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CURVATURE EXTREMA ALONG TRAJECTORIES IN THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

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One approach to summarizing a spacecraft trajectory in a multi-body system involves sampling its periapses and apoapses relative to a reference point. However, in the rotating frame, this trajectory can revolve about one or more primaries, equilibrium points, or other locations. Diverse itineraries create ambiguity in selecting a suitable reference point for a geometrically meaningful apsis. Extrema in the curvature offer an alternative approach for sampling at geometrically meaningful locations and do not require selection of a reference point. This paper performs a preliminary numerical examination of these curvature extrema in the Earth-Moon circular restricted three-body problem.

INTRODUCTION

One approach to describing and analyzing spacecraft trajectories in a multi-body system is to study their apses relative to a specified reference point, e.g., a celestial body. Periapsis corresponds to a locally minimum distance from the reference point whereas apoapsis identifies a location of locally maximum distance. Although apses are critical in assessing path-based and mission constraints, their properties and evolution over time also supply meaningful information about the geometry of a trajectory. When constructing a Poincaré map, apses are also useful for seeding initial conditions and defining a hyperplane for efficient analysis of the phase space.

However, in a three-body system, a spacecraft trajectory, when visualized in a frame that rotates with the primaries, can revolve about one or more celestial bodies, an equilibrium point, or other location in a system. These diverse itineraries can render the task of selecting a reference point for locating a geometrically meaningful apsis ambiguous and time-dependent in some cases. Furthermore, at a high energy level, the gateways associated with the collinear equilibrium points no longer exist. Thus, the distinction between regions of motion where one primary or equilibrium point is better suited to serve as a reference point is no longer clear.

In differential geometry, space curves can be described by their curvature as a function of time. At a single instant of time, the curvature describes the deviation of the path from a straight line within the osculating plane. The osculating plane is the plane formed by three position vectors at successive instants of time that are separated by an infinitesimally small value. For spatial trajectories, an additional quantity labeled torsion (also sometimes labeled the 'second curvature') captures the rotation of the osculating plane in three-dimensional space. Turning points in the curvature along a

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trajectory correspond to geometrically meaningful locations where the shape most or least resembles a straight line. Accordingly, this information has been used in the field of computer graphics for shape interrogation of curves or surfaces [2].

This paper presents a preliminary numerical analysis of the characteristics of extrema in the curvature for states specified in the Earth-Moon rotating frame along spacecraft trajectories in the Earth-Moon circular restricted three-body problem (CR3BP). First, the locations of extrema in the curvature and apses relative to various reference points are compared along selected trajectories in the rotating frame. These examples indicate that maxima may offer a meaningful analog to these apses and successfully capture the geometry of the trajectory. Furthermore, the benefit of using curvature extrema is that they do not require selection of a reference point or lead to mathematically valid but extraneous apses. For planar motion, the number and direction of the velocity vectors that lead to curvature extrema are explored for position vectors sampled across the system at various energy levels. These characteristics are examined and explained based on the geometry of the associated trajectories. This analysis is extended to spatial trajectories where the velocity vectors that produce curvature extrema at a selected initial position vector form one or more curves. Such insight could inform future descriptions and discretizations that meaningfully capture the shape and evolution of spacecraft trajectories within cislunar space and other multi-body gravitational systems, particularly as the paths encompass multiple regions or high energy levels.

DYNAMICAL MODEL

The motion of a spacecraft in cislunar space is approximated using the circular restricted three-body problem (CR3BP). In this dynamical model, the Earth and Moon are modeled using spherically symmetric bodies whereas the spacecraft is assumed to possess a negligibly small mass in comparison to the primary bodies [3]. Furthermore, the Earth and Moon are assumed to travel along circular orbits about their barycenter.

Mass, length, and time quantities are typically nondimensionalized to mitigate the potential for ill-conditioning [3, 1]. Mass quantities are normalized by the total mass of the primary system. Length quantities are normalized by the assumed constant distance between the Earth and Moon. Finally, time quantities are nondimensionalized to set the period of the primary system to 2π.

To improve visualization and analysis, the motion of a spacecraft is typically represented in a synodic or rotating frame [3, 1]. The origin of this frame is defined as the barycenter of the Earth-Moon system. Then, the axes are defined as follows: $\hat{x}$ is directed from the Earth to the Moon, $\hat{z}$ is directed along the orbital angular momentum vector of the primary system, and $\hat{y}$ completes the right-handed and orthogonal triad.

The nondimensional equations of motion for the spacecraft in the CR3BP are expressed in the Earth-Moon rotating frame. The state of the spacecraft is defined as $\bar{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$. In addition, a pseudo-potential function is defined as

$$U^* = \frac{x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{1}$$

With these definitions, the equations of motion governing the spacecraft are written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}, \qquad \ddot{z} = \frac{\partial U^*}{\partial z} \tag{2}$$

where μ is the mass ratio comparing the Moon's mass to the total mass of the system, $r_1 = \sqrt{(x+\mu)^2+y^2+z^2}$, and $r_2 = \sqrt{(x-1+\mu)^2+y^2+z^2}$ [3]. This dynamical model is autonomous but chaotic. However, a constant of motion, labeled the Jacobi constant, exists and is equal to

$$C_J = 2U^* - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \tag{3}$$

At a single value of this quantity, a wide variety of motion can exist throughout the system.

CURVATURE

Differential geometry includes the study of the geometry of curves in three-dimensional space. Concepts such as curvature and the Frenet frame supply a mechanism for capturing how curved the path is, the reference point about which the path curves, and the direction of motion around that reference point. This section presents a brief definition and mathematical computation of these quantities from a position vector $\bar{r}(t) = [x(t), y(t), z(t)]^T$, velocity vector $\dot{\bar{r}}(t) = [\dot{x}(t), \dot{y}(t), \dot{z}(t)]^T$, and acceleration vector $\ddot{\bar{r}}(t) = [\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)]^T$ that evolve over time along the trajectory.

At any location along the curved trajectory, the axes associated with the Frenet frame are composed of the tangent $\hat{T}$, principal normal $\hat{N}$, and binormal $\hat{B}$ vectors. Two concepts are valuable in defining these axes. First, when generated over a time interval $t \in [t_0, t_f]$, the arclength s along a path is equal to [4]

$$s = \int_{t_0}^{t_f} ds = \int_{t_0}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \tag{4}$$

Then, the osculating plane is defined as the plane that passes through three sequential points as the arclength between them approaches an infinitesimally small value [2]. Using these definitions, both $\hat{T}$ and $\hat{N}$ lie in the osculating plane. Specifically, $\hat{T}$ is tangent to the path whereas $\hat{N}$ is directed towards the center of curvature and along the derivative of $\hat{T}$ with respect to the arclength [4]. Finally, $\hat{B}$ completes the orthogonal, right-handed triad.

Two quantities capture how a trajectory curves and twists at any instant of time in three-dimensional space. First, the curvature $\kappa(t)$ reflects the deviation from a straight line in the osculating plane at the associated location along the trajectory. The curvature is calculated at any location as

$$\kappa(t) = \frac{||\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)||}{||\dot{\bar{r}}(t)||^3} = \frac{\sqrt{(\ddot{z}\dot{y} - \ddot{y}\dot{z})^2 + (\ddot{x}\dot{z} - \ddot{z}\dot{x})^2 + (\ddot{y}\dot{x} - \ddot{x}\dot{y})^2}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}} \tag{5}$$

This unsigned quantity can be interpreted as the rate of change of the angle swept out by the tangent to the curve with respect to arclength [2]. As a result, regions of locally high curvature tend to correspond to geometrically meaningful locations along a trajectory. Note, however, that this curvature possesses a singularity, approaching infinity as the speed approaches zero. Second, the torsion $\tau(t)$ reflects the rotation of the osculating plane. The torsion is calculated at any location as

$$\tau(t) = \frac{\bar{r}(t)\dot{\bar{r}}(t)\ddot{\bar{r}}(t)}{||\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)||^2} = \frac{\dddot{x}(\dot{y}\ddot{z} - \ddot{y}\dot{z}) + \dddot{y}(\ddot{x}\dot{z} - \dot{x}\ddot{z}) + \dddot{z}(\dot{x}\ddot{y} - \ddot{x}\dot{y})}{(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2 + (\dot{x}\ddot{z} - \ddot{x}\dot{z})^2 + (\dot{x}\ddot{y} - \ddot{x}\dot{y})^2} \tag{6}$$

This quantity can be nonzero with the sign indicating the direction of twisting over time.

