this expression can be bounded from above by

$$CC_0 (\sum_{l=0}^{L} \sum_{i=1}^{J_l} \|P_i^l u\|_a^2)^{1/2} \|u\|_a.$$

Finally, by using (26), we obtain

$$a(u,u) \ \leq \ CC_0^2 \ \sum_{l=0}^{L} \sum_{i=1}^{J_l} a(P_i^l u, P_i^l u),$$

for sufficiently small H. $\square$

We define the multilevel additive Schwarz operator by

$$T^{(L)} = P_0 + \sum_{l=1}^{L} \sum_{i=1}^{J_l} T_i^l$$

and the multilevel multiplicative Schwarz operator by

$$E_J^{(L)} = \prod_{l=1}^{L} \prod_{i=1}^{J_l} \left(I - T_i^l \right) (I - P_0). \tag{27}$$

To fully analyze the convergence rates of the algorithms based on these operators, we need to estimate the spectral radius of $\mathcal{E}$. Here we can use a result due to Zhang

[37], which provides bounds on the parameters $\varepsilon_{i,j}^{l,k}$ for the subspace decomposition $\{V_i^l\}$ and the mappings $\{P_i^l\}$ considered in this section.

LEMMA 9. *The following strengthened Cauchy-Schwarz inequalities hold:*

$$|a(P_i^l u, P_j^k v)| \leq \varepsilon_{i,j}^{l,k} \|P_i^l u\|_a \|P_j^k v\|_a.$$

Here $0 \leq \varepsilon_{i,j}^{l,k} \leq C(r^d)^{|l-k|}$, *where* $d = 2$ *or* 3 *is the dimension of the space.*

It is now easy to show that $\rho(\mathcal{E}) \leq O(J)$ by using Gershgorin's theorem. By using the fact that $\delta_i^l = O(h_l) = O(Hr^l)$, we find that

$$\sum_{l=0}^{L} \sum_{i=1}^{J_l} \delta_i^l \leq CH \left(1 + \frac{Jr}{1-r}\right).$$

This sum can therefore be made arbitrarily small, and we can therefore satisfy the assumption of Theorem 1. Using the general theory, we obtain

THEOREM 4. *There exists a constant* $H_0 > 0$, *such that for* $H \leq H_0$,

$$\|T^{(L)}\|_a \leq C\,(J+1)\,, \quad \forall v \in V^h,$$

and

$$a(T^{(L)}v, v) \geq cC_0^{-2} \|v\|_a^2, \quad \forall v \in V^h.$$

Here c *and* C *may depend on* H_0 *but they do not depend on* h, H *and* L.

Similarly, we obtain

THEOREM 5. *There exists a constant* $H_0 > 0$, *such that if* $H \leq H_0$, *then*

$$\|E_M^{(L)} v\|_a \leq \sqrt{1 - \frac{c}{(J+1)^2 C_0^2}}\ \|v\|_a, \quad \forall v \in V^h.$$

Here $c > 0$ *may depend on* H_0 *but it does not depend on* H, h *and* L. J *is the maximum number of colors used, on each level, to color the extended subregions.*

In conclusion, we note that there are other multilevel decompositions for which the general framework and abstract theory can be used to obtain new results. Among them are Yserentant's decomposition, cf. Bank, Dupont, and Yserentant [1] and Yserentant [36], and the multilevel diagonal scaling method developed by Zhang [37]. This latter method can be viewed as a generalization of the BPX method due to Bramble, Pasciak, and Xu [5].

REFERENCES

[1] R. E. Bank, T. F. Dupont, and H. Yserentant, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427 – 458.

[2] P. E. Bjørstad and O. B. Widlund, *To overlap or not to overlap: A note on a domain decomposition method for elliptic problems*, SIAM J. Sci. Stat. Comput., 10 (1989), pp. 1053 – 1061.

[3] J. H. Bramble, Z. Leyk, and J. E. Pasciak, *Iterative schemes for non-symmetric and indefinite elliptic boundary value problems*, tech. rep., Cornell University, 1991.

[4] J. H. Bramble, J. E. Pasciak, J. Wang, and J. Xu, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.

[5] J. H. Bramble, J. E. Pasciak, and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.

[6] X.-C. Cai, *Some Domain Decomposition Algorithms for Nonselfadjoint Elliptic and Parabolic Partial Differential Equations*, PhD thesis, Courant Institute of Mathematical Sciences, September 1989. Tech. Rep. 461, Department of Computer Science, Courant Institute.

[7] ——, *An additive Schwarz algorithm for nonselfadjoint elliptic equations*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, , J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1990.

[8] ——, *Additive Schwarz algorithms for parabolic convection-diffusion equations*, Numer. Math., 60 (1991), pp. 41–61.

[9] X.-C. Cai, W. D. Gropp, and D. E. Keyes, *A comparison of some domain decomposition algorithms for nonsymmetric elliptic problems*, in Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Philadelphia, PA, 1992, SIAM. To appear.

