

GEOMETRY-BASED ANALYSIS OF MOTION RELATIVE TO PERIODIC ORBITS IN A MULTI-BODY SYSTEM

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This paper presents an updated approach to using motion primitives to summarize relative motion near a target vehicle on a periodic orbit in the Earth-Moon circular restricted three-body problem. Relative trajectories are generated from initial chaser vehicle displacements near various target states. Each trajectory is geometrically sampled and described using the curve-based Frenet axes. Clustering is used to generate groups of relative trajectories with similar geometry as they approach or depart various states along the periodic orbit. A motion primitive is extracted from each cluster and visualized in the Frenet axes to support intuitive analysis of the solution space.

INTRODUCTION

Exploration of cislunar space is expected to be a priority for multiple space agencies over the next decade.^{1,2} In the Earth-Moon system, bounded motions near the Moon or equilibrium points could be valuable for placing long-term infrastructure such as propellant depots or crewed stations due to their relative stability and/or lunar coverage.³ These destinations may act as a rendezvous hub for spacecraft to replenish consumables or transfer crew.⁴ Designing the trajectories for these visiting vehicles is a critical task for both the far-range approach and near-range rendezvous segments, relying on a sufficient understanding of the underlying solution space.

Orbital rendezvous operations are typically studied via the motion of a chaser spacecraft relative to a target spacecraft. Although the Clohessy-Wiltshire equations provide a useful approximation of this relative motion for a near-circular orbit in a two-body system, they cannot be applied accurately when gravitational effects from additional bodies are present.⁵ Luquette and Sanner presented a linear state-space representation of relative motion in a multi-body system by linearizing the equations of motion of the circular restricted three-body problem (CR3BP) about a target spacecraft state.⁶ However, Franzini and Innocenti demonstrated that this linearization approach is sensitive to perturbations and this approximation rapidly decreases in accuracy as the state deviation from the target vehicle increases.⁷ Alternatively, relative motion with respect to a target traveling along a periodic orbit in the CR3BP has been analyzed through the dynamical structure of its local neighborhood. As an example, linear orbital elements were introduced by Hsiao and Scheeres by employing characteristics of a periodic orbit's eigenspace.⁸ Elliott and Bosanac extended this idea to define local toroidal coordinates, which describe relative motion geometrically using approximations of

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the invariant tori for a periodic orbit possessing a center eigenspace.⁹ However, a more generalized procedure for understanding relative motion in the CR3BP is still an area of significant interest.

Suitable local reference frames centered at the target vehicle can facilitate analysis of relative motion. The local vertical, local horizontal (LVLH) frame has frequently been employed for rendezvous and proximity operations since the space shuttle era.¹⁰ In a LEO environment, R-bar, H-bar, and V-bar axes are defined from the position and orbital angular momentum vectors of a target spacecraft relative to a specified reference point to intuitively study the relative motion of a chaser spacecraft. LVLH axes relative to the Moon were used by Franzini and Innocenti to formulate relative equations of motion for a chaser spacecraft near a target vehicle traveling along an L_2 near rectilinear halo orbit (NRHO) in the Earth-Moon CR3BP.¹¹ However, limitations of using the LVLH axes in the CR3BP include the requirement to specify a single reference point, potential misalignment with the geometry of the target vehicle's orbit, and discontinuities when the target vehicle changes direction of motion relative to the reference point.

This paper continues the authors' ongoing application of a motion primitive approach to summarizing possible relative motions of a chaser spacecraft in a multi-body system. In robotics, motion primitives have been used to summarize the fundamental building blocks of motion.¹² This concept has recently been applied to trajectory analysis and design in multi-body systems. Smith and Bosanac introduced a motion primitive to summarize a group of arcs with a similar geometry for a single spacecraft in the CR3BP; these groups were constructed via clustering.¹³ Gillespie, Miceli, and Bosanac expanded upon this work by 1) grouping periodic orbits along a family using analytical curvature-based criteria, and 2) demonstrating an updated clustering approach to produce more accurate groupings of similar arcs along both natural and thrust-enabled trajectories that approach or depart a periodic orbit.^{14,15} Joyner and Bosanac have more recently applied this concept to relative motion primitives that summarize chaser trajectories that approach or depart a single state along a periodic orbit in the Earth-Moon CR3BP.¹⁶