Shape interrogation methods focus on describing and sampling curves and surfaces to capture their shape [2]. One approach relies on studying the extrema in the curvature and/or torsion. Along a trajectory, stationary points in the curvature must satisfy the condition $\dot{\kappa} = 0$, requiring the calculation of third time derivatives of the position vector. These stationary points locate geometrically meaningfully regions of a trajectory. Stationary points in the torsion, however, must satisfy the condition $\dot{\tau} = 0$, requiring fourth-order derivatives. This paper focuses on studying extrema in κ.

COMPARING CURVATURE EXTREMA ALONG TRAJECTORIES TO APSES

Extrema in the curvature along a trajectory in the rotating frame of a multi-body system are often located close to periapses and apoapses relative to applicable, well-known reference points.

When studying spacecraft trajectories, apses relative to the primaries and/or equilibrium are often employed. These locations are critical in studying locally minimum and maximum distances for assessing path constraints. However, they are also valuable in capturing the geometric evolution of a trajectory and generating a consistent set of initial conditions for sampling a solution space. In these cases, the following challenges in the use of apses can occur:

- Although apses relative to any reference point in a multi-body system can be mathematically calculated, they may not be physically meaningful nor reflect geometrically interesting locations along the trajectory when the spacecraft is not orbiting about that reference point
- When a trajectory passes through multiple regions of a system (e.g., passing from the vicinity of one primary through a libration point gateway to the vicinity of another primary), it can be challenging to determine which reference point the trajectory is revolving about and, therefore, a suitable choice for defining a geometrically meaningful apse
- At high energy levels or low values of the Jacobi constant, it can be challenging to determine which reference point is suitable for capturing these geometrically meaningful points as the motion is complex and libration point gateways do not exist

Extrema in the curvature, which are mathematically calculated using the velocity and acceleration vectors with an observer in the rotating frame, can approximately capture geometrically meaningful locations without reliance on a specified reference point.

To initially connect the location of maxima in the curvature and apses relative to meaningful reference points, consider a planar trajectory that passes through multiple regions in the Earth-Moon CR3BP. Figure 1 displays an example using a black solid curve with the primaries displayed via gray circles (not to scale) as well as L_1 and L_2 identified by red markers. In Figure 1a), maxima (blue circles) and nonzero minima (blue crosses) in the curvature are located along the trajectory. In Figure 1b), apses are located with respect to the Earth (magenta circles), Moon (brown crosses), and L_1 (teal diamonds). When the trajectories revolve around the Moon, maxima in the curvature are often located close to perilunes and apolunes. However, as the trajectory passes through the L_1 gateway, extrema in the curvature are located close to apses relative to L_1: maxima lie close to apoapsis during a revolution around L_1 and when passing through the L_1 gateway whereas minima occur near periapsis during a revolution around L_1. Finally, when the trajectory revolves in the Earth vicinity, the maxima in the curvature are located close to the apses relative to the Earth.

To further delve into the connection between curvature extrema and apses, consider additional planar and spatial trajectories. Figure 2 includes four examples: a) a low prograde orbit around the Moon, b) a distant retrograde orbit around the Moon, c) a spatial trajectory that briefly revolves around L_1 before completing 4 revolutions around the Moon and then departing through the L_1 gateway, and d) a highly out-of-plane trajectory that revolves around the Moon with an apsidal rotation. In each figure, the curvature maxima are located by blue circles and minima using blue crosses. Perilunes and apolunes are indicated by magenta diamonds and crosses, respectively. Relative to L_1, however, periapses and apoapses are located by green diamonds and crosses, respectively.

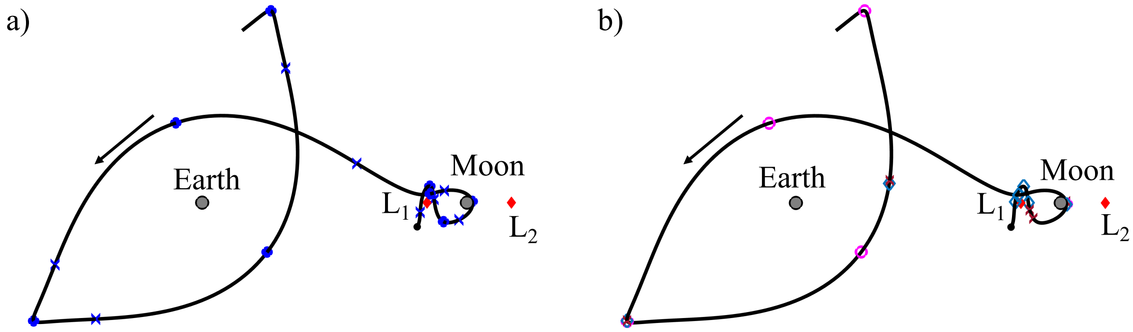

Figure 1. Trajectory in the Earth-Moon CR3BP and rotating frame with a) curvature maxima (blue circles) and minima (blue crosses); and b) apses with respect to the Earth (magenta circles), Moon (brown crosses), and L_1 (teal diamonds).

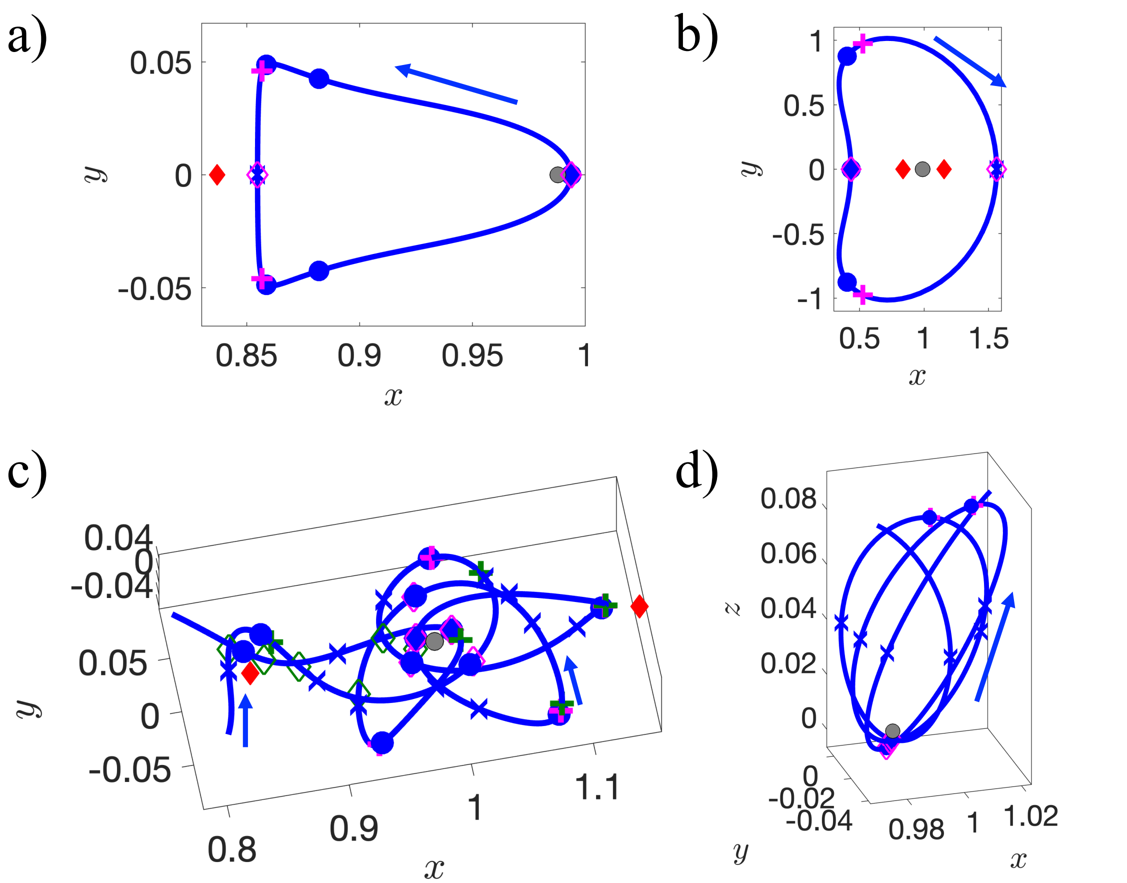

Figure 2. Examples of trajectories with distinct geometries, annotated with: curvature maxima (blue circles), curvature minima (blue crosses), apoapses (magenta crosses) and periapses (magenta diamonds) relative to the Moon, and apoapses (green crosses) and periapses (green diamonds) relative to L_1.

