H^1-NORM ERROR BOUNDS FOR PIECEWISE HERMITE BICUBIC ORTHOGONAL SPLINE COLLOCATION SCHEMES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. Two piecewise Hermite bicubic orthogonal spline collocation schemes are considered for the approximate solution of elliptic, self-adjoint, nonhomogeneous Dirichlet boundary value problems on rectangles. In the first scheme, the nonhomogeneous Dirichlet boundary condition is approximated by means of the piecewise Hermite cubic interpolant, while the piecewise cubic interpolant at the boundary Gauss points is used for the same purpose in the second scheme. The piecewise Hermite bicubic interpolant of the exact solution of the boundary value problem is used as a comparison function to show that the H^1-norm of the error for each scheme is $O(h^3)$.

Key words. Dirichlet boundary value problem, piecewise Hermite bicubics, Gauss points, orthogonal spline collocation, interpolant, H^1-norm error.

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1. Introduction. We consider two piecewise Hermite bicubic orthogonal spline collocation schemes for the solution of the nonhomogeneous Dirichlet boundary value problem

$$\begin{aligned} Lu &= f(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ u &= g(x,y), \quad (x,y) \in \partial\Omega, \end{aligned} \tag{1}$$

where $\partial\Omega$ is the boundary of Ω and L is the elliptic, self-adjoint operator given by

$$Lu = -\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u}{\partial y}\right) + c(x,y)u. \tag{2}$$

In both schemes, the approximate solutions, which are continuously differentiable in $\overline{\Omega}$ and piecewise cubic in x and y, are defined by collocating the differential equation of (1) at the Gauss points. In the first scheme, the approximate solution on $\partial\Omega$ is equal to the piecewise Hermite cubic interpolant of g, while in the second scheme the approximate solution on $\partial\Omega$ is equal to the piecewise cubic interpolant of g at the boundary Gauss points. Taylor's theorem and the Bramble-Hilbert lemma are used to bound the truncation errors for both schemes. Then energy inequalities, derived from the Peano representation of the remainder in the two-point Gauss-Legendre quadrature, are used to establish the uniqueness (and hence existence) of the collocation solutions and their rates of convergence for a sufficiently small mesh size h of the partition of Ω. It is shown that the H^1-norm error bounds for both schemes are $O(h^3)$, provided that the exact solution u of (1) belongs to $H^5(\Omega)$ in the case of the first scheme, and $H^5(\Omega) \cap C^4(\overline{\Omega})$ in the case of the second scheme.

For the homogeneous Dirichlet boundary value problem ((1) with $g = 0$), the L^2 and H^1 norm error analyses of piecewise Hermite bicubic orthogonal spline collocation

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were given in [7] and [8]. However, in [8], assumptions on the existence of the collocation solution and boundedness of partial derivatives of certain divided difference quotients were imposed. In [7], these assumptions were removed and the error analysis was carried out under the assumption that h be sufficiently small. However, the analysis of [7], which makes use of the finite element Galerkin solution as a comparison function, appears to be applicable only to homogeneous Dirichlet boundary value problems (1) with the additional constraint that $a = b$.

In this paper, we use the piecewise Hermite bicubic interpolant $u_{\mathcal{H}}$ of the exact solution u as a comparison function. The success of our approach depends essentially on the rather surprising property of $u_{\mathcal{H}}$, namely that, for u sufficiently smooth,

$$\max_{\xi \in \mathcal{G}} \left| \frac{\partial^2 (u - u_{\mathcal{H}})}{\partial x^{2-i} \partial y^i}(\xi) \right| = O(h^3), \qquad i = 0, 2,$$

where $\mathcal{G}$ is the set of Gauss points in Ω. In comparison, it should be noted that

$$\max_{(x,y) \in \Omega} \left| \frac{\partial^2 (u - u_{\mathcal{H}})}{\partial x^{2-i} \partial y^i}(x, y) \right| = O(h^2), \qquad i = 0, 2,$$

where the exponent 2 on h is known to be optimal [1].

An outline of the paper is as follows. Preliminaries are given in Section 2. The piecewise Hermite bicubic orthogonal spline collocation schemes are defined in Section 3. The error analyses of the first and second schemes are given in Sections 4 and 5, respectively. In Section 6, we consider a class of boundary value problems for which the existence and uniqueness of collocation solutions as well as derivations of the corresponding error bounds require no restrictions on the size of h.

2. Preliminaries. Let $\{x_k\}_{k=0}^{N_x}$ and $\{y_l\}_{l=0}^{N_y}$ be two partitions of $[0, 1]$ such that

$$x_0 = 0 < x_1 < \cdots < x_{N_x - 1} < x_{N_x} = 1, \qquad y_0 = 0 < y_1 < \cdots < y_{N_y - 1} < y_{N_y} = 1.$$

Let $I_k^x = [x_{k-1}, x_k]$, $I_l^y = [y_{l-1}, y_l]$, $h_k^x = x_k - x_{k-1}$, $h_l^y = y_l - y_{l-1}$, and let

$$\underline{h}_x = \min_k h_k^x, \qquad \overline{h}_x = \max_k h_k^x, \qquad \underline{h}_y = \min_l h_l^y, \qquad \overline{h}_y = \max_l h_l^y,$$

$$h = \max(\overline{h}_x, \overline{h}_y).$$

As in [1], it will be assumed that the collection of partitions of Ω generated by $\{x_k\}_{k=0}^{N_x}$ and $\{y_l\}_{l=0}^{N_y}$ is regular, that is, there exist positive constants σ_1, σ_2, and σ_3 such that

$$\sigma_1 \overline{h}_x \le \underline{h}_x, \qquad \sigma_1 \overline{h}_y \le \underline{h}_y, \qquad \sigma_2 \le \frac{\overline{h}_x}{\overline{h}_y} \le \sigma_3.$$

Throughout the paper, C denotes a generic positive constant which may depend on σ_1, σ_2, and σ_3.

Let $\mathcal{M}_x$ and $\mathcal{M}_y$ be spaces of piecewise Hermite cubics defined by

$$\mathcal{M}_x = \{v \in C^1[0,1] : v|_{I_k^x} \in P_3\}, \quad \mathcal{M}_y = \{v \in C^1[0,1] : v|_{I_l^y} \in P_3\},$$

where P_3 denotes the set of polynomials of degree ≤ 3, and let

$$\mathcal{M}_x^0 = \{v \in \mathcal{M}_x : v(0) = v(1) = 0\}, \quad \mathcal{M}_y^0 = \{v \in \mathcal{M}_y : v(0) = v(1) = 0\},$$

$$\mathcal{M} = \mathcal{M}_x \otimes \mathcal{M}_y, \quad \mathcal{M}^0 = \mathcal{M}_x^0 \otimes \mathcal{M}_y^0.$$

In the following, $H^m(\Omega)$ denotes the Sobolev space equipped with the norm

$$||v||_{H^m(\Omega)} = \left(\sum_{0 \leq i+j \leq m} \left|\left| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right|\right|_{L^2(\Omega)}^2 \right)^{1/2},$$

where $|| \cdot ||_{L^2(\Omega)}$ is the standard L^2-norm. Also, $C^m(\overline{\Omega})$ denotes the set of all functions $v(x,y)$ such that $\partial^{i+j} v / \partial x^i \partial y^j$ are continuous in $\overline{\Omega}$ for all $0 \leq i+j \leq m$. Similarly, $C^{m,n}(\overline{\Omega})$ represents the set of all functions $v(x,y)$ such that $\partial^{i+j} v / \partial x^i \partial y^j$ are continuous in $\overline{\Omega}$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. If $v \in C^m(\overline{\Omega})$, then $||v||_{C^m(\overline{\Omega})}$ is defined by

$$||v||_{C^m(\overline{\Omega})} = \max_{0 \leq i+j \leq m} \max_{(x,y) \in \overline{\Omega}} \left| \frac{\partial^{i+j} v}{\partial x^i \partial y^j}(x,y) \right|.$$

For $u \in C^{1,1}(\overline{\Omega})$, let its piecewise Hermite bicubic interpolant $u_{\mathcal{H}} \in \mathcal{M}$ be defined by

$$\frac{\partial^{i+j}(u_{\mathcal{H}} - u)}{\partial x^i \partial y^j}(x_k, y_l) = 0, \quad 0 \leq k \leq N_x, \quad 0 \leq l \leq N_y, \quad 0 \leq i,j \leq 1.$$

It is well known that each $u \in C^{1,1}(\overline{\Omega})$ has a unique Hermite interpolant $u_{\mathcal{H}}$. Moreover, the following approximation result was proved in [1] (see also [2]).

