

AUTOMATICALLY DESCRIBING THE GEOMETRY OF A SPACECRAFT TRAJECTORY IN A MULTI-BODY SYSTEM

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This paper uses differential geometry to automatically describe the geometry and itinerary of a complex spatial trajectory in the Earth-Moon circular restricted three-body problem. A trajectory is first geometrically segmented based on substantial changes in the shape, revealed by minima in the curvature and inflection points. The shape of each segment and the state vector at its maximum curvature point are used to 1) extract a set of geometric descriptors, and 2) hypothesize whether the trajectory revolves or transits through a specific region of the system. This approach is demonstrated using trajectories of various geometries and energy levels.

INTRODUCTION

A large set of spacecraft trajectories generated in a chaotic multi-body gravitational system can exhibit a wide variety of geometries and itineraries. Such complex datasets may be generated during analysis of the solution space, trajectory design, or trajectory prediction. Automatically extracting meaningful descriptions of these geometries and itineraries can support reducing the burden on human analysts, informing decision-makers, and even automating the path-planning process.

Dynamical systems techniques have traditionally been used to describe the itinerary of a trajectory in a multi-body gravitational system. In the circular restricted three-body problem (CR3BP), the phase space can be divided into regions, e.g., near each of the two primaries and the exterior region.¹ Stable and unstable manifolds of libration point orbits are often generated to visualize and analyze the tube-like structures governing natural transport between these regions via the libration point gateways.¹ These manifolds can also intersect to form lobes that describe transitions between localized regions of the phase space.¹ Arcs along or inside specified hyperbolic invariant manifolds can be sequenced in a desired order to design or extract a high-level itinerary of a trajectory.¹ These approaches have been valuable in designing complex spacecraft trajectories, studying resonance transitions, and describing the path of celestial bodies.¹

However, existing approaches may struggle to be generalizable across broader regions of the phase space and dynamical models, or in extracting additional information. For instance, precise definitions of the regions of a system or libration point gateways can be challenging to extend to a wide array of energy levels. Existing approaches also rely on the existence and a priori knowledge of the governing hyperbolic invariant manifold structures. As the fidelity of the dynamical model is

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increased or continuous thrust is incorporated, such structures may be difficult to compute, might not exist, or their finite-time equivalents may be challenging to generate. Furthermore, for spatial motion, analysis becomes substantially more challenging than for planar motion. Finally, these approaches may not supply detailed geometric information, e.g., the evolution of the trajectory in three-dimensional space or even detecting revolutions around an arbitrary region of the phase space.

This paper uses differential geometry to automate describing the geometry and itinerary of a complex spatial trajectory in the Earth-Moon CR3BP. First, a trajectory is geometrically segmented into smaller arcs based on the following locations in the Earth-Moon rotating frame: local minima in the curvature, i.e., where the path curves the least; and states where the curvature is equal to zero. Through this definition, each interior segment contains a maximum in curvature that is used to summarize its average geometry. Geometric descriptors are inspired by the traditional orbital elements from the two-body problem, but calculated using curve-based information from differential geometry that only depends on the position vectors and its time derivatives. These quantities are also used to differentiate between a segment exhibiting revolutions around a reference point or transiting through a region. Then, the sequence of segments is used to extract an approximate itinerary. This procedure is demonstrated and assessed using trajectories of various geometries and energy levels.

DYNAMICAL MODEL

The motion of a spacecraft is approximated via the Earth-Moon circular restricted three-body problem (CR3BP). This model assumes that a spacecraft is influenced only by the gravity of the Earth and Moon, each modeled with the same gravitational field as a point mass with constant mass.² The spacecraft, however, is assumed to possess a comparatively negligible mass. The two celestial bodies are also assumed to follow circular orbits about their barycenter.²

The motion of the spacecraft is typically expressed in nondimensional form in the Earth-Moon rotating frame.² Length, mass, and time quantities are nondimensionalized using characteristic quantities $l^* = 384,400$ km, $m^* \approx 6.045626 \times 10^{24}$ kg, and $t^* \approx 3.7519 \times 10^5$ s to set the distance between the Earth and Moon, system mass, and mean motion of the Earth-Moon system to unity.² Using these definitions, the ratio of the Moon's mass to the system's total mass is labeled as $\mu = 1.215058439471 \times 10^{-2}$. The Earth-Moon rotating frame is defined with the origin at the Earth-Moon barycenter. The axes $\hat{x}, \hat{y}, \hat{z}$ are then defined as follows: $\hat{x}$ points from the center of the Earth to the center of the Moon, $\hat{z}$ is aligned with the orbital angular momentum vector of the Earth-Moon system, and $\hat{y}$ completes the orthogonal right-handed triad.²

The equations of motion for the spacecraft are expressed nondimensionally in the Earth-Moon rotating frame, producing an autonomous dynamical system. The state vector for the spacecraft is defined as $\bar{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$. Then, the equations of motion are written as²

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}, \quad \ddot{z} = \frac{\partial U^*}{\partial z} \quad (1)$$

where the pseudopotential function is defined as

$$U^* = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \quad (2)$$

and $\rho_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}$ and $\rho_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$.² Using these equations, an integral of motion is expressed as $C_J = 2U^* - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$ and is labeled the Jacobi constant.

DIFFERENTIAL GEOMETRY OF SPATIAL CURVES

Differential geometry is useful for analyzing the shape of a curved path in three-dimensional space. At a single instant of time along a nonlinear trajectory in the rotating frame, the position, velocity, and acceleration vectors are defined as $\bar{r}(t) = [x, y, z]^T$, $\dot{\bar{r}}(t) = [\dot{x}, \dot{y}, \dot{z}]^T$, and $\ddot{\bar{r}}(t) = [\ddot{x}, \ddot{y}, \ddot{z}]^T$, respectively. The arclength is the distance traversed along this path from an initial time t_0 to a final time t_f and is calculated as³

$$s = \int_{t_0}^{t_f} ds = \int_{t_0}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \quad (3)$$

This positive quantity increases with time and supports parameterizing a trajectory in a geometric manner that is not significantly influenced by variations in speed.

Measures of Curvature

The curvature reflects the extent to which the trajectory deviates from a straight line in its instantaneous plane of motion, i.e., the osculating plane.³ This nonnegative scalar, $\kappa(t)$, is calculated directly from the position vector and its time derivatives as⁴

$$\kappa(t) = \frac{\|\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)\|}{\|\dot{\bar{r}}(t)\|^3} \quad (4)$$

However, a singularity exists when the speed is zero.

The torsion, or second curvature, captures the extent to which a curve twists out of its osculating plane in three-dimensional space.⁴ This scalar quantity is calculated when $\kappa(t) \neq 0$ as⁴

$$\tau(t) = \frac{\dot{\bar{r}}(t) \cdot \ddot{\bar{r}}(t) \times \ddot{\bar{r}}(t)}{\|\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)\|^2} \quad (5)$$

The sign of this quantity reflects the direction in which the osculating plane twists.