Along the low prograde orbit in Figure 2a), three of the five maxima in curvature occur near perilunes and apolunes. However, two additional maxima occur at locations that are noticeably offset from two other maxima, but do not exist close to apolune or perilune. To explain these two maxima, Figure 3a) displays the curvature as a function of time along this trajectory, starting from perilune, with maxima highlighted in red circles and nonzero minima in red crosses. Traveling in a counterclockwise direction around the orbit from perilune, the first maxima in the unsigned curvature corresponds to the blue circle that is not located near an apolune. In this case, this maxima occurs after the curvature vanishes, equalling zero, and the trajectory changes between convex and concave; equivalently, the center of curvature flips between the interior to exterior of the orbit. The maxima and minima that are located near perilune and apolune are not immediately located between points where the curvature passes through zero.

For the distant retrograde orbit in Figure 2b), three maxima and one minimum in curvature occur near perilunes and apolunes. The perilune locations are closely located near one minimum and one maximum on the x-axis. However, the remaining maxima are slightly offset from apolune. No nonzero minima occur between the three maxima on the righthand side of the orbit likely due to the trajectory changing between concave and convex.

In Figure 2c), the spatial trajectory possesses a complex geometry but still reveals a connection between the location of extrema in the curvature and apses. Early on, the trajectory completes half a revolution around the vicinity of L_1. Here, the curvature maximum occurs near apoapsis relative to L_1 whereas the curvature minima occur near periapsis relative to L_1. As the trajectory then revolves around the Moon, curvature maxima tend to occur close to both perilune and apolune; curvature minima occurring between subsequent maxima. The trajectory then passes through the L_1 gateway with a curvature maximum that is located close to periapsis relative to L_1; here, the trajectory is convex with respect to L_1. Figure 3b) displays the evolution of the curvature along this more complex trajectory.

Finally, consider the high-inclination and high-eccentricity orbit around the Moon in Figure 2d). In this case, the maxima in the curvature are all located near perilune and apolune along each revolution around the Moon. The minima occur roughly halfway between subsequent maxima.

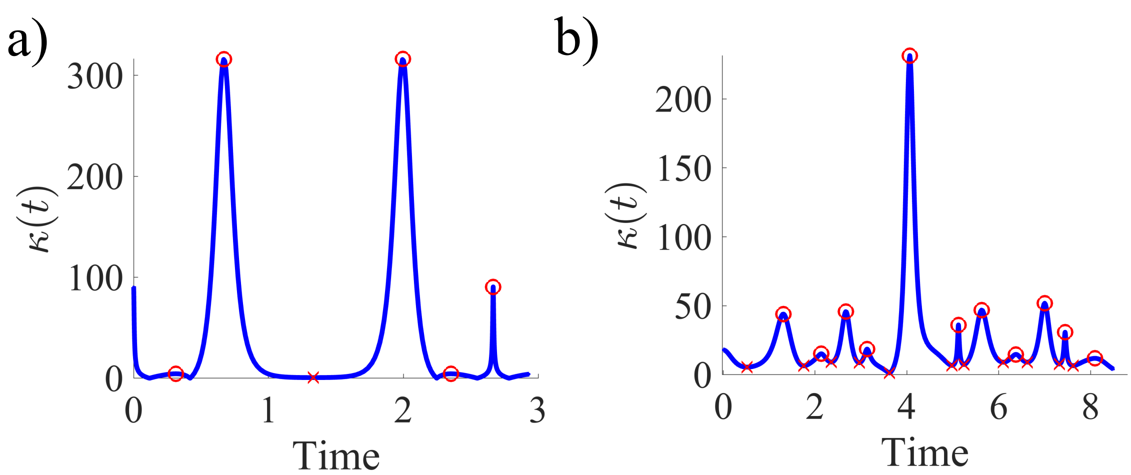

Figure 3. Time history of the curvature along the following trajectories from Figure 1: a) top left trajectory, and b) bottom left trajectory. Maxima are located by red circles and minima identified via red crosses.

EXISTENCE OF CURVATURE EXTREMA FOR PLANAR MOTION

For a specified position vector, the number of curvature extrema and the associated velocity directions along planar trajectories vary throughout the system.

To initially study the existence of extrema in the curvature throughout the Earth-Moon CR3BP, consider planar motion. Position vectors are seeded from a uniform grid within the ranges $x \in [-2, 2]$, $y \in [-2, 2]$ with $z = 0$. For each position vector, velocity vectors are calculated to produce $\dot{\kappa} = 0$. Turning points in the curvature are then classified based on whether they correspond to a minimum or maximum using the sign of $\ddot{\kappa}$. Locations where the unsigned curvature vanishes are not considered as minima.

Calculating the velocity vectors that lead to extrema in the curvature at a single energy level reveals regions with distinct characteristics. Figure 4 displays the number of a) maxima, b) minima, and c) extrema associated with states generated from a coarse grid of position vectors at a Jacobi constant of $C = 3.16$. In addition, Figure 4d) displays the direction of the velocity vector and e) displays the direction to the center of curvature for a coarsely sampled subset of position vectors. In subfigures a)-c), the number of extrema that are computed at each position vector are colored on a dark brown to copper color scale. Then, in subfigures d) and e) vectors associated with maxima are colored blue whereas those associated with minima are plotted in magenta. In all subfigures, the locations of the Earth and Moon are indicated by gray circles, not to scale, and the equilibrium points are identified via red diamonds.

Far beyond the zero velocity curves, two velocity vectors produce extrema at each position vector: one maximum for motion that is instantaneously prograde around the barycenter (usually during a temporary change in direction within a loop along a path that is otherwise retrograde relative to the barycenter) and one minimum for motion that is retrograde about the barycenter. Moving closer to the zero velocity curves, these velocity vectors in both directions produce extrema that are all maxima. Sufficiently close to the exterior of the zero velocity curves, a total of four extrema exist: three or four of these extrema are maxima, depending on the exact location. The two additional extrema that appear as position vectors approach the zero velocity curves correspond to motion that is instantaneously retrograde around the barycenter. These velocity vectors are pointed towards or away from the zero velocity curves, corresponding to trajectories that are about to or have recently bounced off the zero velocity curves.

Within the Earth vicinity, similar trends occur. Near the zero velocity curves, there are four velocity vectors that produce maxima in the curvature. Two extrema correspond to motion that is instantaneously either prograde or retrograde relative to the Earth and close to parallel to the nearby tangent to the zero velocity curves; the retrograde state is associated with a temporary direction change within a loop along a path that is otherwise prograde relative to the Earth. The other two velocity vectors occur immediately before or after a trajectory that is prograde relative to the Earth bounces off the zero velocity curves. Moving further away from the zero velocity curves, these four states shift from maxima to minima in the curvature. Continuing to move closer to the Earth, a ring of position vectors only possesses two velocity vectors that correspond to the maxima in the curvature that are instantaneously prograde or retrograde about the Earth and oriented almost perpendicular to the direction to the Earth. Close to the Earth, position vectors each possess four velocity vectors that produce extrema in the curvature: two velocity vectors that correspond to minima point approximately towards or away from the Earth and tend to lie along highly eccentric paths, whereas two velocity vectors that produce maxima are approximately perpendicular to the