Lemma 2.1. *If $u \in H^4(\Omega)$, then*

$$||u - u_{\mathcal{H}}||_{H^1(\Omega)} \leq C h^3 ||u||_{H^4(\Omega)}. \tag{3}$$

Let $\mathcal{G}_x = \{\xi_{k,i}^x\}_{k,i=1}^{N_x,2}$, $\mathcal{G}_y = \{\xi_{l,j}^y\}_{l,j=1}^{N_y,2}$ be the sets of Gauss points

$$\xi_{k,i}^x = x_{k-1} + h_k^x \xi_i, \quad \xi_{l,i}^y = y_{l-1} + h_l^y \xi_i,$$

where

$$\xi_1 = (3 - \sqrt{3})/6, \quad \xi_2 = (3 + \sqrt{3})/6, \tag{4}$$

and let

$$\mathcal{G} = \{(\xi^x, \xi^y) : \xi^x \in \mathcal{G}_x, \xi^y \in \mathcal{G}_y\}.$$

For u and v defined on $\mathcal{G}$, let $\langle u, v\rangle_{\mathcal{G}}$ and $||u||_{\mathcal{G}}$ be given by

$$\langle u, v\rangle_{\mathcal{G}} = \frac{1}{4}\sum_{k=1}^{N_x}\sum_{l=1}^{N_y} h_k^x h_l^y \sum_{i=1}^{2}\sum_{j=1}^{2}(uv)(\xi_{k,i}^x, \xi_{l,j}^y),$$

and

$$||u||_{\mathcal{G}} = \langle u, u\rangle_{\mathcal{G}}^{1/2}.$$

The formula defining $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ is obtained by applying to $\int\int_{\Omega}(uv)(x,y)\,dxdy$ the composite two-point Gauss-Legendre quadratures with respect to x and y. Clearly,

$$\langle u, v\rangle_{\mathcal{G}} = \sum_{l=1}^{N_y}\frac{h_l^y}{2}\sum_{j=1}^{2}\langle u(\cdot,\xi_{l,j}^y), v(\cdot,\xi_{l,j}^y)\rangle_x = \sum_{k=1}^{N_x}\frac{h_k^x}{2}\sum_{i=1}^{2}\langle u(\xi_{k,i}^x,\cdot), v(\xi_{k,i}^x,\cdot)\rangle_y, \tag{5}$$

where, for u and v defined on $\mathcal{G}_x$ and $\mathcal{G}_y$,

$$\langle u, v\rangle_x = \sum_{k=1}^{N_x}\frac{h_k^x}{2}\sum_{i=1}^{2}(uv)(\xi_{k,i}^x), \quad \langle u, v\rangle_y = \sum_{l=1}^{N_y}\frac{h_l^y}{2}\sum_{j=1}^{2}(uv)(\xi_{l,j}^y).$$

Corollary 5.3 of [7] implies that each $v \in \mathcal{M}^0$ is uniquely defined by its values at all Gauss points $\xi \in \mathcal{G}$. Therefore, if $\langle v, v\rangle_{\mathcal{G}} = 0$ and $v \in \mathcal{M}^0$, then $v = 0$. Hence, $\mathcal{M}^0$ can be regarded as a Hilbert space with $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ as an inner product.

Let Δ_h be the operator from $\mathcal{M}^0$ into $\mathcal{M}^0$ defined by

$$(\Delta_h v)(\xi) = \Delta v(\xi), \quad \xi \in \mathcal{G},$$

where Δ is the Laplacian. The following lemma gives the most important properties of the operator $-\Delta_h$.

LEMMA 2.2. *$-\Delta_h$ is a self-adjoint operator from $\mathcal{M}^0$ into $\mathcal{M}^0$. Moreover,*

$$C||v||_{H^1(\Omega)}^2 \le \langle -\Delta_h v, v\rangle_{\mathcal{G}}, \quad v \in \mathcal{M}^0, \tag{6}$$

$$C||v||_{\mathcal{G}}^2 \le \langle -\Delta_h v, v\rangle_{\mathcal{G}}, \quad v \in \mathcal{M}^0. \tag{7}$$

Proof. The first part of the lemma follows from Lemma 3.1 in [4]. The inequalities (6) and (7) are easily established using (2.6)–(2.8) of [7], and the Poincaré inequality $||v||_{H^1(\Omega)} \le C||\nabla v||_{L^2(\Omega)}$ for all $v \in H^1(\Omega)$ that vanish on $\partial\Omega$. □

3. The piecewise Hermite bicubic orthogonal spline collocation schemes. We consider two piecewise Hermite bicubic orthogonal spline collocation schemes for the boundary value problem (1). The schemes differ in the way the nonhomogeneous Dirichlet boundary condition is approximated. In the first scheme, the collocation solution $u_h^I \in \mathcal{M}$ is defined by requiring that

$$Lu_h^I(\xi) = f(\xi), \quad \xi \in \mathcal{G}, \tag{8}$$

and

$$(9) \qquad \frac{\partial^i (u_h^I - g)}{\partial x^i}(x_k, \alpha) = 0, \quad \alpha = 0,1, \quad 0 \le k \le N_x, \quad i = 0,1,$$

$$(10) \qquad \frac{\partial^i (u_h^I - g)}{\partial y^i}(\alpha, y_l) = 0, \quad \alpha = 0,1, \quad 0 \le l \le N_y, \quad i = 0,1.$$

In the second scheme, the collocation solution $u_h^{II} \in \mathcal{M}$ is defined by requiring that

$$(11) \qquad L u_h^{II}(\xi) = f(\xi), \quad \xi \in \mathcal{G},$$

and

$$(12) \qquad (u_h^{II} - g)(\xi^x, \alpha) = 0, \quad \alpha = 0,1, \quad \xi^x \in \mathcal{G}_x,$$

$$(13) \qquad (u_h^{II} - g)(\alpha, \xi^y) = 0, \quad \alpha = 0,1, \quad \xi^y \in \mathcal{G}_y,$$

$$(14) \qquad (u_h^{II} - g)(\alpha, \beta) = 0, \quad \alpha, \beta = 0,1.$$

Clearly, u_h^I and u_h^{II} on $\partial\Omega$ are the piecewise Hermite cubic interpolant of u and piecewise cubic interpolant of u at the boundary Gauss points, respectively. From a practical point of view, the second collocation scheme is preferable since it does not require the knowledge or evaluation of the first partial derivatives of g. Obviously, if $g = 0$, then both schemes coincide. However, if $g \neq 0$, then, in general, $u_h^I \neq u_h^{II}$. If u_h^I is expanded in terms of Hermite basis functions, then the coefficients in such expansion corresponding to the values of u_h^I on $\partial\Omega$ can be determined independently of all other coefficients. Therefore, after moving these coefficients to the right-hand side in (8), the scheme (8)–(10) with $g \neq 0$ can be reduced, from the computational point of view, to that with $g = 0$. Similar remarks apply also to the scheme (11)–(14), since the coefficients in the Hermite basis expansion of u_h^{II} corresponding to the values of u_h^{II} on $\partial\Omega$ can be first obtained by solving linear systems that typically arise in one-dimensional orthogonal spline collocation.

4. Convergence analysis of the first collocation scheme. First we show that if u is sufficiently smooth and $u_{\mathcal{H}}$ is its piecewise Hermite bicubic interpolant, then the truncation error $\max_{\xi \in \mathcal{G}} |L(u - u_{\mathcal{H}})(\xi)|$ is $O(h^3)$. Then we derive certain energy inequalities for the orthogonal spline collocation operator corresponding to L. Using these two results we are able to obtain an error bound on $||u - u_h^I||_{H^1(\Omega)}$.