Stationary Points in Curvature

Stationary points in the curvature have been used to identify the most geometrically meaningful locations along a spatial curve. In this paper, stationary points of interest include local minima in the curvature where $\dot{\kappa} = 0$ and $\ddot{\kappa} > 0$; local maxima in the curvature where $\dot{\kappa} = 0$ and $\ddot{\kappa} < 0$; and inflection points where $\kappa = 0$ and the path transitions between convex and concave motion.⁴ The time derivatives of the curvature are calculated as

$$\dot{\kappa}(t) = \frac{(\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)) \cdot (\ddot{\bar{r}}(t) \times \ddot{\bar{r}}(t))}{(\|\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)\|)(\|\ddot{\bar{r}}(t)\|^3)} - \frac{3\|\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)\|(\dot{\bar{r}}(t) \cdot \ddot{\bar{r}}(t))}{\|\dot{\bar{r}}(t)\|^5} \quad (6)$$

and

$$\ddot{\kappa}(t) = \kappa \left[\frac{\|\dot{\bar{r}} \times \ddot{\bar{r}}\|^2 + (\dot{\bar{r}} \times \ddot{\bar{r}}) \cdot (\ddot{\bar{r}} \times \ddot{\bar{r}})}{\|\dot{\bar{r}} \times \ddot{\bar{r}}\|^2} - \frac{[(\dot{\bar{r}} \times \ddot{\bar{r}}) \cdot (\ddot{\bar{r}} \times \ddot{\bar{r}})]^2}{\|\dot{\bar{r}} \times \ddot{\bar{r}}\|^4} - \frac{6(\dot{\bar{r}} \times \ddot{\bar{r}}) \cdot (\ddot{\bar{r}} \times \ddot{\bar{r}})(\dot{\bar{r}} \cdot \ddot{\bar{r}})}{\|\dot{\bar{r}} \times \ddot{\bar{r}}\|^2 \|\dot{\bar{r}}\|^2} - \frac{3(\|\ddot{\bar{r}}\|^2 + \dot{\bar{r}} \cdot \ddot{\bar{r}})}{\|\dot{\bar{r}}\|^2} + \frac{15(\dot{\bar{r}} \cdot \ddot{\bar{r}})^2}{\|\dot{\bar{r}}\|^4} \right] \quad (7)$$

In these expressions, singularities occur when 1) the velocity and acceleration vectors are parallel or antiparallel, i.e., when $\kappa = 0$, or 2) when the speed is equal to zero. These locations have been used by Farouki⁵ as well as Cho, Maekawa, and Patrikalakis⁶ to geometrically segment curved trajectories into smaller arcs; this approach is a type of shape interrogation technique.⁴

Integrals of Curvature and Torsion

Integrating the curvature and torsion along the arclength of a trajectory produces the total absolute curvature and total torsion. The total absolute curvature is defined as⁷

$$\kappa_{tot}(t_0, t_f) = \int_{s(t_0)}^{s(t_f)} \kappa(s) ds = \int_{t_0}^{t_f} \kappa(t) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \quad (8)$$

and reflects the angle traced out by the tangent vector. As the trajectory completes a revolution in the rotating frame, this quantity increases by 2π . The total absolute torsion is defined as⁷

$$\tau_{tot}(t_0, t_f) = \int_{s(t_0)}^{s(t_f)} \|\tau(s)\| ds = \int_{t_0}^{t_f} \|\tau(t)\| \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \quad (9)$$

Conceptually, this integral reflects the angular change in the osculating plane over time, increasing by 2π as the osculating plane completes a revolution in three-dimensional space.

Frenet Axes

The Frenet axes are three orthogonal, right-handed axes that evolve with the curvature of a trajectory.⁸ The three axes are $\hat{T}$, which is parallel to the unit velocity vector; $\hat{N}$, which is the principal normal vector that points towards the center of curvature within the osculating plane; and $\hat{B}$, which is the binormal vector and perpendicular to the osculating plane.³ When $\kappa \neq 0$ and the speed is nonzero, these unit vectors are calculated as³

$$\hat{T} = \frac{\dot{\vec{r}}}{\|\dot{\vec{r}}\|} \quad \hat{N} = \hat{B} \times \hat{T} \quad \hat{B} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|} \quad (10)$$

When $\kappa = 0$, $\hat{N}$ and $\hat{B}$ are undefined. On either side of this singularity, these axes flip by 180° .

Center of Curvature

The instantaneous center of curvature is straightforwardly located in terms of the Frenet axes. Specifically, the position vector $\bar{e}$ that locates the center of curvature for a state $\bar{x}(t)$ along a trajectory is calculated as

$$\bar{e} = \bar{r} + \frac{1}{\kappa} \hat{N} \quad (11)$$

When $\kappa = 0$, the center of curvature is undefined as the path instantaneously resembles a line.

The center of curvature, calculated along the entire path, traces out a curve in the configuration space that is labeled as the curve's evolute.⁴ At the locations of local minima or maxima in curvature along the curved path, the associated points along the evolute correspond to cusps. Furthermore, locations of zero curvature along the path correspond to asymptotes to the evolute when the normal vector is undefined.

Gauss Linking Integral

The Gauss linking integral,⁹ also labeled as the linking number, is a scalar quantity in knot theory that captures the degree of linking between two closed curves. The Gauss linking integral is approximated along a discretely represented curve as

$$\ell = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\bar{r}_1 - \bar{r}_2}{\|\bar{r}_1 - \bar{r}_2\|^3} \cdot (d\bar{r}_1 \times d\bar{r}_2) \approx \frac{1}{4\pi} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{(\bar{m}_{1,i} - \bar{m}_{2,j}) \cdot (\Delta \bar{r}_{1,i} \times \Delta \bar{r}_{2,j})}{\|\bar{m}_{1,i} - \bar{m}_{2,j}\|^3} \quad (12)$$

where $\bar{r}_k$ is the position vector of a point on curve γ_k , and $d\bar{r}_k$ is the infinitesimal displacement vector along curve γ_k , for $k = 1, 2$. This integral is approximated as a double sum over the discretized states of the two curves. Representing each curve by N_k segments, $\Delta\bar{r}_{k,i}$ is the vector connecting consecutive states i and $i + 1$ along γ_k , and $\bar{m}_{k,i}$ is the midpoint of this segment, for $k = 1, 2$. This quantity produces an integer value for two closed curves: the magnitude indicates the number of times two curves link with one another, whereas the sign indicates the direction of linking. In astrodynamics, this quantity has been used by Owens and Baresi to construct heteroclinic connections between quasi-periodic orbits:¹⁰ the linking number identified when two closed curves formed by states along their hyperbolic invariant manifolds intersected.

Although the Gauss linking integral is formally defined for closed curves, this quantity has been used to approximate the extent of linking between two open curves. However, for open curves, the Gauss linking integral is no longer expected to produce integer values. For instance, Niemyska et al.¹¹ applied the Gauss linking integral to open curves to reveal a lasso-type topology in protein backbones. This information has also been used to extract local topological information for open curves. Specifically, Panagiotou et al.¹² defined a local periodic linking number that measures the entanglement between segments of chains under periodic boundary conditions. Alternatively, Shen et al.¹³ segmented a curve and computed the Gauss linking integral at multiple scales to capture local structure and connectivity of each curve segment. Although the Gauss linking integral loses its strict topological invariance property when applied to open curves, this quantity can be useful for studying the relative geometry between two curves on global and local scales.