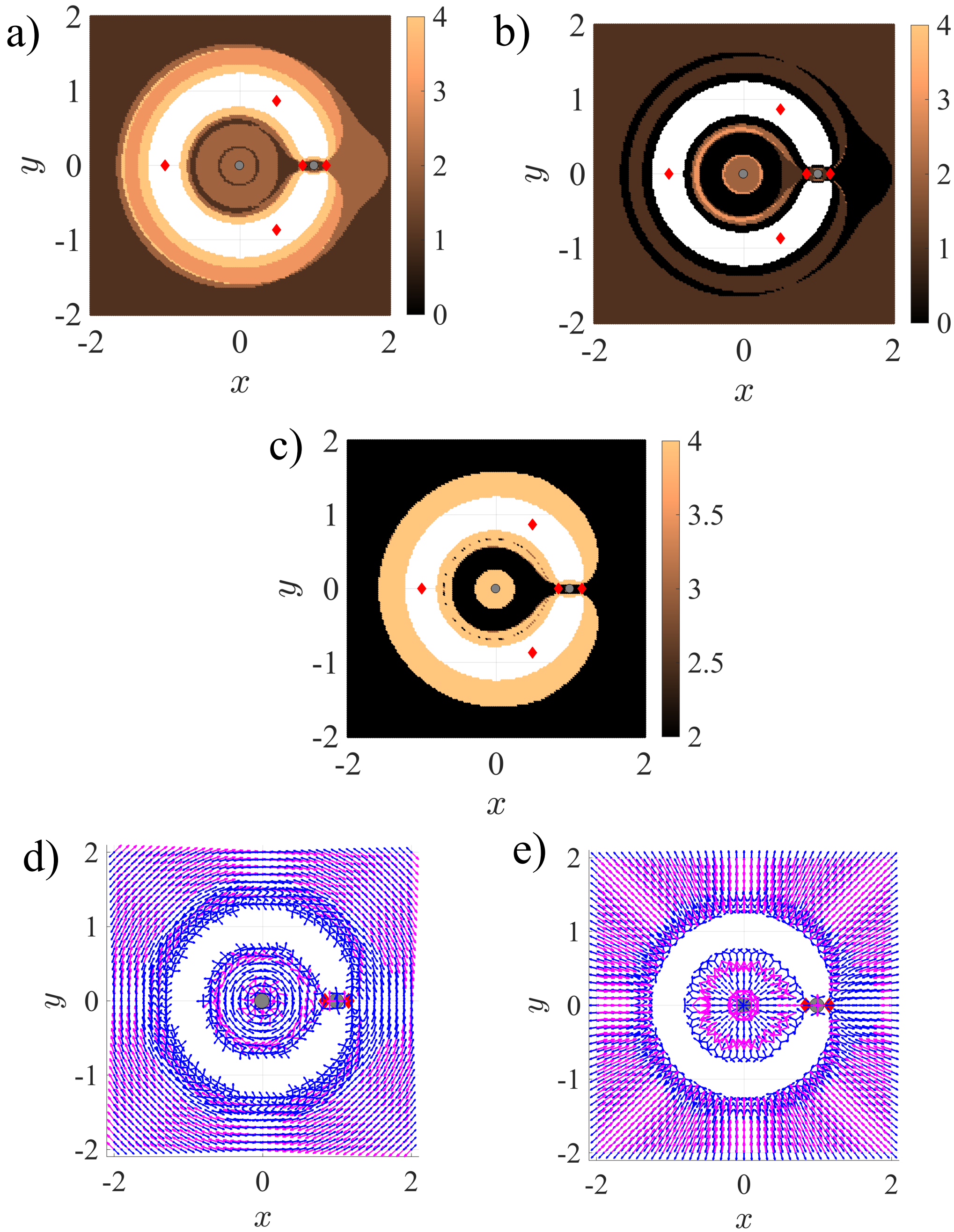

Figure 4. Number of a) maxima, b) minima, c) extrema in the curvature for planar states that pass through each position vector at $C_J = 3.16$ along with d) the associated velocity vectors and e) direction to center of curvature for a subset of these positions.

direction to the Earth.

In the vicinity of the Moon, the existence of extrema and the direction of the associated velocity vectors at each position vector becomes more complicated. Figure 5 displays the same information as Figure 4 for a zoomed-in region near the Moon, including the L_1 and L_2 gateways. In the majority of this region, four maxima exist to capture both motion that is: prograde or retrograde around the Moon, and motion that is about to or has recently bounced off the zero velocity curves. Closer to the Moon, these curvature extrema that produce trajectories bouncing off the zero velocity curves become minima. A small ring of extrema then corresponds to motion that is revolving around the Moon vicinity with velocity vectors that are roughly perpendicular to the direction to the Moon. Sufficiently close to the Moon, additional minima appear again with velocity vectors that are pointed roughly towards or away from the Moon to produce instantaneously retrograde motion. Near the libration points, a region of position vectors is associated with four extrema: two maxima corresponding to velocity vectors that point predominantly in the $+/-\hat{x}$-direction as motion flows through the gateways and two minima with velocity vectors pointed towards the zero velocity curves as the trajectory completes loops in the vicinity of L_1. This region approximately resembles but does not exactly coincide with the L_1 Lyapunov orbit at this Jacobi constant. Similar observations apply in the vicinity of L_2.

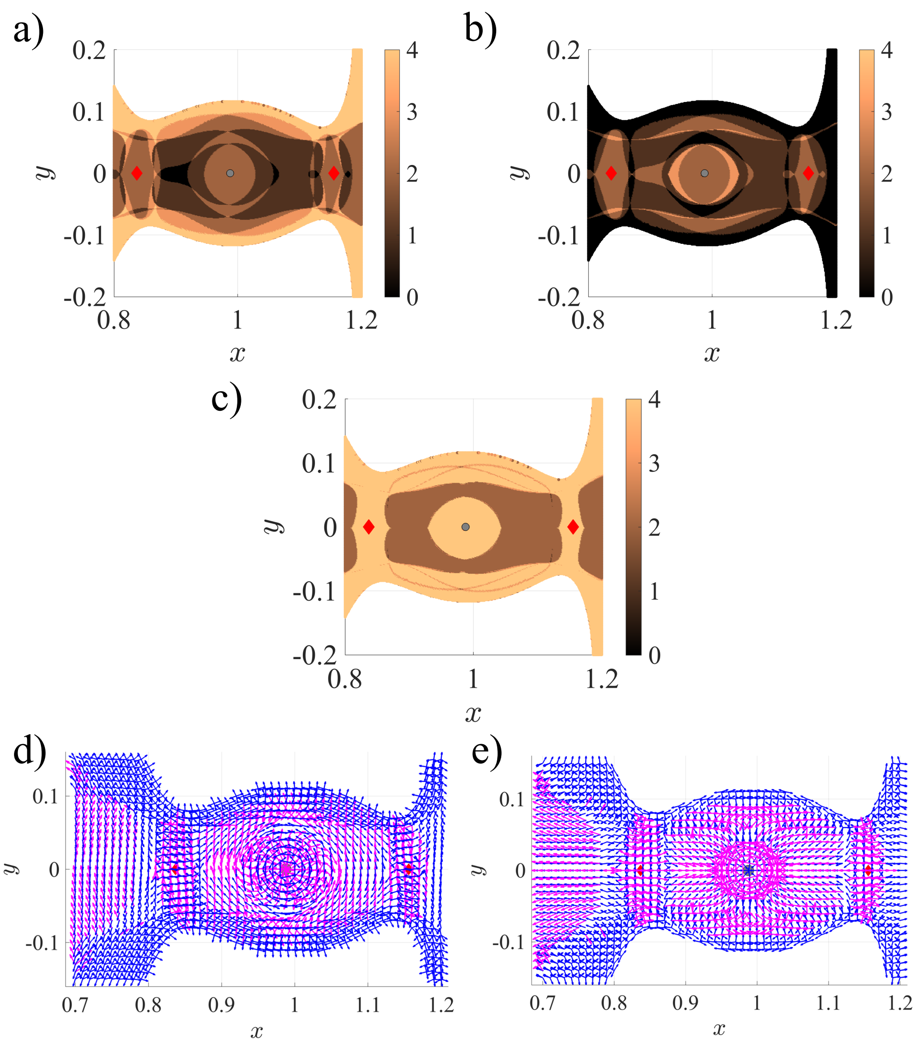

Figure 5. Number of a) maxima, b) minima, c) extrema in the curvature for planar states that pass through each position vector at $C_J = 3.16$ along with d) the associated velocity vectors and e) direction to center of curvature for a subset of these positions.