This paper expands upon prior work generating relative motion primitives by 1) using a curve-based frame to analyze and intuitively visualize trajectories and 2) summarizing trajectories that approach or depart a variety of states along a periodic orbit. First, trajectories are generated for chaser spacecraft relative to a target vehicle initially located at various states along a selected periodic orbit in the Earth-Moon CR3BP. Each trajectory is described and sampled in a geometric manner using a curve-based frame defined using the state of the target vehicle. Next, clustering is used to group relative motion arcs that approach or depart various locations along an orbit with similar geometry. A relative motion primitive is then extracted from each group as a representative trajectory. This process is demonstrated for relative motions that approach or depart an L_2 NRHO.

BACKGROUND

Circular Restricted Three-Body Problem

The CR3BP approximates spacecraft motion in cislunar space using several simplifying assumptions. The primary bodies, i.e., the Earth (P_1) and Moon (P_2), are modeled gravitationally as point masses. Furthermore, P_1 and P_2 follow circular orbits about their mutual barycenter.¹⁷ The mass of the spacecraft is also assumed to be negligible compared to the constant masses of the two primaries.

Spacecraft states and other parameters in the CR3BP are often nondimensionalized using characteristic values for length, mass, and time.¹⁷ The average separation distance between the Earth and Moon forms the characteristic length, set at $l^* = 384,000$ km. The total mass of the two primaries

is set as the characteristic mass, with $m^* \approx 6.046804 \times 10^{24}$ kg. Finally, the characteristic time $t^* \approx 3.751903 \times 10^5$ s sets the mean motion of the primaries to unity.

Trajectories are typically analyzed in a rotating reference frame with an origin at the $P_1 - P_2$ barycenter.¹⁷ The axes of this frame are defined such that $\hat{x}$ lies along the line from P_1 to P_2 , $\hat{z}$ is collinear with the orbital angular momentum vector of the two primaries, and $\hat{y}$ completes the right-handed orthogonal coordinate system.¹⁷

The nondimensional equations of motion for the CR3BP are written in the $P_1 - P_2$ rotating frame. The state vector for a spacecraft is defined using the position and velocity vectors $\bar{r}$ and $\bar{v}$ as $\bar{x} = [\bar{r}^T, \bar{v}^T]^T = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$. The equations of motion¹⁷ are then concisely written as

$$\ddot{x} = 2\dot{y} + \frac{\partial U^*}{\partial x}, \quad \ddot{y} = -2\dot{x} + \frac{\partial U^*}{\partial y}, \quad \ddot{z} = \frac{\partial U^*}{\partial z} \quad (1)$$

where the pseudopotential function U^* is

$$U^* = \frac{1}{2} (x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \quad (2)$$

Additionally, $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$ represent the distance from each primary, whereas μ is the ratio of the mass of the smaller primary to the total mass of both primaries. For the Earth-Moon system, this value is approximately $\mu \approx 0.012151$. An integral of motion C_J is equal to¹⁷

$$C_J = 2U^* - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \quad (3)$$

and labeled as the Jacobi constant.

The autonomous CR3BP features several fundamental solutions.¹⁷ For each multi-body system, there exist five equilibrium points labeled as L_1 - L_5 that remain stationary in the $P_1 - P_2$ rotating frame. Trajectories that repeat their motion in this frame over a minimal time interval are labeled as periodic orbits; these fundamental solutions exist in continuous families. Finally, hyperbolic invariant manifolds are formed by trajectories that either asymptotically depart (unstable manifold) or approach (stable manifold) libration points, periodic orbits, or quasi-periodic orbits.