TECHNICAL APPROACH

To automatically extract a geometric description of a spacecraft trajectory in a multi-body system, concepts from differential geometry are employed. First, the trajectory is segmented using information about its curvature. In this step, the trajectory is assumed to be regular, corresponding to possessing a nonzero speed at all locations over the examined duration; addressing this singularity is an area of future work. Then, each segment is described using geometric information calculated only from its position vector and time derivatives. Finally, common information along sequential segments is used to estimate its itinerary through the system. Each of these three steps is demonstrated using a single trajectory that passes through various regions with an evolving geometry.

Segmenting a Trajectory

Inspired by shape interrogation schemes, a trajectory is first segmented using its curvature. The boundary conditions of each segment are identified as: 1) local minima in curvature and 2) inflection points where the curvature is zero. Thus, each segment contains a local maximum in curvature, which often occurs at locations close to apses relative to meaningful reference points, such as primaries and/or libration points.¹⁴

Local minima and maxima in curvature as well as zero curvature states are identified during numerical integration of the trajectory. Each type of point is used to formulate one or more event functions that is equal to zero when the event is satisfied. Then, the potential for an event to occur is identified through a sign change in one or more associated logic functions between two subsequent states generated during propagation. When a sign change occurs in a desired direction, Brent's root-finding algorithm¹⁵ is used to efficiently identify the event along the trajectory. Brent's method combines the bisection method, secant method, and inverse quadratic interpolation to locate the root of an equation. This algorithm first attempts to use the faster secant method or inverse quadratic

interpolation to find a root, but falls back to the robust bisection method if the interpolated step falls out of the current bracket or fails to reduce the interval sufficiently.¹⁵

To detect local minima and maxima in curvature, a single event function is defined: $f_1 = \kappa(t)$. However, this quantity can possess a wide array of values as the curvature evolves along a trajectory. Thus a root is identified between two states at the time interval boundaries t_i and t_{i+1} when $f_1 \leq \epsilon_1$. This tolerance is calculated as $\epsilon_1 = 10^{-9}(1 + \min(f_1(t_i), f_1(t_{i+1})))$.

Because curvature is a non-negative quantity, the traditional sign change approach for detecting a zero value within a time interval is not applicable. To address this challenge, a set of event functions and associated logic is employed. The potential for a zero curvature event is first identified if the sign of any of the following event conditions changes:

$$f_2 = \dot{z}\ddot{x} - \dot{x}\ddot{z} \quad (13)$$

$$f_3 = \dot{y}\ddot{z} - \dot{z}\ddot{y} \quad (14)$$

$$f_4 = \dot{x}\ddot{y} - \dot{y}\ddot{x} \quad (15)$$

$$f_5 = \dot{y}\dot{U}_z + \ddot{y}U_z + \ddot{z}(2\dot{x} - U_y) + \dot{z}(2\ddot{x} - \dot{U}_y) \quad (16)$$

$$f_6 = \dot{x}\dot{U}_z - \ddot{x}U_z + \ddot{z}(2\dot{y} + U_x) + \dot{z}(2\ddot{y} + \dot{U}_x) \quad (17)$$

$$f_7 = -\ddot{x}(2\dot{x} - U_y) - \dot{x}(2\ddot{x} - \dot{U}_y) - \ddot{y}(2\dot{y} + U_x) - \dot{y}(2\ddot{y} + \dot{U}_x) \quad (18)$$

where U_i is the partial derivative of the pseudopotential function with respect to the position coordinate i . The functions f_2 - f_4 are equal to the individual components of the cross product in the numerator of Eq. 4. However, f_5 - f_7 capture when a component of curvature possesses a turning point that exactly equals zero. If a root of any of these six conditions is identified, then the value of the curvature is calculated at the associated state. If $|\kappa| \leq 10^{-6}$, a zero curvature state is detected.

This segmentation approach is demonstrated for a complex spatial trajectory in the Earth-Moon CR3BP. The initial position and velocity vectors of this trajectory are equal to

$$\bar{r}_0 = [1.026758344732382, -2.102075762073469 \times 10^{-2}, -1.004328718180706 \times 10^{-3}]^T$$

$$\bar{v}_0 = [5.570406652315534 \times 10^{-1}, -1.847357519636186 \times 10^{-1}, 3.043944890530932 \times 10^{-2}]^T$$

and correspond to a Jacobi constant of 3.16. The trajectory, propagated for 13.89528348337054 nondimensional time units, is depicted in Figure 1 in the Earth-Moon rotating frame, with the a) xy -projection and c) spatial view of the entire trajectory; and b) xy -projection and d) spatial view of the region near the Moon. Twelve local minima in curvature are indicated by black unfilled circles whereas the only zero curvature point is located by a black unfilled square that is visible in the bottom left of Figure 1b). These thirteen geometrically sampled locations, along with the initial and final states, result in the trajectory being discretized into fourteen segments. Alternating segments are colored in red and blue for visual clarity. Each segment bounds a local maximum in the curvature, plotted as a black asterisk.

Describing the Geometry of Each Segment

Each segment is described geometrically using information about the evolution of the position vector and its time derivatives. These quantities are inspired by the orbital elements from the two-body problem that capture the size, shape, and orientation of conic sections relative to specified celestial bodies. However, the geometric descriptors used in this paper capture local shape and

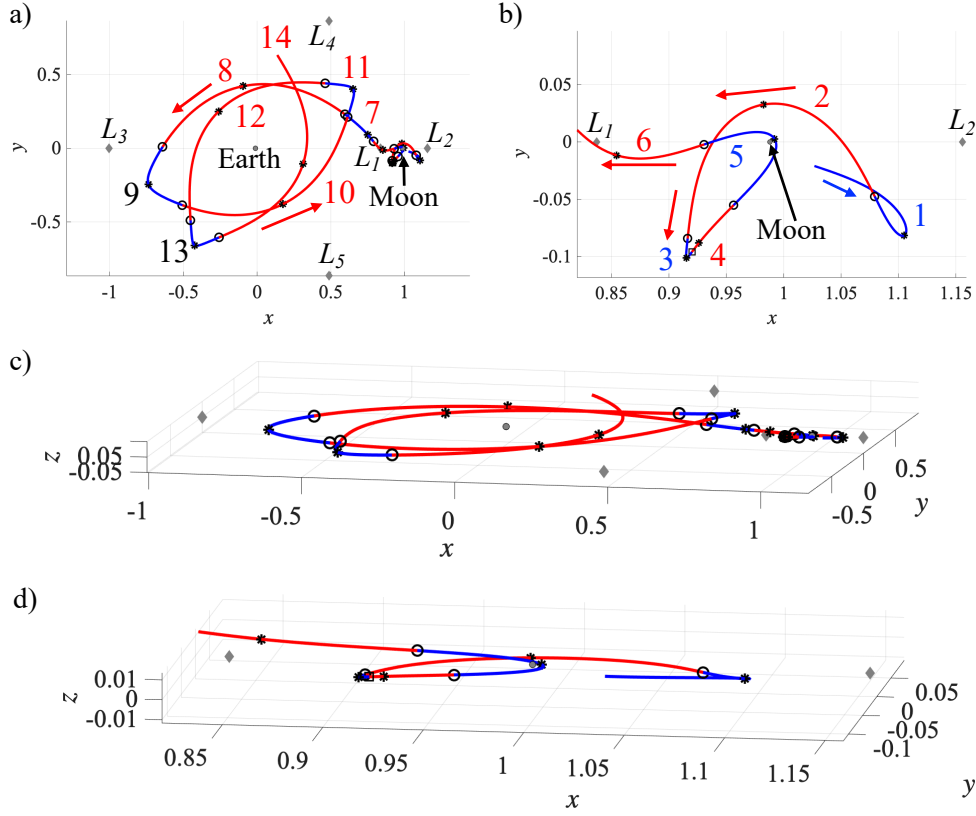


Figure 1. Example trajectory: a) and c) full trajectory; b) and d) zoomed-in view near Moon; a) and b) projection onto xy -plane; and c) and d) spatial view. Segments are defined using boundary conditions at local minima in curvature (black circles) and zero curvature points (black square); alternating segments are colored red and blue. Local maxima in curvature along each segment are located by black asterisks.

orientation information at either the local maximum in curvature or along the segment. The local maximum in curvature is used as a representative state because it is located at the most geometrically meaningful point of the segment.