These initial observations tend to persist across an array of energy levels. Figs. 9-11 display the number of extrema, maxima, and minima for the following Jacobi constants: a) $C_J = 3.20$, b) $C_J = 3.15$, c) $C_J = 3.10$, d) $C_J = 3.05$, e) $C_J = 3.00$, f) $C_J = 2.95$. Across these Jacobi constants, the majority of position vectors throughout the configuration space admit two extrema in the curvature. In the exterior region of the system, one maximum and one minimum exist. Closer to the zero velocity curves, additional curvature extrema appear as trajectories bounce off them. However, even when no zero velocity curves exist in the plane of the primaries, a thin crescent region with four curvature extrema passes through L_3, L_4, L_5. These additional extrema correspond

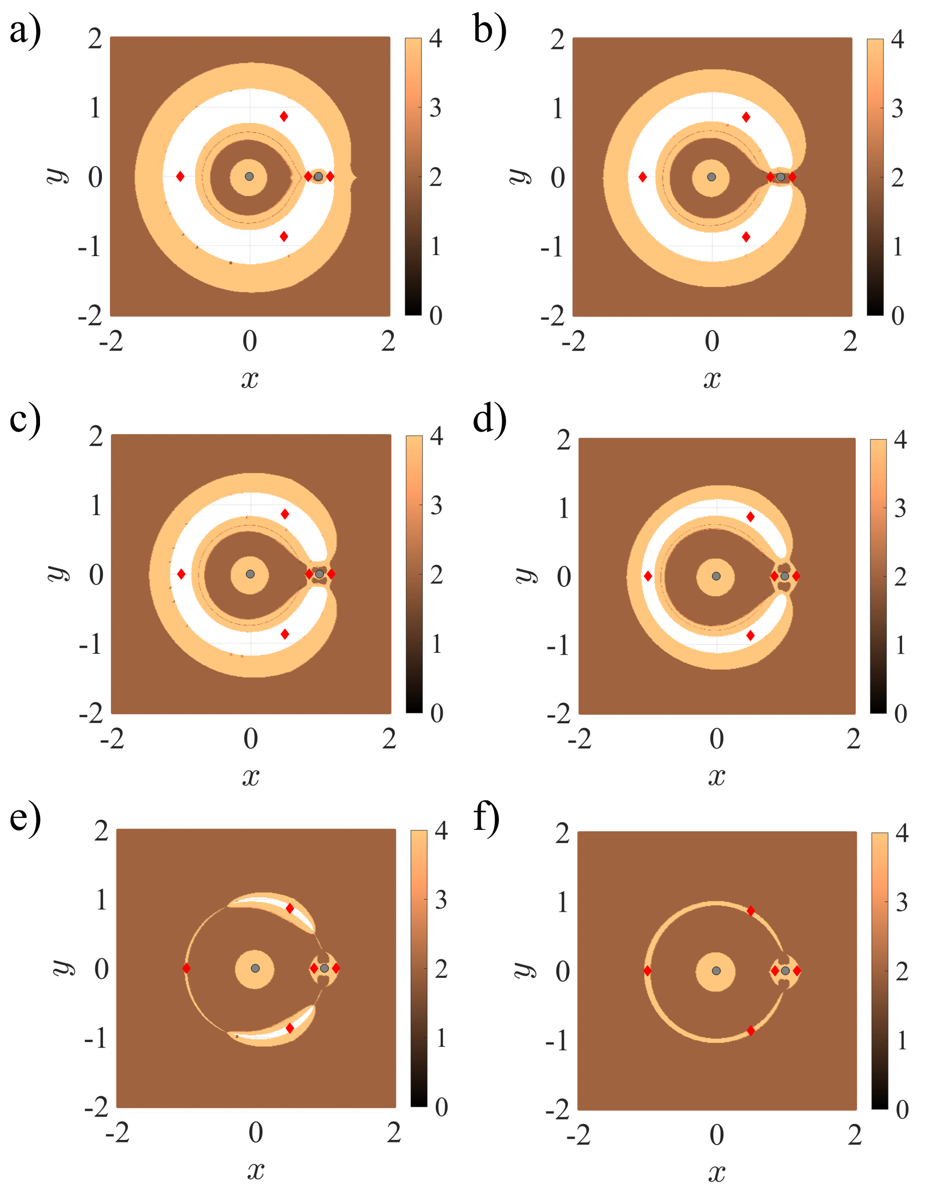

Figure 6. Total number of extrema in the curvature for planar states that pass through each position vector at various Jacobi constants: a) $C_J = 3.20$, **b)** $C_J = 3.15$, **c)** $C_J = 3.10$, **d)** $C_J = 3.05$, **e)** $C_J = 3.00$, **f)** $C_J = 2.95$.

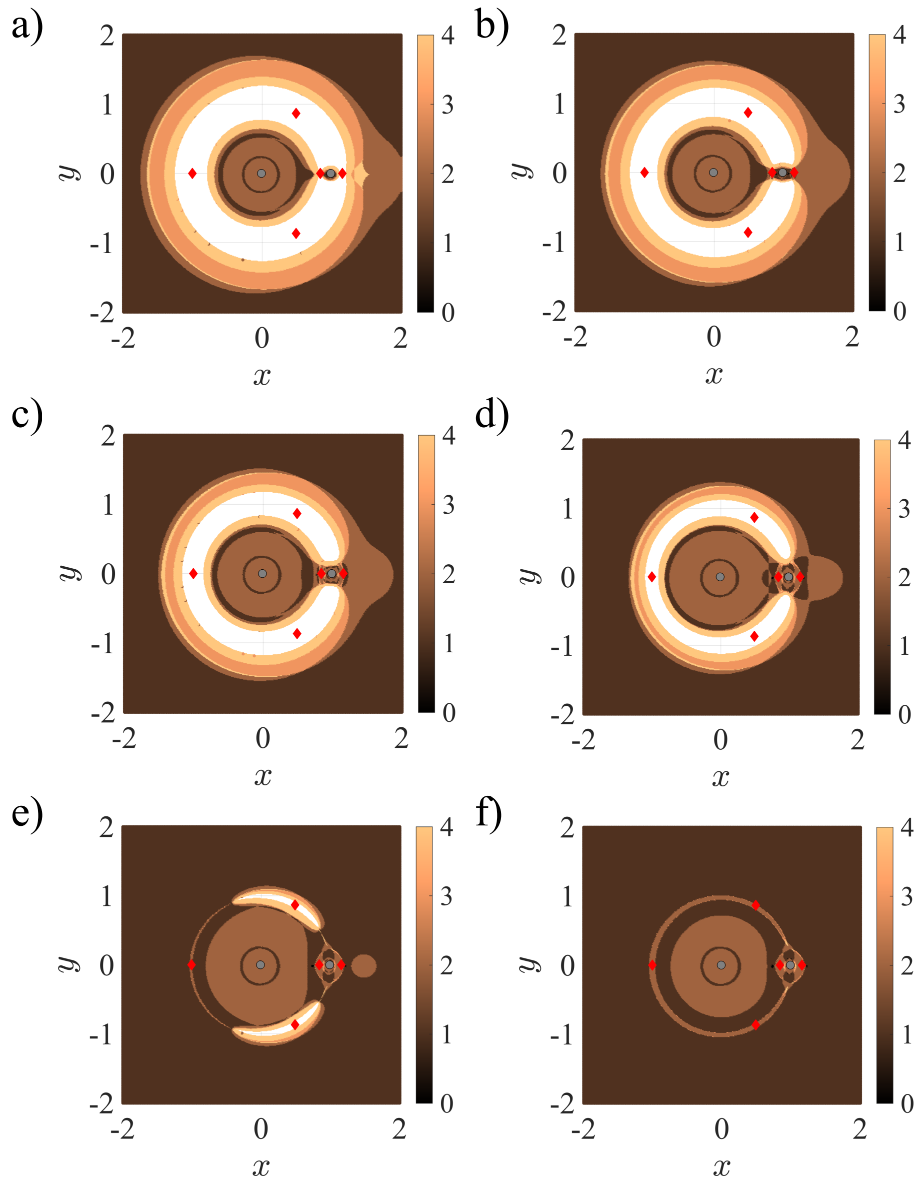

Figure 7. Number of maxima in the curvature for planar states that pass through each position vector at various Jacobi constants: a) $C_J = 3.20$, b) $C_J = 3.15$, c) $C_J = 3.10$, d) $C_J = 3.05$, e) $C_J = 3.00$, f) $C_J = 2.95$.

to trajectories with loops that form in the region, with some temporarily resembling exterior resonant orbits and others the long period motions associated with the libration points. Close to the Earth, a consistently-sized region of four curvature extrema (two maxima and two minima) persists across each energy level and is typically bound by a ring where four curvature minima exist.