Relative Motion in the Circular Restricted Three-Body Problem

When studying relative motion in the CR3BP, the two spacecraft are labeled using the standard target and chaser terminology from the two-body problem.⁵ Specifically, a passive target vehicle (identified with subscript t) travels along a specified orbit whereas an active chaser vehicle (identified with subscript c) approaches or departs at a distance that is much smaller than the distance to any primary bodies. The position, velocity, and acceleration vectors of the chaser relative to the target spacecraft are expressed as

$$\bar{r}_r = \bar{r}_c - \bar{r}_t, \quad \bar{v}_r = \bar{v}_c - \bar{v}_t, \quad \bar{a}_r = \bar{a}_c - \bar{a}_t \quad (4)$$

The absolute state of the target vehicle and the relative state of the chaser vehicle are then combined into a 12-dimensional state vector as

$$\bar{x}_{full} = [x_t, y_t, z_t, \dot{x}_t, \dot{y}_t, \dot{z}_t, x_r, y_r, z_r, \dot{x}_r, \dot{y}_r, \dot{z}_r]^T \quad (5)$$

The relative equations of motion for the chaser spacecraft relative to the target vehicle location are written in the rotating frame as

$$\ddot{x}_r = \left(2\dot{y}_c + \frac{\partial U_c^*}{\partial x_c} \right) - \left(2\dot{y}_t + \frac{\partial U_t^*}{\partial x_t} \right) \quad (6)$$

$$\ddot{y}_r = \left(-2\dot{x}_c + \frac{\partial U_c^*}{\partial y_c} \right) - \left(-2\dot{x}_t + \frac{\partial U_t^*}{\partial y_t} \right) \quad (7)$$

$$\ddot{z}_c = \frac{\partial U_c^*}{\partial z_c} - \frac{\partial U_t^*}{\partial z_t} \quad (8)$$

using U_t^* and U_c^* , the pseudopotential function evaluated at the target and chaser state vectors.¹⁷

Differential Geometry Concepts

Concepts from differential geometry are applied to nonlinear trajectories to concisely summarize their behavior. Each of these concepts is defined using the instantaneous position, velocity, acceleration, and jerk vectors for a curve, respectively defined as $\bar{r}(t) = [x(t), y(t), z(t)]^T$, $\bar{v}(t) = [\dot{x}(t), \dot{y}(t), \dot{z}(t)]^T$, $\bar{a}(t) = [\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)]^T$, and $\bar{j}(t) = [\dddot{x}(t), \dddot{y}(t), \dddot{z}(t)]^T$. In this paper, these vectors are expressed in the Earth-Moon rotating frame. A trajectory traverses a distance s from an initial state over the time interval $t \in [t_0, t_f]$ defined as the arclength, equivalent to¹⁸

$$s = \int_{t_0}^{t_f} ds = \int_{t_0}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt \quad (9)$$

The instantaneous curvature value κ measures the deviation of a curve from a straight line in the osculating plane, i.e., the plane formed by three points on the curve as the limit of the arclength between them approaches zero.¹⁹ The curvature at a state along a trajectory is expressed as

$$\kappa(t) = \frac{\|\bar{v}(t) \times \bar{a}(t)\|}{\|\bar{v}(t)\|^3} = \frac{\sqrt{(\ddot{z}\dot{y} - \ddot{y}\dot{z})^2 + (\ddot{x}\dot{z} - \ddot{z}\dot{x})^2 + (\ddot{y}\dot{x} - \ddot{x}\dot{y})^2}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{3/2}} \quad (10)$$

Although this value of curvature is always positive by definition, an alternative signed version of curvature exists for 2-dimensional or planar curves and is expressed as

$$\kappa_{signed} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \quad (11)$$

Here, a change in sign represents the curve passing through an inflection point where $\kappa = 0$ and a transition from concave to convex or vice versa.¹⁹ Both formulations of curvature possess a singularity when the speed is equal to zero. Finally, the torsion τ represents how sharply a curve is twisting out of the osculating plane and is calculated as

$$\tau(t) = \frac{(\bar{v}(t) \times \bar{a}(t)) \cdot \bar{j}(t)}{\|\bar{v}(t) \times \bar{a}(t)\|^2} = \frac{\dot{x}(\ddot{y}\ddot{z} - \ddot{z}\ddot{y}) - \dot{y}(\ddot{x}\ddot{z} - \ddot{z}\ddot{x}) + \dot{z}(\ddot{x}\ddot{y} - \ddot{y}\ddot{x})}{(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2} \quad (12)$$

with singularities when the speed and curvature are equal to zero.