Descriptors that are calculated along the i th segment over the time interval $t \in [t_i, t_{i+1}]$ include

1. Angle traced out in the osculating plane, $\Delta\theta$:

$$\Delta\theta = \kappa_{tot}(t_i, t_{i+1}) \quad (19)$$

Conceptually, $\Delta\theta$ is comparable to the true anomaly traced out along a conic section relative to a celestial body in the two-body problem. However, in this case, the center of curvature and osculating plane evolve over time. Furthermore, when $\Delta\theta$ passes through each integer multiple of π , the spacecraft completes an additional half-revolution in the rotating frame.

2. Accumulated change in orientation of the osculating plane, Δi :

$$\Delta i = \tau_{tot,mod}(t_i, t_{i+1}) \quad (20)$$

where $\tau_{tot,mod}$ is the total absolute torsion that captures the angle that the binormal vector traces out in three-dimensional space, but slightly modified to accommodate any zero curva-

ture boundary conditions as

$$\tau_{tot,mod}(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} \frac{||\dot{\bar{r}}(t)\ddot{\bar{r}}(t)\ddot{\bar{r}}(t)||}{||\dot{\bar{r}}(t) \times \ddot{\bar{r}}(t)||^2 + 10^{-6}} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \quad (21)$$

This modification includes adding a small constant value of 10^{-6} to the denominator of Eq. 9 to ensure that the singularity is not encountered at zero curvature points.

3. Distance traversed, Δs :

$$\Delta s = s(t_i, t_{i+1}) \quad (22)$$

This quantity captures the size of the arc traced out in the rotating frame.

4. Occurrence of a loop in three-dimensional space: A segment may appear to form a loop as the direction of motion and/or orientation of the osculating plane changes substantially. To identify whether a segment forms a loop, the Gauss linking integral is computed along the segment, described by $\bar{r}(t)$, and its evolute, $\bar{e}(t)$. The linking number is calculated as described in Equation 12. Then, if $|\ell| > 0.85$ the segment is labeled as exhibiting looping behavior; this tolerance is selected empirically.

A set of geometric descriptors are calculated at the curvature maximum located along each segment. These quantities are inspired by the angular orbital elements from the two-body problem that describe the orientation of the orbit plane and the orientation of the orbit within this plane. However, a trajectory in the CR3BP can revolve around or pass near more general reference points and its orbit plane can change over time. This paper defines a representative osculating plane $\mathcal{O}$ as the instantaneous osculating plane evaluated at the local maximum in curvature, $\bar{x}_{\kappa,max}$. This plane is depicted conceptually in Figure 2 for a segment that is colored in blue. In this figure, the curvature maximum is identified by an asterisk and the Frenet axes at that reference state are drawn in red. The equilibrium points are conceptually depicted as red diamonds. Note that this figure is not drawn to scale.

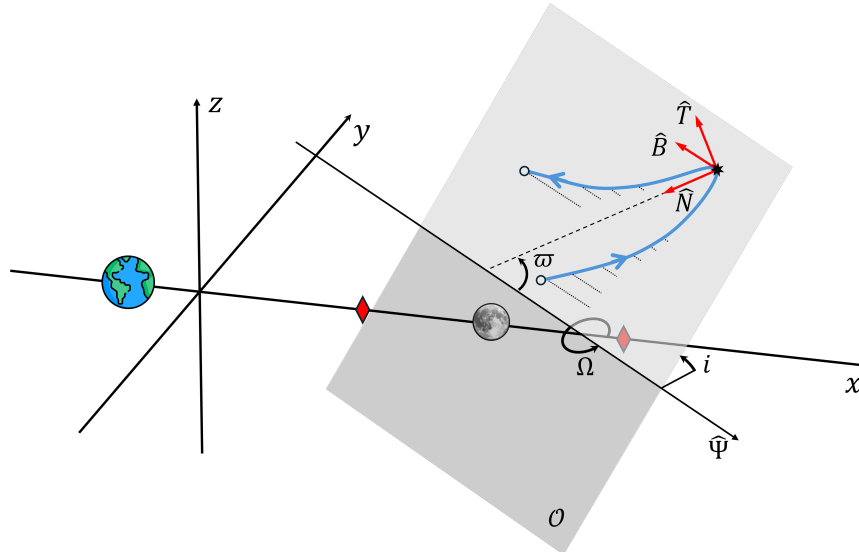


Figure 2. Conceptual definition of geometric descriptors calculated at curvature maxima along a segment.

The geometric descriptors calculated at the maximum in curvature along the i th segment, with the state vector $\bar{x}_{\kappa, \max, i}$, are defined as follows:

1. Elevation angle of the representative osculating plane, i :

$$i = \cos^{-1}(\hat{B}(\bar{x}_{\kappa, \max, i}) \cdot \hat{z}) \quad (23)$$

where $\hat{B}(\bar{x}_{\kappa, \max, i})$ indicates that the binormal vector is calculated at this geometrically meaningful state. The quantity i is inspired by the inclination from the two-body problem, comparing the angle between $\mathcal{O}$ and a reference plane. However, unlike the inclination, i does not require specification of a suitable reference celestial body to compute relative position and orbital angular momentum vectors.

2. Azimuthal angle of the representative osculating plane, Ω : Inspired by the line of nodes from the two-body problem, $\hat{\Psi} = \hat{z} \times \hat{B}(\bar{x}_{\kappa, \max, i})$ is a vector along the line of intersection between the representative osculating plane and the xy -plane. Then, the azimuthal angle is defined as:

$$\Omega = \begin{cases} \cos^{-1}(\hat{\Psi} \cdot \hat{x}), & \hat{\Psi} \cdot \hat{y} \geq 0 \\ 2\pi - \cos^{-1}(\hat{\Psi} \cdot \hat{x}), & \hat{\Psi} \cdot \hat{y} < 0 \end{cases} \quad (24)$$

Inspired by the right ascension of the ascending node from the two-body problem, this angle orients the representative osculating plane relative to the $\hat{x}$ reference direction.