[10] X.-C. Cai and O. Widlund, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Statist. Comput., 13 (1992), pp. 243–258.

[11] X.-C. Cai and J. Xu, *A preconditioned GMRES method for nonsymmetric or indefinite problems*, Math. Comp., 59 (1992). To appear in the October issue.

[12] M. Dryja, *An additive Schwarz algorithm for two- and three-dimensional finite element elliptic problems*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, PA, 1989, SIAM.

[13] M. Dryja, B. F. Smith, and O. B. Widlund, *Schwarz analysis of iterative substructuring algorithms for problems in three dimensions*, tech. rep., Department of Computer Science, Courant Institute, 1992. To appear.

[14] M. Dryja and O. B. Widlund, *An additive variant of the Schwarz alternating method for the case of many subregions*, Tech. Rep. 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[15] ——, *On the optimality of an additive iterative refinement method*, in Proceedings of the Fourth Copper Mountain Conference on Multigrid Methods, Philadelphia, PA, 1989, SIAM, pp. 161–170.

[16] ——, *Some domain decomposition algorithms for elliptic problems*, in Iterative Methods for Large Linear Systems, L. Hayes and D. Kincaid, eds., San Diego, California, 1989, Academic Press, pp. 273–291. Proceeding of the Conference on Iterative Methods for Large Linear Systems held in Austin, Texas, October 19 - 21, 1988, to celebrate the sixty-fifth birthday of David M. Young, Jr.

[17] ——, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20-22, 1989, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1990.

[18] ——, *Multilevel additive methods for elliptic finite element problems*, in Parallel Algorithms for Partial Differential Equations, Proceedings of the Sixth GAMM-Seminar, Kiel, January 19–21, 1990, W. Hackbusch, ed., Braunschweig, Germany, 1991, Vieweg & Son.

[19] ——, *Additive Schwarz methods for elliptic finite element problems in three dimensions*, in Fifth Conference on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Philadelphia, PA, 1992, SIAM. To appear.

[20] S. C. Eisenstat, H. C. Elman, and M. H. Schultz, *Variational iterative methods for nonsymmetric systems of linear equations*, SIAM J. Numer. Anal., 20 (2) (1983), pp. 345–357.

[21] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Publishing, Boston, 1985.

[22] P. L. Lions, *On the Schwarz alternating method. I.*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, PA, 1988, SIAM.

[23] T. P. Mathew, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretisations of Elliptic Problems*, PhD thesis, Courant Institute of Mathematical Sciences, September 1989. Tech. Rep. 463, Department of Computer Science, Courant Institute.

[24] ——, *Schwarz alternating, iterative refinement and Schur complement based methods for mixed formulations of elliptic problems, part II: Convergence theory*, tech. rep., UCLA, 1991. Submitted to Numer. Math.

[25] ——, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and numerical results*, Numerische Mathematik, (1992). To appear.

[26] A. M. Matsokin and S. V. Nepomnyaschikh, *A Schwarz alternating method in a subspace*, Soviet Mathematics, 29(10) (1985), pp. 78–84.

[27] S. V. Nepomnyaschikh, *Domain Decomposition and Schwarz Methods in a Subspace for the Approximate Solution of Elliptic Boundary Value Problems*, PhD thesis, Computing Center of the Siberian Branch of the USSR Academy of Sciences, Novosibirsk, USSR, 1986.

[28] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[29] P. Oswald, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, in Proc. Int. Conf. Theory of Functions, Varna 91, 1991. To appear.

[30] H. A. Schwarz, *Gesammelte Mathematische Abhandlungen*, vol. 2, Springer, Berlin, 1890, pp. 133–143. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, volume 15, 1870, pp. 272–286.

[31] B. F. Smith, *Domain Decomposition Algorithms for the Partial Differential Equations of Linear Elasticity*, PhD thesis, Courant Institute of Mathematical Sciences, September 1990. Tech. Rep. 517, Department of Computer Science, Courant Institute.

[32] O. B. Widlund, *Optimal iterative refinement methods*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, PA, 1989, SIAM.

[33] ——, *Some Schwarz methods for symmetric and nonsymmetric elliptic problems*, in Fifth Conference on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, D. E. Keyes, G. A. Meurant, J. S. Scroggs, and R. G. Voigt, eds., Philadelphia, PA, 1992, SIAM. To appear.

[34] J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Review, 34 (1992).

[35] ——, *A new class of iterative methods for nonselfadjoint or indefinite problems*, SIAM J. Numer. Anal., 29 (1992), pp. 303–319.

[36] H. Yserentant, *On the multi-level splitting of finite element spaces*, Numer. Math., 49 (1986), pp. 379–412.

[37] X. Zhang, *Multilevel Schwarz methods*, Tech. Rep. CS-TR-2894 (UMIACS-TR-92-52), University of Maryland, Department of Computer Science, May 1992. To appear in Numer. Math. A preliminary version appeared as Courant Institute report, 582, in September 1991.