Locations where maxima in curvature occur are correlated with geometrically significant areas along a curve.¹⁹ These curvature maxima are identified where $\dot{\kappa} = 0$ and $\ddot{\kappa} < 0$. For trajectories in multi-body systems, maxima in curvature are often found near apsides with respect to points of interest in the CR3BP such as primary bodies or libration points.²⁰ In the relative CR3BP, curvature maxima can exist near locations of local extrema of the relative distance of the chaser spacecraft with respect to the target vehicle.¹⁶

Curve-Based Reference Frames

Curve-based reference frames, such as the Frenet frame, are defined using the local geometry of a curve. The axes of the Frenet frame are defined as a tangent vector $\hat{T}$ directed along the unit velocity vector, a normal vector $\hat{N}$ pointed directly towards the center of curvature in the osculating plane, and a binormal vector $\hat{B}$ normal to the osculating plane that completes the right-handed coordinate frame.¹⁸ These basis vectors are calculated as

$$\hat{T} = \frac{\bar{v}}{\|\bar{v}\|} = \frac{\dot{\bar{r}}}{\|\dot{\bar{r}}\|}, \quad \hat{N} = \hat{B} \times \hat{T}, \quad \hat{B} = \frac{\bar{v} \times \bar{a}}{\|\bar{v} \times \bar{a}\|} = \frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{\|\dot{\bar{r}} \times \ddot{\bar{r}}\|}$$

The directions of the Frenet axes are governed by a set of differential equations called the Frenet-Serret formulas.¹⁹ These differential equations are expressed in terms of the curvature and torsion of a curve at the current state as

$$\frac{d}{dt} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \|\dot{\bar{r}}\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix}$$

These expressions are only valid at states that do not produce a singularity in the curvature or torsion.

Density-Based Clustering

Clustering algorithms are used to automatically identify natural groupings of similar members in a dataset.²¹ Each individual data point is summarized by selecting relevant attributes and assembling these values into a finite-dimensional feature vector. Using a specified distance metric, the clustering algorithm sorts individual members into groups based on the relative proximity of their respective feature vectors. This paper employs Density-Based Spatial Clustering of Applications with Noise (DBSCAN),²² which is well-suited for identifying irregularly-shaped clusters without a priori knowledge of the number of clusters present. Originally developed by Ester, Kriegel, Sander, and Xu,²² the algorithm identifies groups within dense regions of the dataset featuring overlapping neighborhoods. DBSCAN employs two user-specified parameters: 1) the radius ϵ to define neighborhood size, and 2) the value m_{pts} that supports identifying sufficiently dense neighborhoods. If at least m_{pts} neighbors lie within a radius of ϵ , that member is classified as a core point. If fewer than m_{pts} neighbors lie within the radius ϵ for a data point but a core point is within the ϵ -neighborhood, that member is classified as a border point. If the criteria for either a core point or a border point are not met, the data point is classified as noise. Core points and their corresponding border points within density-connected neighborhoods are then grouped together as clusters. The DBSCAN algorithm is useful when appropriate values for ϵ and m_{pts} can be identified for a given data set. In this paper, DBSCAN is implemented using the *dbscan* function in MATLAB.²³

GENERATING RELATIVE MOTION PRIMITIVES

Analysis of rendezvous trajectories that approach or depart a target vehicle in cislunar space is complicated by the multi-body gravitational dynamics governing relative motion in this region. Families of periodic orbits in the CR3BP are more geometrically diverse than the near-circular orbits frequently encountered in rendezvous and proximity operations near targets in LEO. To address this challenge, this paper continues to build on previous work by Smith and Bosanac²⁴ as well as Miceli and Bosanac²⁵ by applying the motion primitive concept for trajectory design to studying

relative motion in multi-body systems. In this paper, the prior efforts by Joyner and Bosanac¹⁶ are refined and expanded upon through the incorporation of geometric information in describing relative motion trajectories. An updated technical approach uses the Frenet frame to propagate and sample trajectories to capture their geometry in feature vectors. Furthermore, these relative motions are summarized over an array of target states along the desired periodic orbit.