3. Argument of the curvature maximum, ϖ :

$$\varpi = \begin{cases} \cos^{-1}(-\hat{N}(\bar{x}_{\kappa, \max, i}) \cdot \hat{\Psi}), & \hat{N}(\bar{x}_{\kappa, \max, i}) \cdot \hat{z} < 0 \\ 2\pi - \cos^{-1}(-\hat{N}(\bar{x}_{\kappa, \max, i}) \cdot \hat{\Psi}), & \hat{N}(\bar{x}_{\kappa, \max, i}) \cdot \hat{z} > 0 \end{cases} \quad (25)$$

Analogous to the argument of periapsis from the two-body problem, this quantity captures the angle between 1) the vector from the center of curvature to the curvature maximum, i.e., the negative of the principal normal direction, and 2) the reference $\hat{\Psi}$ direction. Thus, this angle approximates the orientation of the segment within $\mathcal{O}$. However, an ambiguity arises when two segments with similar $\hat{N}(\bar{x}_{\kappa, \max, i})$ are located above versus below the xy -plane. To resolve this ambiguity, the angle ϖ is also accompanied by the sign of the z -coordinate of the curvature maxima, listed in the subscript.

To demonstrate calculation of the two sets of descriptors, consider the trajectory depicted in Figure 1. Each of the quantities listed in this subsection is calculated and summarized in Table 1 along each segment. Note that some angular values are normalized by π to aid interpretation. An interpretable description of each segment is then straightforwardly extracted from each row of this table. For instance, segments 1-3 and 5 each complete nearly half a revolution in the rotating frame, indicating that the spacecraft may revolve within one or more region/s of the system. However, segment 4, which is bound by one zero curvature point, does not. Along segment 1, a loop is detected; this geometry is consistent with a substantial change in the direction of motion and osculating plane between segments 1 and 2. A loop is also detected along segment 5, which is consistent with a sharp geometric change produced by its close lunar flyby; the velocity at the curvature maxima far exceeds that at its endpoints, which concentrates the turning behavior into a short interval of time, giving rise to the loop-like behavior. Segments 6 and 7, however, trace less than half a revolution and are not bounded by any zero curvature points, indicating that they are likely not revolving around a region in the system. However, the direction of motion is changing substantially through large changes in i and large values of Δi . Finally, segments 8-14 tend to trace out a quarter of a revolution in the

rotating frame with little variation in the osculating plane and no loops. Segments in this range, alternate between tracing small and large distances. These assessments, directly derived from the quantities in Table 1 are consistent with visual inspection of the trajectory in Figure 1.

Table 1. Geometric descriptors calculated along each segment for the example trajectory in Figure 1

Segment	Along segment				At curvature maximum		
	$\Delta\theta/\pi$	$\Delta i/\pi$	Δs	Loop? $ \ell $, (Y/N)	i (deg)	Ω (deg)	ϖ (deg)
1	1.253	0.460	0.153	-0.9749 (Y)	170.41	347.55	51.24 ₊
2	0.820	0.070	0.280	-0.0004 (N)	5.92	188.53	271.35 ₋
3	0.830	0.066	0.024	-0.7676 (N)	6.88	165.10	84.79 ₊
4	0.0423	0.470	0.055	-0.0051 (N)	172.64	352.30	210.86 ₊
5	0.850	0.970	0.136	-0.9929 (Y)	5.91	95.85	296.20 ₋
6	0.391	0.744	0.167	0.0042 (N)	171.36	338.07	84.83 ₊
7	0.085	0.287	0.270	0.0057 (N)	14.15	150.13	251.15 ₊
8	0.524	0.111	1.442	-0.0445 (N)	1.77	126.25	335.08 ₋
9	0.574	0.147	0.555	0.1088 (N)	6.03	27.62	171.17 ₊
10	0.510	0.191	1.461	-0.0576 (N)	1.72	315.17	341.33 ₋
11	0.585	0.358	0.393	0.2308 (N)	8.56	217.85	176.03 ₊
12	0.548	0.405	1.564	-0.0865 (N)	1.65	147.94	347.21 ₋
13	0.621	0.702	0.351	0.4473 (N)	10.67	61.24	177.21 ₊
14	0.551	0.502	1.571	-0.0923 (N)	1.64	354.73	347.53 ₋

Extracting the Trajectory Itinerary

An estimate for the trajectory itinerary is extracted using differential geometry to supply a high-level description of the regions visited by the trajectory, and its behaviors in these regions. In this paper, five points of interest are considered: the Earth; the Moon; and the Lagrange points L_1 , L_2 , L_3 , L_4 , and L_5 . Two types of orbiting behaviors are also defined: loitering, i.e., revolving around a common region; and transiting, i.e., passing through a region without completing a substantial portion of a revolution. Assigning each point of interest and the behavioral type is a complex problem as the center of curvature and/or reference osculating plane may not coincide with the selected points of interest. Thus, a heuristic formulation is employed in this paper.

Recall that due to the segmentation scheme, a maximum in curvature exists along each segment. Recent work by Bosanac¹⁴ noted that 1) a stationary point in the curvature can exist close to a traditional apse from the two-body problem measured relative to a specified reference point when the trajectory is revolving around that point; and 2) when the spacecraft is revolving near multiple reference points, the velocity vector at the curvature maximum may lie between the velocity vectors of the apsides relative to those reference points. These observations have motivated the formulation of

a procedure for estimating a suitable reference point to assign to each segment by using the location of the curvature maxima relative to the apses with respect to the selected points of interest within the segment.

When propagating each segment, any apses with respect to each of the points of interest are identified. Consider a point of interest with a position vector $\bar{r}_P$; for the selected points of interest in this paper, their velocity in the rotating frame is zero. An apse with the state vector $\bar{x}_{apse} = [\bar{r}_{apse}^T, \bar{v}_{apse}^T]^T$ relative to this point of interest satisfies the following criterion: $(\bar{r}_{apse} - \bar{r}_P) \cdot \bar{v}_{apse} = 0$. A single segment could contain zero or more apses with respect to each point of interest.

The location of the curvature maximum along the segment is then compared to any apses to identify a suitable point of interest. To perform this comparison, the relative arclength between the curvature maximum and the closest apse to reference point j is defined in percentage form as

$$\rho_j = 100 \left(\frac{s(\bar{x}_{\kappa, max}) - s(\bar{x}_{apse, j})}{\Delta s} \right) \quad (26)$$

A positive (or negative) sign indicates that the apse location occurs before (or after) the curvature maxima along the segment. Furthermore, the alignment between the direction to the center of curvature and the position vector to point of interest j is calculated as

$$\beta_j = \frac{\bar{r}_{apse, j} - \bar{r}_P}{\|\bar{r}_{apse, j} - \bar{r}_P\|} \cdot \hat{N}(\bar{x}_{\kappa, max}) \quad (27)$$

These two quantities are then processed to estimate zero or more suitable reference points to describe the point of interest near a segment:

1. If $\rho_j \leq 10$, the segment is estimated to likely be loitering about or transiting through a region around the point of interest defined by reference point j .
2. If apses measured relative to multiple reference points satisfy the condition $\rho_j \leq 10$, an additional check is performed to determine which reference point is best aligned with the direction to the center of curvature. Specifically, the largest, positive value of β_j identifies the most likely estimate of a suitable reference point.
3. If $\rho_j \geq 10$ for the only apse, the segment is estimated to be likely be transiting near reference point j , but with a low confidence in this assessment.
4. If none of the above criteria are satisfied, the assessments of the adjacent segments are used. If both the adjacent segments are traveling near the same point of interest, there is a high likelihood that the current segment is also traveling in the same region. If however, both adjacent segments are mapped to different points of interest, then there is a high confidence that the segment in question may be a connection between two distinct types of motion.