The vicinity of the Moon, L_1, and L_2, exhibits an interesting variation in the number of curvature extrema. Figures 9-11 reveal that in a region around the Moon, there are typically four curvature extrema. A subset of this region lies within a smooth diamond-like boundary where the number of minima and maxima vary. Within this boundary, the direction to the center of curvature of all

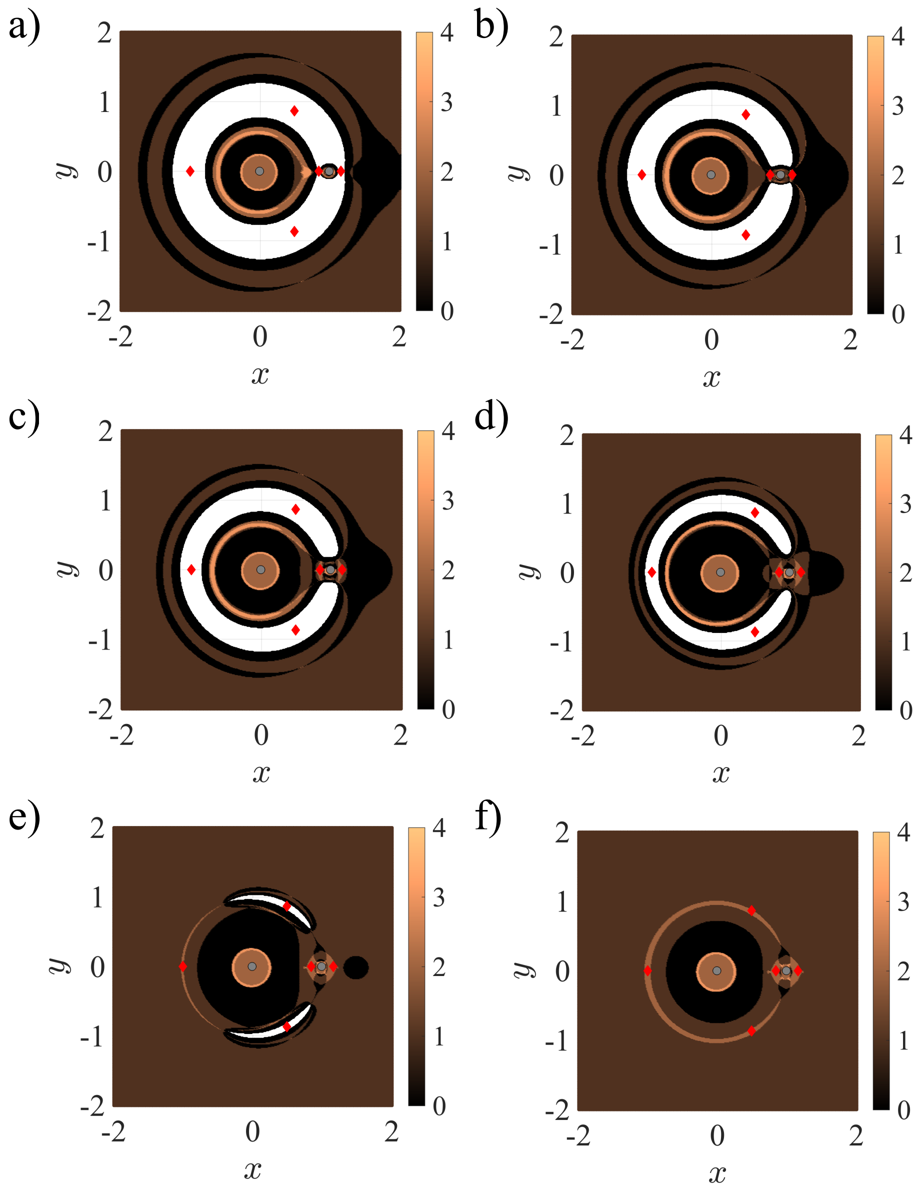

Figure 8. Number of minima in the curvature for planar states that pass through each position vector at various Jacobi constants: a) $C_J = 3.20$**, b)** $C_J = 3.15$**, c)** $C_J = 3.10$**, d)** $C_J = 3.05$**, e)** $C_J = 3.00$**, f)** $C_J = 2.95$**.**

extrema points points either towards the Moon or perpendicular to it; there are no extrema in the curvature in this region that correspond to concave motion around the Moon. There are also regions above and below the Moon where only two curvature extrema exist. As the Jacobi constant is decreased within the examined range, these regions grow and shift further from the Moon.

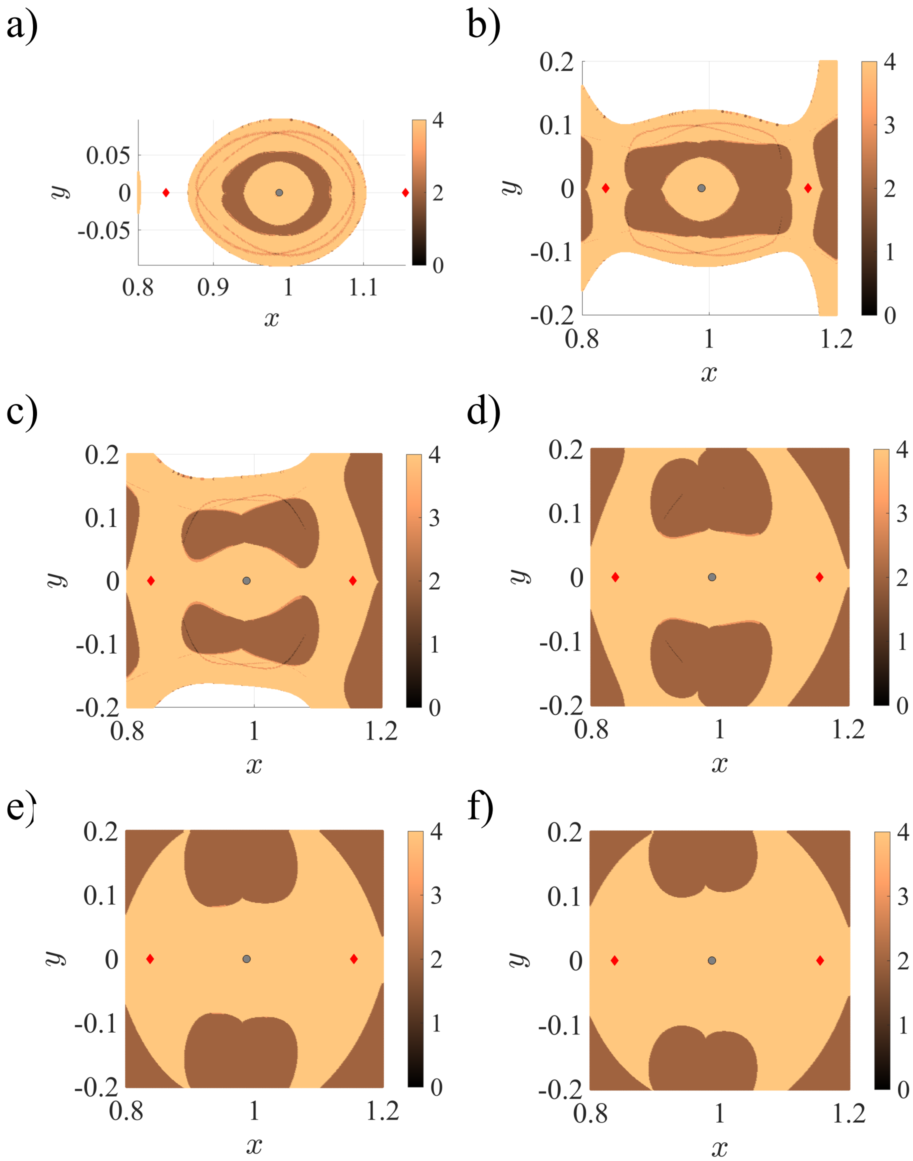

Figure 9. Total number of extrema in the curvature for planar states that pass through each position vector near the Moon at various Jacobi constants: a) $C_J = 3.20$**, b)** $C_J = 3.15$**, c)** $C_J = 3.10$**, d)** $C_J = 3.05$**, e)** $C_J = 3.00$**, f)** $C_J = 2.95$**.**

EXISTENCE OF CURVATURE EXTREMA FOR SPATIAL MOTION

For a specified position vector, the velocity vectors that lead to spatial curvature extrema form one or more one-parameter curves.