Step 1: Generating initial target and chaser spacecraft states

Consider a target spacecraft traveling along a periodic orbit in the Earth-Moon CR3BP and defined by the state vector $\bar{r}_t$. This periodic orbit is selected from a continuous family of similar orbits. Next, $N_{node} = 60$ reference states $\bar{r}_r$ are sampled at nodes separated by intervals of equal arclength along the orbit to sufficiently capture the full evolution of the dynamics over a single period.²⁶ This sampling approach avoids concentrating samples near low-speed regions around apolune and undersampling high-speed regions near perilune inherent to sampling at intervals of equal time. The Frenet axes are propagated and sampled along with the state vector. A chaser vehicle with an absolute state of $\bar{r}_c$ state is then defined relative to the target state as $\bar{r}_c$, shown conceptually in Figure 1.

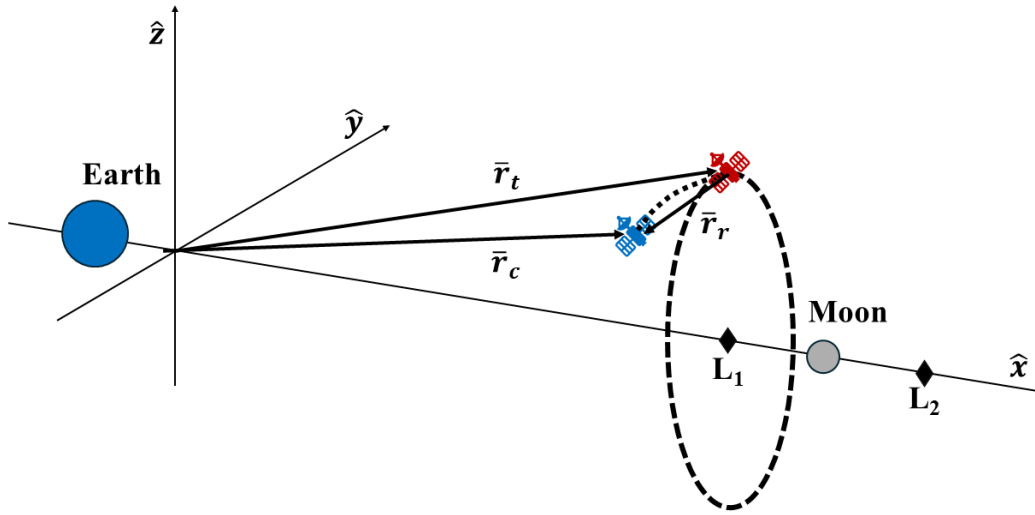


Figure 1. Conceptual diagram of a target spacecraft traversing a periodic orbit about L_1 with a nearby chaser spacecraft in the Earth-Moon CR3BP.

A set of chaser spacecraft states is generated relative to each of the individual target states. Chaser state vectors are defined in the local Frenet frame for the respective target location along a periodic orbit. This approach leverages the strong dependence of relative chaser displacement with respect to the tangent velocity vector of the target spacecraft on relative motion behavior observed in previous investigations.^{27,28} To sufficiently sample the design space, a Fibonacci lattice algorithm is employed to generate chaser state vectors. This algorithm provides a mathematically idealized version of the optimal packing patterns found in nature, generating any desired odd number of approximately equidistant sample points along the surface of a sphere.²⁹ Output sample points are given in latitude and longitude, then converted to Cartesian coordinates and scaled to a specified value.

The Fibonacci lattice algorithm is employed to sample the possible reference position and velocity displacement vectors for the chaser spacecraft. A total of $N_{pos} = 201$ chaser relative position

Table 1. Summary of quantities used to generate reference relative chaser position and velocity vectors.

	$\Delta\theta_{avg}$	Total Samples	Magnitude
Relative Position $d\bar{r}$	12.5°	201	10 km
Relative Velocity $d\bar{v}$	10.2°	301	1 m/s

vectors are generated and scaled to $\|d\bar{r}\| = 10$ km, a value taken from the rendezvous sphere concept used in collision avoidance analysis for the Lunar Gateway station.³⁰ A total of $N_{vel} = 301$ relative velocity vectors are similarly generated with a displacement of $\|d\bar{v}\| = 1$ m/s based on typical values for rendezvous with targets located in periodic orbits.²⁷ Although chaser displacement magnitudes for $\|d\bar{r}\|$ and $\|d\bar{v}\|$ are specified in *km* and *m/s* for clarity, these values are converted to nondimensional units using the characteristic length and time values for the Earth-Moon system. Finally, each unique combination of relative position and velocity displacement vectors for the chaser spacecraft is used to form 60,501 relative state vectors for each reference state along a periodic orbit. These relative state vectors are summarized in Table 1 in terms of the average angle between neighboring samples, total number of samples, and magnitude. Although the same set of chaser displacement states defined in the Frenet frame is used for each target state sample, these chaser states are not equivalent in the rotating relative Earth-Moon frame.