For the example in Figure 1, Table 2 displays the values of ρ_j for each segment with respect to each of the points of interest, except L_4 and L_5 . In this table, a dash indicates that the segment does not possess an apse relative to this point of interest. The blue highlights locate any values of ρ that satisfy the first two criteria listed in the prior paragraph. The identified point of interest is listed, along with the confidence in this estimate (high or low) in the final column.

To differentiate between loitering and transiting behavior, the total curvature along each segment is employed. Recall that $\Delta\theta$ captures the angle traced out around an evolving center of curvature. In this paper, a segment is classified as producing loitering behavior within a region when $\Delta\theta \geq 0.5\pi$. Otherwise, the segment is considered a transiting segment. This information is added to the second

Table 2. Relative locations of curvature maxima and apses, used to extract behavior estimates. POI indicates point of interest, type indicates loitering (L) vs transit (T) behavior, and conf. indicates confidence in the assessment as high (H) or low (L).

Segment	ρ_E	ρ_M	ρ_{L_1}	ρ_{L_2}	ρ_{L_3}	POI	Conf.	β_{POI}	Type
1	2.94	0.95	1.87	-3.25	3.15	Moon	H	0.8797	L
2	-	-0.1	-30.84	45.29	-	Moon	H	0.999	L
3	41.43	0.36	-3.0	1.67	14.52	Moon	H	0.9653	L
4	-	-	-17.06	-	-	Moon	L	-0.0631	T
5	4.54	10^{-4}	4.76	4.32	4.52	Moon	H	1.0	L
6	-	-	-13.01	-	-	L_1	L	0.3258	T
7	-	-	-	-	-	-	H	-	T
8	-0.34	-	-	-	-	Earth	H	1.0	L
9	5.2×10^{-4}	2.70	2.51	2.88	21.73	Earth	H	0.999	L
10	-0.22	-46.98	-43.90	-50.12	-	Earth	H	1.0	L
11	0.39	-	-	-	3.26	Earth	H	0.9989	L
12	-0.10	-	-	-	-25.84	Earth	H	1.0	L
13	4.7×10^{-2}	4.27	3.74	4.84	-	Earth	H	0.999	L
14	-0.22	-9.35	-9.03	-9.62	-16.34	Earth	H	1.0	L

last column of Table 2 with loitering labeled as “L” and transiting labeled as “T”. Each assessment uses the information previously listed in Table 1.

The behaviors of each segment are then used to extract an itinerary. The first five segments have a common reference point and the sum of their angles traced out in the evolving osculating plane is $\Delta\theta \approx 3.753\pi - 3.7953\pi$, depending on how segment 4 is counted. Thus, at the beginning of the trajectory, the spacecraft predominantly loiters around the Moon for nearly two revolutions. Then, the spacecraft possibly transits through the L_1 gateway. Segments 8-14 have a common reference point and the sum of their angles traced out in the evolving osculating plane is $\Delta\theta \approx 3.913\pi$. Thus, at the end of the trajectory, the spacecraft loiters around the Earth for approximately two revolutions.

RESULTS

This section presents additional examples of trajectories where the geometry and/or itinerary are extracted. Each example is designed to demonstrate the presented technical approach and highlight any interesting challenges. The general structure of the figures and tables follows the conventions demonstrated throughout the technical approach section.

Case 1

Consider a complex spatial trajectory that remains between the L_1 and L_2 gateways. The initial position and velocity vectors of this trajectory are equal to

$$\bar{r}_0 = [9.878494156052903 \times 10^{-1}, -5 \times 10^{-2}, -2 \times 10^{-2}]^T$$

$$\bar{v}_0 = [4.75394 \times 10^{-1}, -1.999903381 \times 10^{-2}, 1.295 \times 10^{-1}]^T$$

These state components correspond to $C_J \approx 3.15968$. When propagated for 17.893 nondimensional time units, the resulting trajectory is plotted in Figure 3 via a) a projection onto the xy -plane and b) a spatial view. Applying the segmentation procedure to this trajectory yields 21 segments. For conciseness, only the even-numbered segments are labeled in Figure 3.

The geometric descriptors for each segment of this trajectory are summarized in Table 3. Segments 4-7 in the vicinity of L_2 have a high elevation angle of the representative osculating plane, in the range of 111° - 115° . Similar behavior is also observed for segments 18-20 in the vicinity of L_1 . Furthermore, segments 6, 7, 15 and 17 are characterized as producing loops in three-dimensional space based on the values of the Gauss integral for these segments; segments 6 and 7 correspond

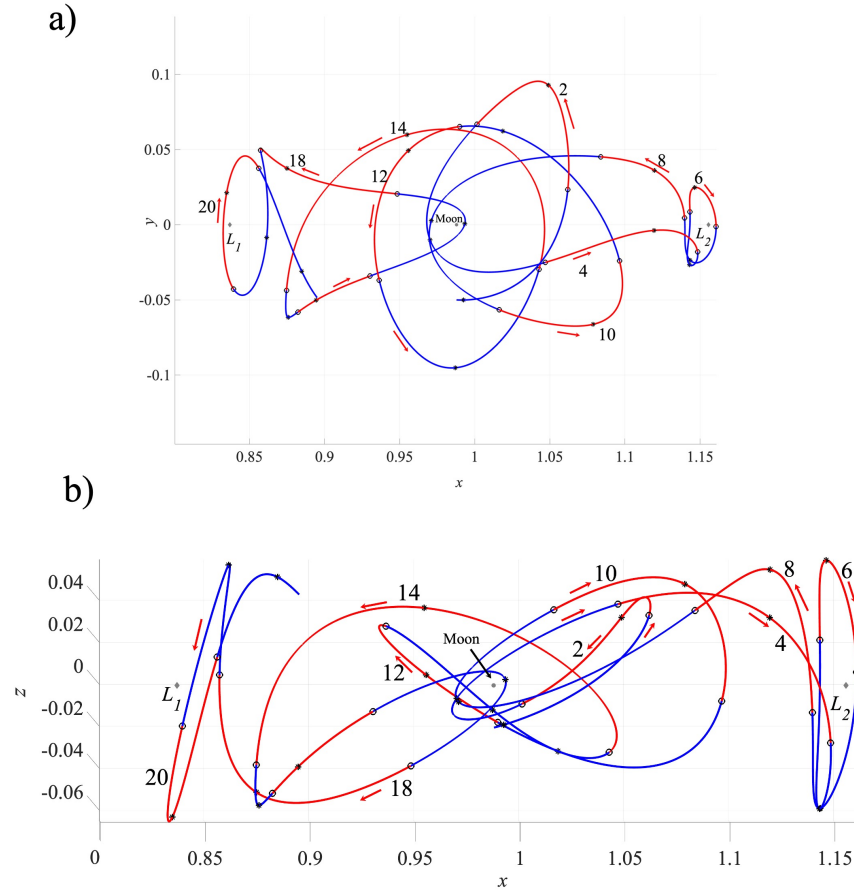


Figure 3. Selected spatial trajectory for case 1 that remains between the L_1 and L_2 gateways: a) projection onto the xy -plane and b) spatial view

to the high inclination revolution near L_2 whereas segments 15 and 17 occur on the left of Figure 3. Across the entire trajectory, the orientation of the representative osculating plane as well as the orientation of the curvature maximum within $\mathcal{O}$ vary substantially along the trajectory, consistent with the evolving geometry of this trajectory that is visually observed in Figure 3.