A discrete approximation of the curves of velocity vectors that lead to spatial curvature extrema are calculated using a root-finding approach. First, each velocity vector is rewritten in terms of two angles, θ_{xy} and θ_z, as

$$\bar{v} = v\cos(\theta_{xy})\cos(\theta_z)\hat{x} + v\sin(\theta_{xy})\cos(\theta_z)\hat{y} + v\sin(\theta_z)\hat{z} \tag{7}$$

where v corresponds to the speed at a desired Jacobi constant and a given position vector. Once a

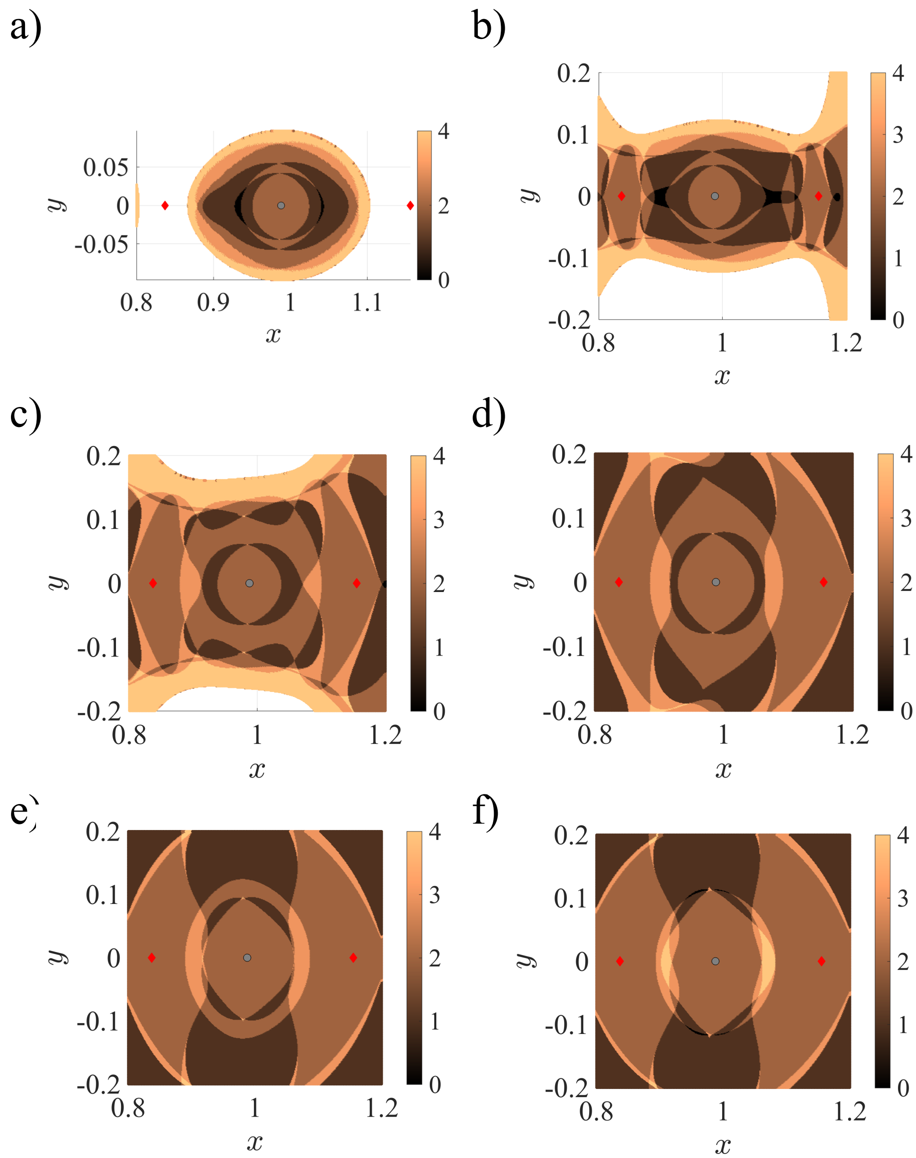

Figure 10. Total number of maxima in the curvature for planar states that pass through each position vector near the Moon at various Jacobi constanst: a) $C_J = 3.20$**, b)** $C_J = 3.15$**, c)** $C_J = 3.10$**, d)** $C_J = 3.05$**, e)** $C_J = 3.00$**, f)** $C_J = 2.95$**.**

value of $\theta_z \in [-90°, 90°]$ is selected, θ_{xy} is varied between 0 and 360 degrees in steps of 1 degree to construct various combinations of $\dot{x}$ and $\dot{y}$. The associated velocity vectors are then used to identify initial guesses for turning points in the curvature at fixed angles θ_z: if two adjacent velocity vectors produce distinct signs in $\dot{\kappa}$, one is used to supply an initial guess for a turning point. This initial guess is then corrected to produce a velocity vector that satisfies the following two conditions:

$$\frac{d\kappa(\bar{x})}{dt} = 0, \quad \text{and} \quad \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{v^2} - 1 = 0 \tag{8}$$

given fixed values of x, y, z, and $\dot{z}$. This root-finding problem is solved using the *fsolve* function in MATLAB to produce a turning point, if one exists. The value of $\ddot{\kappa}$ is then calculated to identify

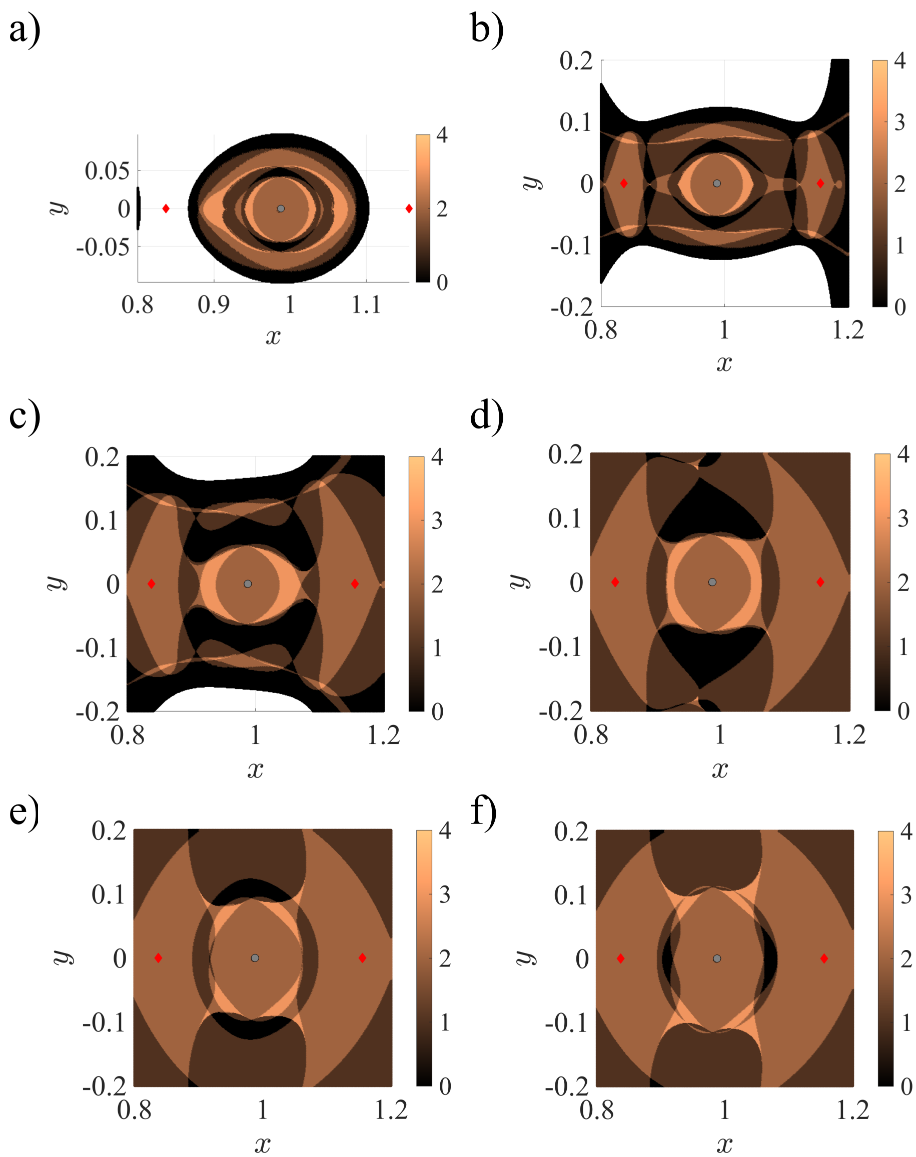

Figure 11. Total number of minima in the curvature for planar states that pass through each position vector near the Moon at various Jacobi constants: a) $C_J = 3.20$**, b)** $C_J = 3.15$**, c)** $C_J = 3.10$**, d)** $C_J = 3.05$**, e)** $C_J = 3.00$**, f)** $C_J = 2.95$**.**

whether the solutions are minima or maxima in the curvature. This process is repeated for various values of θ_z to produce discrete approximations of one or more one-parameter families of velocity vectors that supply extrema in the curvature at a single position vector.