4.1. Truncation error. The following lemmas are essential in the estimation of the truncation error.

LEMMA 4.1. *Let $v(x, y) = x^m y^n$, where m, n are nonnegative integers such that $m + n \le 4$. Let $\tilde{v}$ be the Hermite bicubic interpolant of v on Ω, that is, $v \in P_3 \otimes P_3$ and*

$$(15) \qquad \frac{\partial^{i+j} (\tilde{v} - v)}{\partial x^i \partial y^j}(\alpha, \beta) = 0, \quad \alpha, \beta = 0,1, \quad 0 \le i, j \le 1.$$

Then

$$\frac{\partial^2(\tilde{v}-v)}{\partial x^{2-i}\partial y^i}(\xi_p,\xi_q)=0, \quad p,q=1,2, \quad i=0,2, \tag{16}$$

where ξ_1 and ξ_2 are given by (4).

Proof. We verify (16) for $i = 0$ only, since the proof for $i = 2$ is similar. Set $f(x) = x^m$ and $g(y) = y^n$ so that $v(x,y) = f(x)g(y)$. Clearly, $\tilde{v}(x,y) = \hat{f}(x)\hat{g}(y)$, where $\hat{f}$ and $\hat{g}$ are the Hermite cubic interpolants of f and g on $[0,1]$, respectively, that is, $\hat{f}, \hat{g} \in P_3$, and

$$\hat{f}^{(i)}(\alpha)=f^{(i)}(\alpha), \quad \hat{g}^{(i)}(\alpha)=g^{(i)}(\alpha), \quad \alpha=0,1, \quad i=0,1.$$

Therefore,

$$\frac{\partial^2\tilde{v}}{\partial x^2}(\xi_p,\xi_q)=\hat{f}''(\xi_p)\hat{g}(\xi_q)=\frac{\partial^2 v}{\partial x^2}(\xi_p,\xi_q),$$

since $\hat{f}''(\xi_p) = f''(\xi_p)$ by (2.5) of [4] and since $\hat{g} = g$ for $n \le 3$, and $\partial^2\tilde{v}/\partial x^2 = \partial^2 v/\partial x^2 = 0$ if $n = 4$. $\square$

LEMMA 4.2. *Assume that $u \in H^4(\Omega)$, and let $u_{\mathcal{H}}$ be its piecewise Hermite bicubic interpolant. Then*

$$\left\|\frac{\partial^{i+j}(u-u_{\mathcal{H}})}{\partial x^i\partial y^j}\right\|_{\mathcal{G}} \le Ch^{4-i-j}||u||_{H^4(\Omega)}, \quad 0 \le i+j \le 1. \tag{17}$$

Moreover, if $u \in H^5(\Omega)$, then

$$\left\|\frac{\partial^2(u-u_{\mathcal{H}})}{\partial x^{2-i}\partial y^i}\right\|_{\mathcal{G}} \le Ch^3||u||_{H^5(\Omega)}, \quad i=0,2. \tag{18}$$

Proof. First we verify (17). Let $l^{i,j}_{p,q}$, $p,q = 1,2$, $0 \le i+j \le 1$, be the linear functional on $H^4(\Omega)$ such that

$$l^{i,j}_{p,q}v=\frac{\partial^{i+j}(v-\tilde{v})}{\partial x^i\partial y^j}(\xi_p,\xi_q),$$

where $\tilde{v}$ is the Hermite bicubic interpolant of v defined by (15), and ξ_1, ξ_2 are given by (4). By the Sobolev embedding theorem (see, for example, [2]), $l^{i,j}_{p,q}$ is a well defined bounded functional on $H^4(\Omega)$. Moreover, $l^{i,j}_{p,q}v = 0$ for all polynomials v of degree ≤ 3, since then $v = \tilde{v}$. Therefore, it follows from the Bramble-Hilbert lemma (see, for example, [2]) that

$$|l^{i,j}_{p,q}v| \le C|v|_{4,\Omega}, \quad v \in H^4(\Omega), \tag{19}$$

where

$$|v|^2_{4,\Omega}=\int\int_\Omega\sum_{m=0}^4\left|\frac{\partial^4 v}{\partial x^{4-m}\partial y^m}(x,y)\right|^2\,dxdy.$$

Assume now that $u \in H^4(\Omega)$, and let $u_{\mathcal{H}}$ be its piecewise Hermite bicubic interpolant. Let

$$\alpha_{k,l}^{i,j} = \left\{ \frac{h_k^x h_l^y}{4} \sum_{p=1}^{2} \sum_{q=1}^{2} \left[\frac{\partial^{i+j}(u - u_{\mathcal{H}})}{\partial x^i \partial y^j}(\xi_{k,p}^x, \xi_{l,q}^y) \right]^2 \right\}^{1/2},$$

$$\beta_{k,l} = |u|_{4, I_k^x \times I_l^y},$$

and let $v \in H^4(\Omega)$ be defined by $v(x,y) = u(x_{k-1} + x h_k^x, y_{l-1} + y h_l^y)$. Clearly,

$$\frac{\partial^{i+j}(u - u_{\mathcal{H}})}{\partial x^i \partial y^j}(\xi_{k,p}^x, \xi_{l,q}^y) = (h_k^x)^{-i} (h_l^y)^{-j} l_{p,q}^{i,j} v,$$

and hence (19) and a simple change of variables give

$$\alpha_{k,l}^{i,j} \leq C h^{4-i-j} \beta_{k,l}.$$

Therefore, (17) follows, since

$$\left|\left| \frac{\partial^{i+j}(u - u_{\mathcal{H}})}{\partial x^i \partial y^j} \right|\right|_{\mathcal{G}}^2 = \sum_{k=1}^{N_x} \sum_{l=1}^{N_y} (\alpha_{k,l}^{i,j})^2,$$

and

$$\sum_{k=1}^{N_x} \sum_{l=1}^{N_y} \beta_{k,l}^2 = |u|_{4,\Omega}^2 \leq ||u||_{H^4(\Omega)}^2.$$

The proof of (18) is similar, since by Lemma 4.1, in place of (19), we have

$$|l_{p,q}^{i,j} v| \leq C |v|_{5,\Omega}, \qquad v \in H^5(\Omega). \qquad \square$$

The following theorem gives a bound on the truncation error in the first collocation scheme.

Theorem 4.1. *Let L be given by (2), where $a \in C^{1,0}(\overline{\Omega})$, $b \in C^{0,1}(\overline{\Omega})$, $c \in C(\overline{\Omega})$. Assume that $u \in H^5(\Omega)$, and let $u_{\mathcal{H}}$ be its piecewise Hermite bicubic interpolant. Then*

$$||L(u - u_{\mathcal{H}})||_{\mathcal{G}} \leq C h^3 ||u||_{H^5(\Omega)}. \tag{20}$$

Proof. Inequality (20) follows easily from the triangle inequality for $||\cdot||_{\mathcal{G}}$, (17), and (18). $\square$

4.2. Energy inequalities. Let L_h be the operator from $\mathcal{M}^0$ into $\mathcal{M}^0$ defined by

$$(L_h v)(\xi) = Lv(\xi), \quad \xi \in \mathcal{G}, \tag{21}$$

where L is given by (2). The next result shows that L_h can be bounded from below, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, by the operator $-\Delta_h$.

THEOREM 4.2. *Assume that $a \in C^{5,0}(\overline{\Omega})$, $b \in C^{0,5}(\overline{\Omega})$, $c \in C(\overline{\Omega})$, and that*

$$a(x,y), b(x,y) > 0, \quad c(x,y) \geq 0, \quad (x,y) \in \overline{\Omega}.$$

Then

$$(c_1 - Cc_2 h)\langle -\Delta_h v, v\rangle_{\mathcal{G}} \leq \langle L_h v, v\rangle_{\mathcal{G}}, \quad v \in \mathcal{M}^0, \tag{22}$$

where the positive constants c_1, c_2 are given by

$$c_1 = \min_{(x,y)\in\overline{\Omega}} [a(x,y), b(x,y)], \quad c_2 = \max_{1\leq l \leq 5} \max_{(x,y)\in\overline{\Omega}} \left[\left| \frac{\partial^l a}{\partial x^l}(x,y) \right|, \left| \frac{\partial^l b}{\partial y^l}(x,y) \right| \right].$$