Step 2: Propagate and sample trajectories

Augmented state vectors comprised of the target state, Frenet frame basis vectors, and relative chaser state are propagated using their respective equations of motion. Each of these state vectors is propagated twice: 1) forward in time to represent a chaser spacecraft departing from the target vehicle from an initial state and 2) backward in time to simulate approaching the target to reach a final state. Propagation of relative motion trajectories is halted when the chaser spacecraft reaches a critical range from the target spacecraft: 1) a terminal outer range $\|\bar{r}_{r,out}\| = 2000$ km and 2) a terminal inner range $\|\bar{r}_{r,in}\| = 0.2$ km. The quantity $\|\bar{r}_{r,out}\|$ demarcates a region around a target spacecraft where relative motion is applicable while allowing for overlap with far-range trajectories for future work. $\|\bar{r}_{r,in}\|$ defines a keep-out sphere around the target as used in Gateway studies.³¹ An upper limit on propagation time is set at 10 nondimensional time units, as some relative motion trajectories near more stable periodic orbits may indefinitely linger in the target vicinity.

The basis axes of the Frenet frame are propagated using the Frenet-Serret formulas for spatial periodic orbits. However, the planar orbits necessitate a modified approach. In cases such as the L_1 and L_2 Lyapunov families, the instantaneously calculated normal and bi-normal axes of the Frenet frame abruptly “flip” 180° as the target spacecraft passes through a zero curvature point before “flipping” back to their previous orientation. This behavior is observed in the region on the right of the L_1 Lyapunov orbit in Fig. 2a). If the Frenet-Serret formulas are employed to propagate the unit axes instead of calculating them instantaneously, the derivatives cause the axes to evolve in the opposite direction than expected in these regions. This behavior occurs when a planar periodic orbit possesses regions where the center of curvature points in the opposite direction from the rest of the orbit, analogous to concave and convex sections along a two-dimensional curve. These sections are demarcated by locations where $\kappa = 0$, but both possess positive curvature when using the unsigned definition. Using the signed definition of curvature, these regions are distinguished by either positive or negative curvature. Incorporating the signed definition in the Frenet-Serret formulas produces a

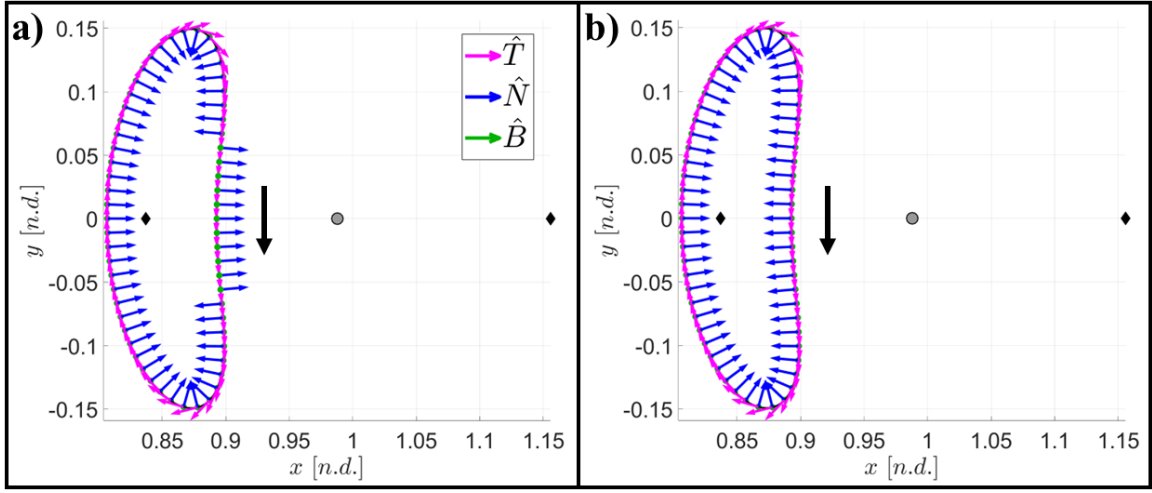


Figure 2. Evolution of Frenet unit axes along an L_1 Lyapunov orbit computed with a) unsigned curvature and b) signed curvature to avoid discontinuities.