Table 4 lists and processes the ρ_{apse} values for each of the segments with respect to the identified points of interest, except L_4 and L_5 . The behaviors of some segments are readily discernable, marked as having a high confidence in the final column. However, in some other cases, the confidence may be low or a point of interest may not even be clear.

Based on the information in Table 4, an itinerary is extracted. First, the spacecraft loiters around

Table 3. Geometric descriptors calculated along each segment for case 1 from Figure 3

Segment	Along segment				At curvature maximum		
	$\Delta\theta/\pi$	$\Delta i/\pi$	Δs	Loop? (Y/N)	i (deg)	Ω	ϖ (deg)
1	0.459	0.150	0.128	-0.049 (N)	31.21	330.87	304.28 ₋
2	0.880	0.400	0.148	0.453 (N)	52.34	272.86	149.77 ₊
3	0.877	0.353	0.187	-0.332 (N)	33.80	239.51	326.24 ₋
4	0.615	0.426	0.149	-0.063 (N)	113.24	203.14	129.86 ₊
5	1.052	1.077	0.123	0.338 (N)	111.45	133.13	259.36 ₋
6	1.017	0.614	0.105	-0.954 (Y)	115.89	190.93	84.07 ₊
7	1.042	0.732	0.129	-0.954 (Y)	115.18	129.56	259.47 ₋
8	0.684	0.839	0.123	0.255 (N)	74.81	333.33	78.87 ₊
9	0.943	0.363	0.233	-0.531 (N)	31.53	225.54	301.60 ₋
10	0.768	0.370	0.144	0.188 (N)	60.42	176.72	120.14 ₊
11	0.593	0.177	0.162	-0.082 (N)	37.42	134.31	304.41 ₋
12	0.549	0.121	0.141	-0.052 (N)	31.77	108.14	22.08 ₊
13	0.907	0.226	0.190	0.158 (N)	40.66	75.54	194.92 ₋
14	1.122	0.247	0.338	-0.419 (N)	42.40	22.52	81.40 ₊
15	0.807	0.300	0.036	0.977 (Y)	49.53	347.28	261.86 ₋
16	0.104	0.341	0.067	0.023 (N)	127.70	171.15	37.13 ₋
17	0.918	0.600	0.146	0.865 (Y)	46.02	215.25	141.35 ₊
18	0.733	0.579	0.143	-0.260 (N)	105.73	159.78	300.89 ₋
19	1.054	0.198	0.185	-0.188 (N)	104.85	86.87	99.11 ₊
20	1.040	0.103	0.176	0.157 (N)	102.56	84.43	290.22 ₋
21	0.530	0.292	0.111	0.124 (N)	71.80	111.40	110.92 ₊

Table 4. Relative locations of curvature maxima and closest apses, used to extract behavioral estimates, for case 1 in Figure 3. POI indicates point of interest, the type indicates loitering (L) vs transit (T) behavior, and conf. indicates confidence in the assessment as high (H) or low (L).

Seg.	ρ_E	ρ_M	ρ_{L_1}	ρ_{L_2}	ρ_{L_3}	POI	Conf.	β_{POI}	Type
1	-	-5.508	-	-84.261	-	M	H	0.990	T
2	35.755	1.789	12.911	38.507	-	M	H	0.955	L
3	7.42	0.031	6.67	8.702	7.425	M	H	1.000	L
4	-	-	-	-28.60	-				L
5	-17.017	3.25	10.24	-0.017	-12.303	L_2	H	0.998	L
6	-60.104	-17.855	-54.857	-0.046	-	L_2	H	0.999	L
7	-	-	-	-0.051	56.557	L_2	H	0.9989	L
8	-	18.486	-	57.109	-	M	L	0.646	L
9	-4.284	-0.0184	-3.919	-4.860	-4.320	M	H	1.000	L
10	-35.749	-10.45	-25.68	-	-36.97	M	L	0.898	L
11	-	-	-	79.511	-				L
12	-48.2	-18.54	-43.69	-57.21	-48.65	M	L	0.990	L
13	-	-0.077	-	-	-	M	H	0.998	L
14	43.4	31.11	-34.58	41.264	43.21				L
15	-	1.653	-2.569	4.358	49.96	M	H	0.792	L
16	-	-	-5.54	-	-				T
17	-1.6182	-2.232e-04	-1.6809	-1.5474	-1.6129	M	H	1.0	L
18	-41.568	-33.071	-	-38.535	-41.57	M	L	0.804	L
19	7.96	17.641	2.582	11.113	8.267	L_1	H	0.988	L
20	18.112	8.372	1.093	11.195	16.769	L_1	H	0.989	L
21	-	-	-	-	-				L

the Moon for one revolution as the sum of $\Delta\theta \approx 2.22\pi$ along the first three segments. Then, the spacecraft loiters around L_2 for 1.5 revolutions. Next, the spacecraft loiters around the Moon for up to four revolutions. The spacecraft transits to the vicinity of L_1 where it completes approximately 1 revolution. This itinerary is consistent with a visual inspection of the trajectory in Figure 3.

Case 2

This example cases focuses on using the Gauss linking integral to identify distinct looping geometries along a trajectory. The initial position and velocity vectors of this trajectory are

$$\bar{r}_0 = [1.030018316748033, 6.241835260713430 \times 10^{-4}, 1.252171757718679 \times 10^{-4}]^T$$

$$\bar{v}_0 = [-5.513921356410 \times 10^{-3}, 7.04189981485092 \times 10^{-1}, 1.55674687397916 \times 10^{-1}]^T$$

This initial states produces $C_J \approx 3.012762$. Propagating for 79.98662 nondimensional time units produces the spatial trajectory plotted in Figure 4 via its projection onto the xy -plane. Three regions along this trajectory are examined in detail to differentiate between whether a loop occurs in three-dimensional space and/or the type of looping geometry. These three regions are highlighted by black boxes and labeled a-c. A zoomed-in view of each region of interest is plotted in Figure 5. Furthermore, the blue to cyan colored arc is the trajectory segment whereas the orange arc is a portion of its evolute.

The first highlighted segment, region a, along the trajectory is used to examine whether the value of the Gauss linking integral predicts the existence of a loop in three-dimensional space. This segment of the trajectory is plotted in Figure 5a) and d), with the Gauss linking integral approaching a value of -0.9. In Figure 5a), the contribution to the Gauss linking integral as a function of truncated segment length is also displayed using the colorbar. Starting from the full length of the segment, and its corresponding evolute, sub-segments are symmetrically removed from both ends in successive steps, which progressively shrinks the length over which the Gauss linking integral is evaluated. The integral value remains close to -0.9 until the truncation reaches the point where the curve and its evolute are closely linked. This behavior supports the connection between a high magnitude of the Gauss integral calculated along an open curve with the identification of looping behavior.