Proof. We prove the theorem by adapting the approach of Cooper and Prenter (see proof of Theorem 4.4 in [3]). Assume that $\xi^y \in \mathcal{G}_y$. The Peano representation of the remainder in the two-point Gauss-Legendre quadrature (see, for example, Section 4.2 in [5]) and Leibnitz' formula give

$$\langle -\frac{\partial}{\partial x}\left(a \frac{\partial v}{\partial x}\right)(\cdot, \xi^y), v(\cdot,\xi^y)\rangle_x = I_1(a,v,\xi^y) + I_2(a,v,\xi^y), \tag{23}$$

where

$$\begin{aligned} I_1(a,v,\xi^y) &= \int_0^1 \left[a \left(\frac{\partial v}{\partial x}\right)^2 \right](x,\xi^y)\,dx \\ &\quad + 4\sum_{k=1}^{N_x} (h_k^x)^4 \int_{I_k^x} \left[a \left(\frac{\partial^3 v}{\partial x^3}\right)^2 \right](x,\xi^y) K\left(\frac{x - x_{k-1}}{h_k^x}\right)\,dx, \end{aligned}$$

$$I_2(a,v,\xi^y) = \sum_{l=1}^{5} \sum_{\substack{i+j=6-l \\ 0\leq i,j\leq 3}} \alpha_{i,j}^{(l)} \sum_{k=1}^{N_x} (h_k^x)^4 \int_{I_k^x} \left[\frac{\partial^l a}{\partial x^l} \frac{\partial^i v}{\partial x^i} \frac{\partial^j v}{\partial x^j}\right](x,\xi^y) K\left(\frac{x-x_{k-1}}{h_k^x}\right)\,dx,$$

the constants $\alpha_{i,j}^{(l)}$ are independent of h, and

$$0 \leq K(t) = \frac{1}{24}\left\{ (1-t)^4 - 2[(\xi_1 - t)_+^3 + (\xi_2 - t)_+^3]\right\} \leq C, \quad t \in [0,1]. \tag{24}$$

Since $I_2(1, v, \xi^y) = 0$, we find that

$$c_1^x \langle -\frac{\partial^2 v}{\partial^2 x}(\cdot,\xi^y), v(\cdot,\xi^y)\rangle_x \leq I_1(a,v,\xi^y), \tag{25}$$

where

$$c_1^x = \min_{(x,y)\in\overline{\Omega}} a(x,y).$$

On the other hand, (24) and the Cauchy Schwarz inequality in $L^2(I_k^x)$ give

$$|I_2(a,v,\xi^y)| \le C c_2^x \sum_{l=1}^{5} \sum_{\substack{i+j=6-l\\ 0\le i,j\le 3}} \sum_{k=1}^{N_x} (h_k^x)^4 \left\|\frac{\partial^i v}{\partial x^i}(\cdot,\xi^y)\right\|_{L^2(I_k^x)} \left\|\frac{\partial^j v}{\partial x^j}(\cdot,\xi^y)\right\|_{L^2(I_k^x)},$$

where

$$c_2^x = \max_{1\le l\le 5} \max_{(x,y)\in\overline{\Omega}} \left|\frac{\partial^l a}{\partial x^l}(x,y)\right|.$$

Hence, using the inverse inequality (cf., for example, [2])

$$(26) \qquad ||u^{(i)}||_{L^2(I_k^x)} \le C(h_k^x)^{l-i}||u^{(l)}||_{L^2(I_k^x)}, \quad 0\le l\le i\le 3, \quad u\in P_3,$$

with $l=1$, $2\le i\le 3$, the Cauchy Schwarz inequality in $\mathcal{R}^{N_x}$, and the Poincaré inequality $||u||_{L^2(0,1)} \le C||u'||_{L^2(0,1)}$ for $u\in\mathcal{M}_x^0$, we get

$$|I_2(a,v,\xi^y)| \le C c_2^x \overline{h}_x \left\|\frac{\partial v}{\partial x}(\cdot,\xi^y)\right\|_{L^2(0,1)}^2.$$

Further, Lemma 3.3 of [4] implies that

$$(27) \qquad |I_2(a,v,\xi^y)| \le C c_2^x \overline{h}_x \langle -\frac{\partial^2 v}{\partial^2 x}(\cdot,\xi^y), v(\cdot,\xi^y)\rangle_x.$$

Therefore, (23), (25), (27), and (5) yield

$$(28) \qquad (c_1^x - C c_2^x \overline{h}_x)\langle -\frac{\partial^2 v}{\partial^2 x}, v\rangle_{\mathcal{G}} \le \langle -\frac{\partial}{\partial x}\left(a\frac{\partial v}{\partial x}\right), v\rangle_{\mathcal{G}}.$$

Hence, (22) follows from (28) and the similar inequality for $\langle -\partial(b\partial v/\partial y)/\partial y, v\rangle_{\mathcal{G}}$. □

COROLLARY 4.1. *Let the functions a, b, and c satisfy the assumptions of Theorem 4.2. If h is sufficiently small, then L_h is an invertible operator from $\mathcal{M}^0$ onto $\mathcal{M}^0$.*

Proof. It is enough to show that if $L_h v = 0$ and $v\in\mathcal{M}^0$, then $v=0$. But this follows easily from (22) and (7), since $c_1 - Cc_2 h > 0$ for h sufficiently small. □

4.3. Error bound. Using results established in previous sections we prove the following theorem.

THEOREM 4.3. *Let the functions a, b, and c satisfy the assumptions of Theorem 4.2. Then, for h sufficiently small, there exists a unique collocation solution $u_h^I \in \mathcal{M}$ satisfying (8)–(10). Moreover, if $u\in H^5(\overline{\Omega})$ is a solution of the boundary value problem (1), then*

$$(29) \qquad ||u-u_h^I||_{H^1(\Omega)} \le Ch^3||u||_{H^5(\overline{\Omega})}.$$

Proof. The existence and uniqueness of u_h^I for sufficiently small h follow from Corollary 4.1. To show (29), we set $v = u_h^I - u_{\mathcal{H}}$, where $u_{\mathcal{H}}$ is the piecewise Hermite bicubic interpolant of u. Equations (1), (8) and inequality (20) give

$$||Lv||_{\mathcal{G}} = ||L(u - u_{\mathcal{H}})||_{\mathcal{G}} \leq Ch^3||u||_{H^5(\Omega)}. \tag{30}$$

Since $v \in \mathcal{M}^0$, (22), the Cauchy Schwarz inequality for $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, (21), and (30) imply

$$C\langle -\Delta_h v, v\rangle_{\mathcal{G}} \leq \langle L_h v, v\rangle_{\mathcal{G}} \leq Ch^3||u||_{H^5(\Omega)}||v||_{\mathcal{G}}.$$

Consequently, by (7) and (6),

$$C||v||_{H^1(\Omega)} \leq \langle -\Delta_h v, v\rangle_{\mathcal{G}}^{1/2} \leq Ch^3||u||_{H^5(\Omega)},$$

and hence (29) follows from the triangle inequality and (3). $\square$

5. Convergence analysis of the second collocation scheme. Since, in general, $u_h^{II} - u_{\mathcal{H}} \notin \mathcal{M}^0$, the piecewise Hermite bicubic interpolant $u_{\mathcal{H}}$ cannot be used directly as a comparison function in the error analysis of the second collocation scheme of Section 3. Therefore, we first introduce the piecewise bicubic Gauss-Hermite interpolant $u_{\mathcal{GH}}$ of u (so that $u_h^{II} - u_{\mathcal{GH}} \in \mathcal{M}^0$) and prove some approximation results for $u - u_{\mathcal{GH}}$. Then using an analysis similar to that of Section 4, we establish a bound on $||u - u_h^{II}||_{H^1(\Omega)}$.

5.1. Piecewise cubic Gauss and piecewise bicubic Gauss-Hermite interpolants. For $u \in C[0,1]$, let its piecewise cubic Gauss interpolant $u_{\mathcal{G}} \in \mathcal{M}_x$ be defined by

$$(u_{\mathcal{G}} - u)(\xi^x) = 0, \quad \xi^x \in \mathcal{G}_x, \qquad (u_{\mathcal{G}} - u)(\alpha) = 0, \quad \alpha = 0, 1.$$

The existence and uniqueness of the Gauss interpolant $u_{\mathcal{G}}$ for each $u \in C[0,1]$ are proved in Lemma 2.3 of [4]. Moreover, we have the following approximation result.