continuous evolution of the unit axes without any “flips”, as seen in Fig. 2b), allowing for consistent analysis and visualization of relative motion in the Frenet frame. This process is dependent on an initial definition of the Frenet axes, as using an initial state in the flipped region will result in a consistent orientation being maintained for the entire period. Note that spatial trajectories can possess zero curvature points; however, these cases are not encountered for the examples explored in this paper and are an avenue of future work.

Relative trajectories for the chaser vehicle are propagated using the Frenet axes to enable direct sampling and visualization in this frame. First, the equations of relative motion are expressed in terms of the curvature, torsion, and Frenet unit vectors of the target spacecraft and their associated derivatives. To simplify these expressions, consider the following substitutions:

$$\bar{C} = \bar{v}_t \times \bar{a}_t \quad \dot{\bar{C}} = \bar{v}_t \times \bar{j}_t \quad H = \|\dot{\bar{r}}_t\| \quad \dot{H} = \frac{\dot{\bar{r}}_t \cdot \dot{\bar{a}}_t}{H} \quad (13)$$

Using these definitions, the target trajectory curvature and its derivative are rewritten as

$$\kappa_t = \frac{\|\bar{C}\|}{H^3} \quad \dot{\kappa}_t = \frac{\bar{C} \cdot \dot{\bar{C}}}{\kappa_t H^6} - \frac{3\kappa_t(\bar{v}_t \cdot \bar{a}_t)}{H^2} \quad (14)$$

In addition, the torsion and its derivative are rewritten as

$$\tau_t = \frac{\bar{C} \cdot \bar{j}_t}{\bar{C} \cdot \bar{C}} \quad \dot{\tau}_t = \frac{\bar{C} \cdot \bar{s}_t - \dot{\bar{C}} \cdot \bar{j}_t - 2\tau_t(\bar{C} \cdot \dot{\bar{C}})}{\bar{C} \cdot \bar{C}} \quad (15)$$

where $\bar{s}_t = \dot{\bar{j}}_t$. From the Frenet-Serret formulas, the following matrices are defined:

$$[\mathbf{M}]_t = \begin{bmatrix} 0 & \kappa_t & 0 \\ -\kappa_t & 0 & \tau_t \\ 0 & -\tau_t & 0 \end{bmatrix}, \quad [\dot{\mathbf{M}}]_t = \begin{bmatrix} 0 & \dot{\kappa}_t & 0 \\ -\dot{\kappa}_t & 0 & \dot{\tau}_t \\ 0 & -\dot{\tau}_t & 0 \end{bmatrix} \quad (16)$$

By assembling the unit axes of the Frenet frame into the rotation matrix $[\mathbf{R}]$, their second derivative can then be written as

$$[\ddot{\mathbf{R}}] = \dot{H}[\mathbf{M}]_t[\mathbf{R}] + H[\dot{\mathbf{M}}]_t[\mathbf{R}] + H[\mathbf{M}]_t[\dot{\mathbf{R}}] \quad (17)$$

The rotation matrix is used to calculate the cross product matrix $[\boldsymbol{\Omega}]$ and its time derivative as

$$[\boldsymbol{\Omega}] = [\dot{\mathbf{R}}][\mathbf{R}]^T \quad [\dot{\boldsymbol{\Omega}}] = [\dot{\mathbf{R}}][\dot{\mathbf{R}}]^T + [\ddot{\mathbf{R}}][\mathbf{R}]^T \quad (18)$$