The Gauss linking integral reveals that the segment highlighted as region b in Figure 4 does not produce a loop in three-dimensional space. This segment is plotted in Figure 5b) and produces a Gauss linking integral that is equal to -0.7288. The magnitude of this value does not exceed the tolerance used to identify a loop, consistent with visual inspection of the trajectory and its evolute.

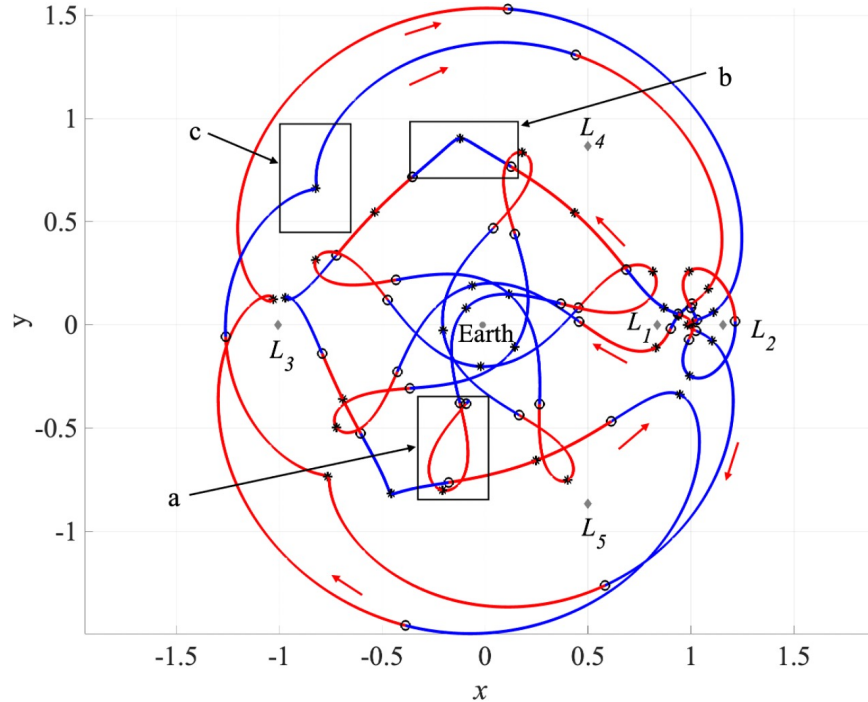


Figure 4. Spatial trajectory in case 2 with three regions highlighted.

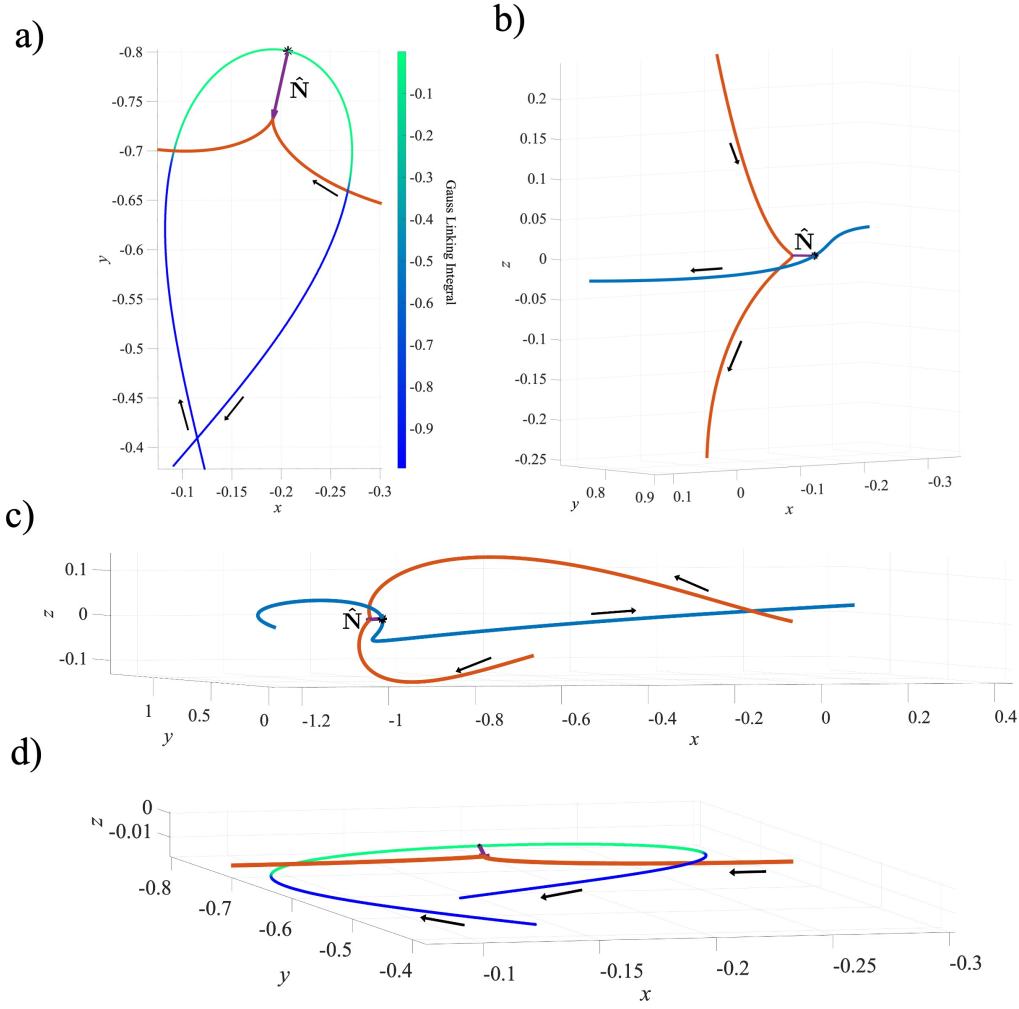


Figure 5. Segments exhibiting distinct geometries: a) and d) the same trajectory from region a, b) from region b, and c) from region c.

A loop is also identified along a geometrically distinct segment of the trajectory. Consider the segment highlighted as region c in Figure 4. This segment is plotted in Figure 5c). Although this segment appears to resemble a smooth cusp, the three-dimensional view demonstrates that a loop is forming as the orientation of the osculating plane evolves. This loop is detected via a value of the Gauss linking integral that is equal to 0.9710.

CONCLUSIONS

This paper focused on automatically extracting a description of the geometry of a complex spacecraft trajectory within the Earth-Moon circular restricted three-body problem. To automate this process, concepts from differential geometry were employed. First, a trajectory is discretized into segments that are bound by minima in curvature and any zero curvature points. Within each segment, the maximum in curvature supplies the most geometrically meaningful state. At the maximum in curvature along a segment, the orientation of its osculating plane is used to summarize the orientation of the segment. Furthermore, integrals of the curvature and torsion, as well as the arclength

are used to describe the shape and angle traced out by the path and its osculating plane over the segment. Then, through a comparison between the curvature maximum and apsides measured relative to meaningful reference points, some segments are labeled as revolving around a primary or equilibrium point. Furthermore, the accumulated angle is used to identify whether a segment corresponds to motion that is transiting through or loitering in a region of this system. This information is combined to estimate the itinerary of the trajectory.

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