Lemma 5.1. *If $u \in C^4[0,1]$, then*

$$||(u - u_{\mathcal{G}})^{(j)}||_{C[0,1]} \leq C(\overline{h}_x)^{4-j}||u||_{C^4[0,1]}, \quad j = 0, 1. \tag{31}$$

Proof. Let $u_{\mathcal{H}}$ be the piecewise Hermite cubic interpolant of u. Then it follows from (2.17) in [1] (see also [2]) that

$$||(u - u_{\mathcal{H}})^{(j)}||_{C[0,1]} \leq C(\overline{h}_x)^{4-j}||u||_{C^4[0,1]}, \quad j = 0, 1. \tag{32}$$

Corollary 5.3 and Lemma 5.4 of [7] imply that any $v \in \mathcal{M}_x^0$ can be written in the form

$$v = \sum_{k=1}^{N_x}\sum_{i=1}^{2} v(\xi_{k,i}^x)\theta_{k,i}^x,$$

where the basis functions $\theta^x_{k,i}$ satisfy

$$||\theta^x_{k,i}||_{C(I^x_l)} \le C3^{-|k-l|}, \quad i = 1, 2, \;\; 1 \le k, l \le N_x.$$

Therefore, for any $x \in [0,1]$,

$$|(u_{\mathcal{G}} - u_{\mathcal{H}})(x)| = \left| \sum_{k=1}^{N_x} \sum_{i=1}^{2} (u - u_{\mathcal{H}})(\xi^x_{k,i}) \theta^x_{k,i}(x) \right| \le C||u - u_{\mathcal{H}}||_{C[0,1]}. \tag{33}$$

Hence (31) follows from the triangle inequality, (32), (33), and the inverse inequality

$$||v'||_{C[0,1]} \le C(\overline{h}_x)^{-1}||v||_{C[0,1]}, \quad v \in \mathcal{M}_x,$$

applied to $v = u_{\mathcal{G}} - u_{\mathcal{H}}$. $\square$

For $u \in C^{1,1}(\overline{\Omega})$, let its piecewise bicubic Gauss-Hermite interpolant $u_{\mathcal{GH}} \in \mathcal{M}$ be defined by

$$\frac{\partial^{i+j}(u_{\mathcal{GH}} - u)}{\partial x^i \partial y^j}(x_k, y_l) = 0, \;\; 1 \le k \le N_x - 1, \;\; 1 \le l \le N_y - 1, \;\; 0 \le i, j \le 1,$$

$$\frac{\partial^{1+i}(u_{\mathcal{GH}} - u)}{\partial x^i \partial y}(x_k, \alpha) = 0, \;\; \alpha = 0, 1, \;\; 1 \le k \le N_x - 1, \;\; i = 0, 1,$$

$$\frac{\partial^{1+j}(u_{\mathcal{GH}} - u)}{\partial x \partial y^j}(\alpha, y_l) = 0, \;\; \alpha = 0, 1, \;\; 1 \le l \le N_y - 1, \;\; j = 0, 1,$$

$$(u_{\mathcal{GH}} - u)(\alpha, \beta) = 0, \quad \frac{\partial^2(u_{\mathcal{GH}} - u)}{\partial x \partial y}(\alpha, \beta) = 0, \quad \alpha, \beta = 0, 1,$$

$$(u_{\mathcal{GH}} - u)(\xi^x, \alpha) = 0, \;\; \xi^x \in \mathcal{G}^x, \quad (u_{\mathcal{GH}} - u)(\alpha, \xi^y), \;\; \xi^y \in \mathcal{G}^y, \quad \alpha = 0, 1.$$

Clearly, on any side of $\overline{\Omega}$, $u_{\mathcal{GH}}$ is equal to the piecewise cubic Gauss interpolant of u with respect to x or y. Also, $u_{\mathcal{GH}} = u_{\mathcal{H}}$ on all interior cells of the partition of Ω (a cell $I^x_k \times I^y_l$ is interior if its boundary does not have common points with $\partial\Omega$). However, in general, $u_{\mathcal{GH}} \ne u_{\mathcal{H}}$ on boundary cells of the partition of Ω (a cell $I^x_k \times I^y_l$ is a boundary cell if its boundary has common points with $\partial\Omega$). It is easy to show that each $u \in C^{1,1}(\overline{\Omega})$ has a unique Gauss-Hermite interpolant $u_{\mathcal{GH}}$. In addition, we have the following approximation result.

Lemma 5.2. *If $u \in C^4(\overline{\Omega})$, then*

$$\left|\left| \frac{\partial^{i+j}(u - u_{\mathcal{GH}})}{\partial x^i \partial^j} \right|\right|_{C(\overline{\Omega})} \le Ch^{4-i-j}||u||_{C^4(\overline{\Omega})}, \quad 0 \le i + j \le 1. \tag{34}$$

Proof. Let $u_{\mathcal{H}}$ be the piecewise Hermite bicubic interpolant of u. Then for $(x,y) \in I_1^x \times I_1^y$,

$$\begin{aligned}(u_{\mathcal{H}} - u_{\mathcal{GH}})(x,y) &= \frac{\partial(u-u_{\mathcal{GH}})}{\partial x}(x_0,y_0)\psi_0^x(x)\phi_0^y(y) + \frac{\partial(u-u_{\mathcal{GH}})}{\partial y}(x_0,y_0)\phi_0^x(x)\psi_0^y(y) \\ &\quad + (u-u_{\mathcal{GH}})(x_0,y_1)\phi_0^x(x)\phi_1^y(y) + \frac{\partial(u-u_{\mathcal{GH}})}{\partial y}(x_0,y_1)\phi_0^x(x)\psi_1^y(y) \\ &\quad + (u-u_{\mathcal{GH}})(x_1,y_0)\phi_1^x(x)\phi_0^y(y) + \frac{\partial(u-u_{\mathcal{GH}})}{\partial x}(x_1,y_0)\psi_1^x(x)\phi_0^y(y),\end{aligned} \tag{35}$$

where ϕ_k^x, ψ_k^x, ϕ_l^y, and ψ_l^y are defined on $[0,1]$ (see, for example, Section 1.7 in [9]) by

$$\phi_k^x(x) = \begin{cases} \phi\left(\dfrac{x-x_k}{h_k^x}\right), x \le x_k, \\ \phi\left(\dfrac{x-x_k}{h_{k+1}^x}\right), x \ge x_k, \end{cases} \qquad \psi_k^x(x) = \begin{cases} h_k^x\psi\left(\dfrac{x-x_k}{h_k^x}\right), x \le x_k, \\ h_{k+1}^x\psi\left(\dfrac{x-x_k}{h_{k+1}^x}\right), x \ge x_k, \end{cases} \tag{36}$$

$$\phi_l^y(y) = \begin{cases} \phi\left(\dfrac{y-y_l}{h_l^y}\right), y \le y_l, \\ \phi\left(\dfrac{y-y_l}{h_{l+1}^y}\right), y \ge y_l, \end{cases} \qquad \psi_l^y(l) = \begin{cases} h_l^y\psi\left(\dfrac{y-y_l}{h_l^y}\right), y \le y_l, \\ h_{l+1}^y\psi\left(\dfrac{y-y_l}{h_{l+1}^y}\right), y \ge y_l, \end{cases} \tag{37}$$

$$\phi(t) = \begin{cases} (1+2|t|)(1-|t|)^2, & |t| \le 1, \\ 0, & |t| > 1, \end{cases} \qquad \psi(t) = \begin{cases} t(1-|t|)^2, & |t| \le 1, \\ 0, & |t| > 1. \end{cases} \tag{38}$$

It follows from Lemma 5.1 that

$$\begin{aligned} |(u-u_{\mathcal{GH}})(x_0,y_1)|, \quad |(u-u_{\mathcal{GH}})(x_1,y_0)| &\le Ch^4\|u\|_{C^4(\overline{\Omega})}, \\ \left|\frac{\partial(u-u_{\mathcal{GH}})}{\partial x}(x_0,y_0)\right|, \quad \left|\frac{\partial(u-u_{\mathcal{GH}})}{\partial y}(x_0,y_0)\right| &\le Ch^3\|u\|_{C^4(\overline{\Omega})}, \\ \left|\frac{\partial(u-u_{\mathcal{GH}})}{\partial y}(x_0,y_1)\right|, \quad \left|\frac{\partial(u-u_{\mathcal{GH}})}{\partial x}(x_1,y_0)\right| &\le Ch^3\|u\|_{C^4(\overline{\Omega})}. \end{aligned} \tag{39}$$