As an example, Figure 12 depicts velocity unit vectors that lead to curvature extrema for two position vectors in the Earth-Moon CR3BP at a Jacobi constant of 3.165 on the gray unit sphere: a) located close to the Moon at x=0.98784, y=0.08, z=0 and b) located close to L_1 at x=0.86, y=0.05, z=0. Figure 12a) displays the velocity unit vectors that lead to curvature maxima (blue) and apses with respect to the Moon (magenta) for a state that is located close to the Moon. In this case, the velocity unit vectors associated with apses with respect to the Moon closely aligns with one family of velocity unit vectors that produce maxima in the curvature. Figure 12b) displays three families of velocity unit vectors that lead to curvature maxima (blue) and minima (teal) as well as apses with respect to the Moon (magenta), L_1 (red), and L_2 (brown) for a state that is located close to L_1. For this position vector, there are two small curves that are solely composed of curvature minima, where one lies near the two planes traced out by apses relative to the Moon and L_2. In addition, part of one curve formed by maxima and minima in the curvature extrema lies close to the plane formed by apses relative to L_1.

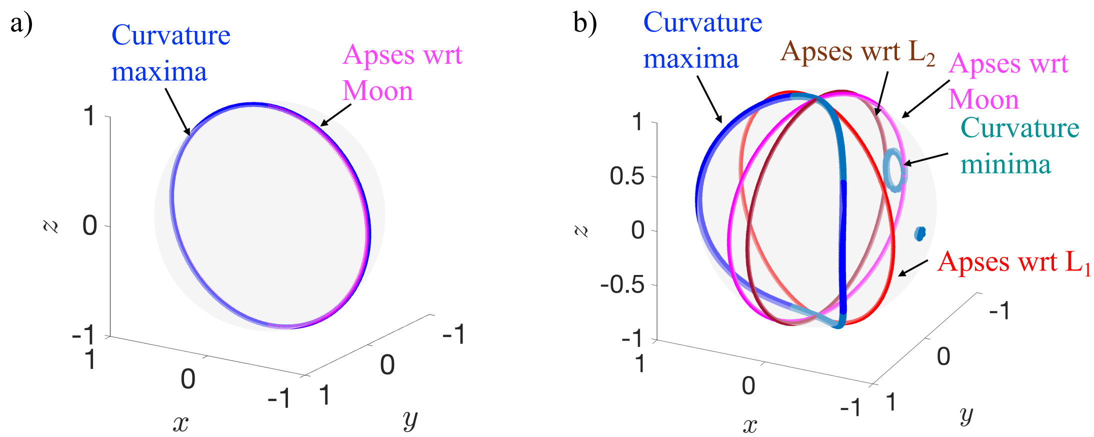

Figure 12. Velocity unit vectors that lead to extrema in the curvature and apses for two position vectors in the Earth-Moon CR3BP at a Jacobi constant of 3.165: a) located close to the Moon at x=0.98784, y=0.08, z=0 and b) located close to L_1 at x=0.86, y=0.05, z=0.

As the position vector is varied, families of velocity unit vectors evolve in their number, direction, and region of existence on the unit sphere. In some cases, multiple distinct, closed curves meet and form a single, more complex curve via bifurcations. To demonstrate this complex evolution, consider an array of position vectors with the common values of $x = 0.86$ and $z = 0$, placing the spacecraft in the vicinity of L_1 an in the plane of the primaries. As the y-coordinate is varied in the range $y \in [0, 0.095]$, the curves formed by velocity unit vectors that produce curvature extrema are depicted in Fig. 13 using two three-dimensional perspectives. Closest to the x-axis, two curves exist and are each composed of minima and maxima. As y is increased, the two curves grow and when they meet, a bifurcation occurs, swapping branches of the curves. Increasing y further, the smallest curve that consists of velocity vectors points approximately towards L_1 shrinks until a bifurcation where two smaller closed curves form. These small curves shrink and then start to shift away from each other. The curves that are pointed towards the zero velocity curves above the x-axis then meet and result in another bifurcation. At $y = 0.95$ when the spacecraft is closest to the zero velocity curves, one large curve of velocity unit vectors lies close to apses relative to L_1, capturing motion that revolves in its vicinity. Two smaller curves of velocity unit vectors are located almost symmetrically around that plane and correspond to paths that are about to or have recently bounced off the zero velocity curves. Generalizing these observations on the evolution of these bifurcations across the phase space is an area of ongoing work.

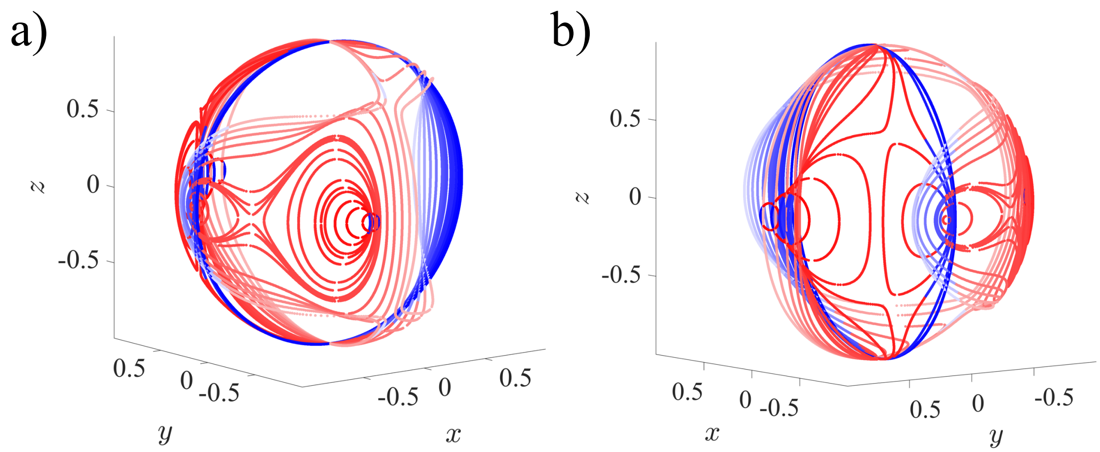

Figure 13. Two three-dimensional perspectives of the families of velocity unit vectors that lead to extrema in the curvature for position vectors with $x = 0.86$ and $z = 0$ but $y \in [0.01, 0.095]$ in the Earth-Moon CR3BP at a Jacobi constant of 3.15: darker colors indicated larger values of y whereas shades of blue located maxima and shades of red locate minima.

CONCLUDING REMARKS

This paper presented a preliminary numerical exploration of extrema in the curvature in the rotating frame for the Earth-Moon circular restricted three-body problem. The curvature captures the deviation of a path at some instant of time from a line. Thus, curvature extrema offer insight into geometrically meaningful locations along a trajectory. These extrema can exist close to apses defined relative to meaningful reference points when the spacecraft is orbiting about them, but offer the benefit of not requiring specification of the reference point. In the planar circular restricted three-body problem, the number and properties of extrema vary across regions throughout the system, with additional extrema appearing as trajectories bounce off the zero velocity curves or perform loops near the libration points. As the energy level varies, these regions shift. This analysis was then extended to spatial motion with selected examples of the families of curves formed by velocity unit vectors that produce curvature extrema for specified position vectors. Visualizing the curves of velocity unit vectors that produce extrema reveals a complex variation where bifurcations occur.

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REFERENCES

[1] W. S. Koon et al. *Dynamical Systems, the Three Body Problem and Space Mission Design*. New York: Marsden Books, 2011.

[2] N.M. Patrikalakis, T. Maekawa, and W. Cho. *Shape Interrogation for Computer Aided Design and Manufacturing*. 2009.

[3] V. Szebehely. *Theory of Orbits: The Restricted Problem of Three Bodies*. London: Academic Press, 1967.

[4] K.L. Wardle. *Differential Geometry*. Mineola, NY: Dover Publications, Inc., 2008.