Using (36)–(38), we also find that

$$\begin{aligned} |\phi_0^x(x)|, |\phi_1^x(x)| \le C, \quad |[\phi_0^x]'(x)|, |[\phi_1^x]'(x)| \le Ch^{-1}, \quad x \in I_1^x, \\ |\psi_0^x(x)|, |\psi_1^x(x)| \le Ch, \quad |[\psi_0^x]'(x)|, |[\psi_1^x]'(x)| \le C, \quad x \in I_1^x. \end{aligned} \tag{40}$$

The functions ϕ_0^y, ϕ_1^y and ψ_0^y, ψ_1^y satisfy similar inequalities on I_1^y. Therefore, (35), (39), and (40) yield

$$\left\|\frac{\partial^{i+j}(u_{\mathcal{H}} - u_{\mathcal{GH}})}{\partial x^i \partial^j}\right\|_{C(I_1^x \times I_1^y)} \le Ch^{4-i-j}\|u\|_{C^4(\overline{\Omega})}, \quad 0 \le i+j \le 1.$$

Since similar inequalities also hold for all remaining boundary cells and since $u_{\mathcal{GH}} = u_{\mathcal{H}}$ on all interior cells, (34) follows from the triangle inequality and the error bound (see, for example, [1] or [2])

$$\left\|\frac{\partial^{i+j}(u - u_{\mathcal{H}})}{\partial x^i \partial y^j}\right\|_{C(I_k^x \times I_l^y)} \le Ch^{4-i-j}\|u\|_{C^4(I_k^x \times I_l^y)}, \quad 0 \le i+j \le 2, \tag{41}$$

for all $1 \le k \le N_x$, $1 \le l \le N_y$. $\square$

5.2. Truncation error. Let $\mathcal{G}^b$ be the subset of $\mathcal{G}$ consisting of all those Gauss points in Ω which are located in the boundary cells of the partition of Ω. The following results are counterparts of Lemma 4.2 and Theorem 4.1.

LEMMA 5.3. *Assume that $u \in C^4(\overline{\Omega})$, and let $u_{\mathcal{GH}}$ be its piecewise bicubic Gauss-Hermite interpolant. Then*

$$\max_{\xi \in \mathcal{G}^b} \left| \frac{\partial^2 (u - u_{\mathcal{GH}})}{\partial x^{2-i} \partial y^i}(\xi) \right| \le C h^2 ||u||_{C^4(\overline{\Omega})}, \qquad i = 0, 2. \tag{42}$$

Proof. We prove (42) for $i = 0$ only since the proof for $i = 2$ is similar. Consider $\xi = (\xi_{1,1}^x, \xi_{1,1}^y)$. It follows from (35) and (36)–(38) that

$$\begin{aligned} \frac{\partial^2 (u_{\mathcal{H}} - u_{\mathcal{GH}})}{\partial x^2}(\xi) &= A_{0,0}^{1,0} h_1^{-1} \frac{\partial (u - u_{\mathcal{GH}})}{\partial x}(x_0, y_0) + A_{0,0}^{0,1} h_1^{-2} h_2 \frac{\partial (u - u_{\mathcal{GH}})}{\partial y}(x_0, y_0) \\ &+ A_{0,1}^{0,0} h_1^{-2} (u - u_{\mathcal{GH}})(x_0, y_1) + A_{0,1}^{0,1} h_1^{-2} h_2 \frac{\partial (u - u_{\mathcal{GH}})}{\partial y}(x_0, y_1) \\ &+ A_{1,0}^{0,0} h_1^{-2} (u - u_{\mathcal{GH}})(x_1, y_0) + A_{1,0}^{1,0} h_1^{-1} \frac{\partial (u - u_{\mathcal{GH}})}{\partial x}(x_1, y_0), \end{aligned}$$

where $h_1 = h_1^x$, $h_2 = h_1^y$, and the coefficients $A_{r,s}^{p,q}$ are independent of h. Therefore, by (39),

$$\left| \frac{\partial^2 (u_{\mathcal{H}} - u_{\mathcal{GH}})}{\partial x^2}(\xi) \right| \le C h^2 ||u||_{C^4(\overline{\Omega})}.$$

Similar inequalities are satisfied for the other three Gauss points in $I_1^x \times I_1^y$ and all remaining Gauss points in boundary cells. Hence (42) for $i = 0$ follows from (41) and the triangle inequality. $\square$

THEOREM 5.1. *Let L be given by (2), where $a \in C^{1,0}(\overline{\Omega})$, $b \in C^{0,1}(\overline{\Omega})$, and $c \in C(\overline{\Omega})$. Assume that $u \in C^4(\overline{\Omega})$, and let $u_{\mathcal{GH}}$ be its piecewise Gauss-Hermite interpolant. Then*

$$\max_{\xi \in \mathcal{G}^b} |L(u - u_{\mathcal{GH}})(\xi)| \le C h^2 ||u||_{C^4(\overline{\Omega})}. \tag{43}$$

Proof. Inequality (43) follows easily from (34) and (42). $\square$

5.3. Error bound. To bound $||u - u_h^{II}||_{H^1(\Omega)}$, we rewrite $u - u_h^{II}$ in the form

$$u - u_h^{II} = u - u_{\mathcal{GH}} + u_{\mathcal{GH}} - u_h^{II}. \tag{44}$$

Clearly, (34) provides a bound on $||u - u_{\mathcal{GH}}||_{H^1(\Omega)}$. To bound $||u_{\mathcal{GH}} - u_h^{II}||_{H^1(\Omega)}$, assume that L_h, given by (21), is an invertible operator from $\mathcal{M}^0$ onto $\mathcal{M}^0$ (cf. Corollary 4.1), and consider η, $\eta^b \in \mathcal{M}^0$ defined as follows

$$(L_h \eta)(\xi) = \begin{cases} 0, & \xi \in \mathcal{G}^b, \\ L(u - u_{\mathcal{GH}})(\xi), & \xi \in \mathcal{G} \setminus \mathcal{G}^b, \end{cases} \tag{45}$$

$$(L_h\eta^b)(\xi) = \begin{cases} L(u-u_{\mathcal{GH}})(\xi), & \xi \in \mathcal{G}^b, \\ 0, & \xi \in \mathcal{G}\setminus\mathcal{G}^b. \end{cases} \tag{46}$$

Since $u_h^{II} - u_{\mathcal{GH}} \in \mathcal{M}^0$, and since by (1), (11), (45), and (46),

$$(L_h[u_h^{II} - u_{\mathcal{GH}}])(\xi) = L(u-u_{\mathcal{GH}})(\xi) = (L_h[\eta+\eta^b])(\xi), \quad \xi \in \mathcal{G},$$

it follows that

$$u_h^{II} - u_{\mathcal{GH}} = \eta + \eta^b. \tag{47}$$

Therefore, in order to find a bound on $||u_{\mathcal{GH}} - u_h^{II}||_{H^1(\Omega)}$ it is sufficient to bound $||\eta||_{H^1(\Omega)}$ and $||\eta^b||_{H^1(\Omega)}$.

Lemma 5.4. *Assume that $a \in C^{5,0}(\overline{\Omega})$, $b \in C^{0,5}(\overline{\Omega})$, $c \in C(\overline{\Omega})$,*

$$a(x,y), b(x,y) > 0, \quad c(x,y) \geq 0, \quad (x,y) \in \overline{\Omega}.$$

Then, for h sufficiently small, there exists a unique $\eta \in \mathcal{M}^0$ satisfying (45). Moreover, if $u \in H^5(\Omega)$ is a solution of the boundary value problem (1), then

$$||\eta||_{H^1(\Omega)} \leq Ch^3||u||_{H^5(\Omega)}. \tag{48}$$

Proof. For h sufficiently small, Corollary 4.1 implies the existence and uniqueness of η. Since $u_{\mathcal{GH}} = u_{\mathcal{H}}$ on the interior cells of the partition of Ω, it follows from (45) and (20) that

$$||L_h\eta||_{\mathcal{G}} \leq Ch^3||u||_{H^5(\Omega)}.$$

Hence (48) is easily obtained by repeating the proof of Theorem 4.3 with η in place of v. □

When applied to η^b, the approach used in the proof of Lemma 5.4 yields only $||\eta^b||_{H^1(\Omega)} \leq Ch^{2.5}||u||_{C^4(\overline{\Omega})}$. However, negative energy inequalities, which were used for finite differences in [6], can also be applied in the context of orthogonal spline collocation. In order to prove Lemma 5.7, which shows that $||\eta^b||_{H^1(\Omega)} \leq Ch^3||u||_{C^4(\overline{\Omega})}$, we need two additional results.

Lemma 5.5. *Assume that $v \in \mathcal{M}^0$ and $w \in \mathcal{M}$ are such that*

$$v(\xi) = \frac{\partial w}{\partial x}(\xi), \quad \xi \in \mathcal{G}. \tag{49}$$

Then

$$\langle(-\Delta_h)^{-1}v, v\rangle_{\mathcal{G}} \leq C\sum_{l=1}^{N_y}\frac{h_l^y}{2}\sum_{j=1}^{2}||w(\cdot,\xi_{l,j}^y)||^2_{L^2(0,1)}. \tag{50}$$

Proof. Setting $z = (-\Delta_h)^{-1}v$, and using (49) and (5), we get

$$\langle(-\Delta_h)^{-1}v, v\rangle_{\mathcal{G}} = \langle z, v\rangle_{\mathcal{G}} = \langle z, \frac{\partial w}{\partial x}\rangle_{\mathcal{G}} = \sum_{l=1}^{N_y}\frac{h_l^y}{2}\sum_{j=1}^{2}\langle z(\cdot,\xi_{l,j}^y), \frac{\partial w}{\partial x}(\cdot,\xi_{l,j}^y)\rangle_x. \tag{51}$$

With $\xi^y = \xi^y_{l,j}$, the Peano representation of the remainder in the two-point Gauss-Legendre quadrature gives (cf. (23))

$$
\begin{aligned}
\langle z(\cdot,\xi^y), \frac{\partial w}{\partial x}(\cdot,\xi^y)\rangle_x &= \int_0^1 \left(z\frac{\partial w}{\partial x}\right)(x,\xi^y)\,dx \\
&\quad - \sum_{k=1}^{N_x} (h_k^x)^4 \int_{I_k^x} \frac{\partial^4}{\partial x^4}\left(z\frac{\partial w}{\partial x}\right)(x,\xi^y) K\left(\frac{x-x_{k-1}}{h_k^x}\right)\,dx,
\end{aligned} \tag{52}
$$

where K is given by (24). Interchanging of z and w in (52), we also have

$$
\begin{aligned}
\langle \frac{\partial z}{\partial x}(\cdot,\xi^y), w(\cdot,\xi^y)\rangle_x &= \int_0^1 \left(w\frac{\partial z}{\partial x}\right)(x,\xi^y)\,dx \\
&\quad - \sum_{k=1}^{N_x} (h_k^x)^4 \int_{I_k^x} \frac{\partial^4}{\partial x^4}\left(w\frac{\partial z}{\partial x}\right)(x,\xi^y) K\left(\frac{x-x_{k-1}}{h_k^x}\right)\,dx.
\end{aligned} \tag{53}
$$

Since $z(\cdot,\xi^y) \in \mathcal{M}_x^0$ and $w \in \mathcal{M}_x$, equations (52), (53), and Leibnitz' formula give

$$
\langle z(\cdot,\xi^y), \frac{\partial w}{\partial x}(\cdot,\xi^y)\rangle_x = -\langle \frac{\partial z}{\partial x}(\cdot,\xi^y), w(\cdot,\xi^y)\rangle_x - I, \tag{54}
$$

where

$$
I = 10\sum_{k=1}^{N_x}(h_k^x)^4 \int_{I_k^x}\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^3 z}{\partial x^3}\right)(x,\xi^y) K\left(\frac{x-x_{k-1}}{h_k^x}\right)\,dx.
$$

Using the Cauchy-Schwarz inequality for $\langle\cdot,\cdot\rangle_x$, and Lemma 3.2 and (3.4) of [4], we obtain

$$
\left|\langle \frac{\partial z}{\partial x}(\cdot,\xi^y), w(\cdot,\xi^y)\rangle_x\right| \le C\left|\left|\frac{\partial z}{\partial x}(\cdot,\xi^y)\right|\right|_{L^2(0,1)} ||w(\cdot,\xi^y)||_{L^2(0,1)}. \tag{55}
$$

Similarly, using (24), the Cauchy-Schwarz inequality in $L^2(I_k^x)$, the inverse inequality (26), and then the Cauchy-Schwarz inequality in $\mathcal{R}^{N_x}$, we get

$$
|I| \le C\left|\left|\frac{\partial z}{\partial x}(\cdot,\xi^y)\right|\right|_{L^2(0,1)} ||w(\cdot,\xi^y)||_{L^2(0,1)}. \tag{56}
$$

Therefore, it follows from (54)–(56) and (3.2) of [4] that

$$
\langle z(\cdot,\xi^y), \frac{\partial w}{\partial x}(\cdot,\xi^y)\rangle_x \le C\langle -\frac{\partial^2 z}{\partial x^2}(\cdot,\xi^y), z(\cdot,\xi^y)\rangle_x^{1/2} ||w(\cdot,\xi^y)||_{L^2(0,1)}. \tag{57}
$$

Finally, (51), (57), and the Cauchy-Schwarz inequality in $\mathcal{R}^{2N_y}$ yield

$$
\langle(-\Delta_h)^{-1}v, v\rangle_{\mathcal{G}} \le C\langle -\frac{\partial^2 z}{\partial x^2}, z\rangle_{\mathcal{G}}^{1/2}\left[\sum_{l=1}^{N_y}\frac{h_l^y}{2}\sum_{j=1}^{2}||w(\cdot,\xi_{l,j}^y)||^2_{L^2(0,1)}\right]^{1/2},
$$

and hence (50) follows, since $\langle -\partial^2 z/\partial x^2, z\rangle_{\mathcal{G}} \le \langle -\Delta_h z, z\rangle_{\mathcal{G}} = \langle(-\Delta_h)^{-1}v, v\rangle_{\mathcal{G}}$. □

Lemma 5.6. *Let $v \in \mathcal{M}_x^0$ be such that $v(\xi_{k,i}^x) = 0$, $2 \le k \le N_x$, $i = 0, 1$, and let $w \in \mathcal{M}_x$ be given by*

$$w = \alpha \psi_0^x + \beta \sum_{k=1}^{N_x} \phi_k^x,$$

where the functions ψ_0^x, ϕ_k^x are defined in (36), and

$$\alpha = \sqrt{3}[v(\xi_{1,1}^x) - v(\xi_{1,2}^x)], \qquad \beta = \frac{h_1^x}{2}[v(\xi_{1,1}^x) + v(\xi_{1,2}^x)].$$

Then $w'(\xi^x) = v(\xi^x)$, $\xi^x \in \mathcal{G}_x$, and

$$||w||_{L^2(0,1)} \le C h \max_{i=1,2} |v(\xi_{1,i}^x)|.$$

Proof. The first part of the lemma follows by a simple verification, since

$$[\psi_0^x]'(\xi_{1,i}^x) = \frac{1}{2\sqrt{3}} \begin{cases} 1, i = 1, \\ -1, i = 2, \end{cases} \qquad [\phi_k^x]'(\xi_{j,i}^x) = \frac{1}{h_k^x} \begin{cases} 1, k = j, i = 1, 2, \\ -1, k = j-1, i = 1, 2, \\ 0, \text{otherwise}. \end{cases}$$

To prove the second part of the lemma, observe that

$$||w||_{L^2(0,1)}^2 = \alpha^2 \int_0^1 [\psi_0^x]^2 \, dx + 2\alpha\beta \int_0^1 \psi_0^x \phi_1^x \, dx + \beta^2 \sum_{k=1}^{N_x} \sum_{\substack{l=k-1 \\ l \ne 0, N_x+1}}^{k+1} \int_0^1 \phi_k^x \phi_l^x \, dx.$$

Therefore, the desired inequality is obtained using (36) and (38). $\square$

We are now in a position to prove the following result.

Lemma 5.7. *Let the functions a, b, and c satisfy the assumptions of Lemma 5.4. Then, for h sufficiently small, there exists a unique $\eta^b \in \mathcal{M}^0$ satisfying (46). Moreover, if $u \in C^4(\overline{\Omega})$ is a solution of the boundary value problem (1), then*

$$||\eta^b||_{H^1(\Omega)} \le C h^3 ||u||_{C^4(\overline{\Omega})}. \tag{58}$$

Proof. Corollary 4.1 implies the existence and uniqueness of η^b for h sufficiently small. Let $\mathcal{G}_i^b$, $1 \le i \le 4$, be subsets of $\mathcal{G}^b$ such that

$$\mathcal{G}_1^b = \{(\xi^x, \xi^y) \in \mathcal{G}^b : \xi^x \in I_1^x\}, \qquad \mathcal{G}_2^b = \{(\xi^x, \xi^y) \in \mathcal{G}^b : \xi^x \in I_{N_x}^x\},$$

$$\mathcal{G}_3^b = \{(\xi^x, \xi^y) \in \mathcal{G}^b : x_1 \le \xi^x \le x_{N_x-1}, \xi^y \in I_1^y\},$$

$$\mathcal{G}_4^b = \{(\xi^x, \xi^y) \in \mathcal{G}^b : x_1 \le \xi^x \le x_{N_x-1}, \xi^y \in I_{N_y}^y\},$$

and let $v_i \in \mathcal{M}^0$ be defined by

$$v_i(\xi) = \begin{cases} L(u - u_{\mathcal{GH}})(\xi), & \xi \in \mathcal{G}_i^b, \\ 0, & \xi \in \mathcal{G} \setminus \mathcal{G}_i^b. \end{cases}$$

It follows from (43) that

$$\max_{\xi \in \mathcal{G}_i^b} |v_i(\xi)| \le Ch^2 ||u||_{C^4(\overline{\Omega})}, \quad 1 \le i \le 4. \tag{59}$$

Let $\eta_i \in \mathcal{M}^0$ be a solution of $L_h \eta_i = v_i$, $1 \le i \le 4$. Clearly $\eta^b = \sum_{i=1}^4 \eta_i$, and hence it is enough to show that

$$||\eta_i||_{H^1(\Omega)} \le Ch^3 ||u||_{C^4(\overline{\Omega})}, \quad 1 \le i \le 4. \tag{60}$$

Here, we verify (60) for $i = 1$ only since all other cases can be treated similarly. Using the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, we get

$$\langle L_h \eta_1, \eta_1 \rangle_{\mathcal{G}} = \langle (-\Delta_h)^{-1/2} v_1, (-\Delta_h)^{1/2} \eta_1 \rangle_{\mathcal{G}} \le \langle (-\Delta_h)^{-1} v_1, v_1 \rangle_{\mathcal{G}}^{1/2} \langle -\Delta_h \eta_1, \eta_1 \rangle_{\mathcal{G}}^{1/2},$$

and hence (cf. proof of Theorem 4.3)

$$||\eta_1||_{H^1(\Omega)} \le C \langle (-\Delta_h)^{-1} v_1, v_1 \rangle_{\mathcal{G}}^{1/2}. \tag{61}$$

Let ψ_0^x, ϕ_k^x, $1 \le k \le N_x$ be defined by (36), (38), and let $\theta_{l,j}^y$, $1 \le l \le N_y$, $j = 1, 2$, be basis functions for $\mathcal{M}_y^0$ (cf. Corollary 5.3 of [7]) such that

$$\theta_{l,j}^y(\xi_{k,i}^y) = \delta_{l,k}\delta_{j,i}, \quad 1 \le k \le N_y, \ \ i = 1, 2. \tag{62}$$

If $w \in \mathcal{M}$ is defined by

$$w(x, y) = \sum_{l=1}^{N_y} \sum_{j=1}^{2} \left(\alpha_{l,j} \psi_0^x(x) + \beta_{l,j} \sum_{k=1}^{N_x} \phi_k^x(x) \right) \theta_{l,j}^y(y),$$

where

$$\alpha_{l,j} = \sqrt{3}[v_1(\xi_{1,1}^x, \xi_{l,j}^y) - v_1(\xi_{1,2}^x, \xi_{l,j}^y)], \quad \beta_{l,j} = \frac{h_1^x}{2}[v_1(\xi_{1,1}^x, \xi_{l,j}^y) + v_1(\xi_{1,2}^x, \xi_{l,j}^y)],$$

then it follows from (62), Lemma 5.6, and (59) that

$$v_1(\xi) = \frac{\partial w}{\partial x}(\xi), \ \ \xi \in \mathcal{G}, \quad ||w(\cdot, \xi^y)||_{L^2(0,1)} \le Ch^3 ||u||_{C^4(\overline{\Omega})}, \ \ \xi^y \in \mathcal{G}_y. \tag{63}$$

Therefore, (61), (63) and Lemma 5.5 imply (60) for $i = 1$. □

The following result is a counterpart of Theorem 4.3.

Theorem 5.2. *Let the functions $a, b,$ and c satisfy the assumptions of Lemma 5.4. Then, for h sufficiently small, there exists a unique collocation solution $u_h^{II} \in \mathcal{M}$ satisfying (11)–(14). Moreover, if $u \in H^5(\Omega) \cap C^4(\overline{\Omega})$ is a solution of the boundary value problem (1), then*

$$||u - u_h^{II}||_{H^1(\Omega)} \le Ch^3(||u||_{H^5(\Omega)} + ||u||_{C^4(\overline{\Omega})}).$$

Proof. For h sufficiently small, the existence and uniqueness of u_h^{II} follow from Corollary 4.1. The desired error bound is easily obtained from the triangle inequality using (44), (34), (47), (48), and (58). □

6. Hermite orthogonal spline collocation for separable boundary value problems. In this section we apply Hermite orthogonal spline collocation to a class of boundary value problems for which the existence and uniqueness of collocation solutions as well as derivations of the corresponding error bounds do not require any conditions on the size of h.

Consider the boundary value problem

$$\begin{aligned} &\tilde{L}u = \tilde{f}(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ &u = g(x,y), \quad (x,y) \in \partial\Omega, \end{aligned} \tag{64}$$

where

$$\tilde{L}u = -a_1(x)a_2(y)\frac{\partial^2 u}{\partial x^2} - b_1(x)b_2(y)\frac{\partial^2 u}{\partial y^2} + \tilde{c}(x,y)u.$$

Let $u_h^I \in \mathcal{M}$ be the collocation solution of (64) such that

$$\tilde{L}u_h^I(\xi) = \tilde{f}(\xi), \quad \xi \in \mathcal{G}, \tag{65}$$

and such that (9) and (10) are satisfied. Similarly, the collocation solution $u_h^{II} \in \mathcal{M}$ is required to satisfy

$$\tilde{L}u_h^{II}(\xi) = \tilde{f}(\xi), \quad \xi \in \mathcal{G}, \tag{66}$$

and (12)–(14).

Theorem 6.1. *Assume that $a_i, b_i \in C[0,1]$, $i = 1,2$, $\tilde{c} \in C(\overline{\Omega})$, and that*

$$a_i(t), b_i(t) > 0, \quad t \in [0,1], \quad i = 1,2, \qquad \tilde{c}(x,y) \geq 0, \quad (x,y) \in \Omega.$$

Then, for arbitrary h, there exist unique collocation solutions $u_h^I, u_h^{II} \in \mathcal{M}$ satisfying (65), (9), (10), and (66), (12)–(14), respectively. Moreover, if $u \in H^5(\overline{\Omega})$ is a solution of the boundary value problem (64), then

$$||u - u_h^I||_{H^1(\Omega)} \leq Ch^3||u||_{H^5(\Omega)}.$$

Similarly, if $u \in H^5(\Omega) \cap C^4(\overline{\Omega})$, then

$$||u - u_h^{II}||_{H^1(\Omega)} \leq Ch^3(||u||_{H^5(\Omega)} + ||u||_{C^4(\overline{\Omega})}).$$

Proof. Let L and f be defined by

$$Lu = -\frac{\partial}{\partial x}\left(\frac{a_2(y)}{b_2(y)}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{b_1(x)}{a_1(x)}\frac{\partial u}{\partial y}\right) + \frac{\tilde{c}(x,y)}{a_1(x)b_2(y)}, \quad f(x,y) = \frac{\tilde{f}}{a_1(x)b_2(y)}.$$

Clearly, (65) and (66) are equivalent to (8) and (11), respectively. Hence, the required error bounds follow easily from the results of Sections 4 and 5 on Hermite orthogonal spline collocation schemes (8)–(10) and (11)–(14). In particular, the constant c_2 in (22) is equal to 0, which implies the existence and uniqueness of u_h^I and u_h^{II} for arbitrary value of h. $\square$

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