Using these definitions, the equations of relative motion for a chaser spacecraft in the Frenet frame are written as

$$\bar{r}_{r,Fren} = [\mathbf{R}]\bar{r}_{r,xyz} \quad (19)$$

$$\bar{v}_{r,Fren} = [\mathbf{R}]\bar{v}_{r,xyz} + [\boldsymbol{\Omega}]\bar{r}_{r,Fren} \quad (20)$$

$$\bar{a}_{r,Fren} = [\mathbf{R}]\bar{a}_{r,xyz} + 2[\boldsymbol{\Omega}]\bar{v}_{r,Fren} - [\boldsymbol{\Omega}][\boldsymbol{\Omega}]\bar{r}_{r,Fren} + [\dot{\boldsymbol{\Omega}}]\bar{r}_{r,Fren} \quad (21)$$

through successive applications of the transport theorem for transformation to a rotating frame.³²

To summarize relative motion trajectories and succinctly capture their geometry in a discrete feature vector for clustering, a curve-based procedure based on work by Gillespie, Miceli, and Bosanac¹⁴ is employed. Chaser trajectory states are sampled at curvature maxima events where $\dot{\kappa} = 0$ as computed in the Frenet frame. This approach has distinct advantages over sampling curvature maxima in the relative $\hat{x}\hat{y}\hat{z}$ frame in previous work.¹⁶ More curvature maxima events are generally observed in the Frenet frame, eliminating the need to gather additional samples. Relative motion states sampled in the $\hat{x}\hat{y}\hat{z}$ frame depend on the target and chaser location along the overall periodic orbit. Capturing samples in the Frenet frame removes this dependency, producing more generalizable results that allows for more direct comparison of relative motion behavior over the entire orbit. The inclusion of the Frenet unit vectors in each state vector output also enables translation back to the Earth-Moon rotating frame, if desired.

Samples collected at curvature maxima are compiled into two finite-dimensional feature vectors to summarize each relative motion trajectory.¹⁵ For the i th sampled state, two feature vectors are constructed: a position-based vector defined as

$$\bar{f}_{p,i} = [x_{r,i}, y_{r,i}, z_{r,i}]$$

and the shape-based vector defined as

$$\bar{f}_{v,i} = [\tilde{x}_{r,i}, \tilde{y}_{r,i}, \tilde{z}_{r,i}]$$

where $\tilde{x}$ signifies normalized unit velocity components. Normalization of the velocity vector ensures that the shape of each trajectory is described by quantities on the same order of magnitude regardless of the overall speed.

Step 3: Generate local clusters for each target state and identify primitives

Clusters of relative motion trajectories with a similar geometry are identified for each initial target state and summarized with a representative motion primitive. Trajectories generated through the chaser vehicle departing or approaching the target are evaluated separately. The data is also pre-sorted into partitions based on the number of curvature maxima observed to compare only feature

vectors of equivalent dimension. A two-step clustering technique developed by Bosanac,³³ and slightly updated in Gillespie, Bosanac, and Bodin¹⁴ is used in this paper.

Both clustering steps are based on a convoy detection approach from the trajectory clustering literature.³⁴ Convoy detection schemes analyze each set of sample states along a trajectory sequentially and only groups those that remain density-connected for their entire duration. The i th sample along each set of paths is described in a selected feature vector space and then clustered. The convoy detection process is repeated for all the N_s sample states, and trajectories are clustered together if each of their associated sample states is grouped in all clustering results. This process ensures that clusters are composed of trajectories that follow a similar path. First adapted to trajectories in the CR3BP by Bosanac,³³ this approach is used to group similar relative motion paths.

Departing and approaching relative motion trajectories are first grouped using the convoy detection approach by evaluating the shape-based feature vectors using the DBSCAN clustering algorithm.¹⁵ This step supports an initial identification of paths with similar shapes. The feature vectors $\bar{f}_{v,i}$ for the i samples along each trajectory are clustered using a Euclidean distance metric with DBSCAN parameters set to $m_{min,clust} = 5$ and $m_{pts} = 4$ to select smaller localized clusters.¹⁵ A minimum neighborhood size is heuristically defined as $\epsilon_{shape} = 2\sin(\Delta\theta/2)$ ¹⁵ to represent two unit vectors separated by an angle $\Delta\theta = 20^\circ$. In addition to clustering with DBSCAN, this threshold value is used to determine if each pair of feature vectors lies within a mutual neighborhood by comparing their separation distance. By repeating this process for each set of sample states, this initial course clustering step identifies groups of trajectories featuring roughly similar shape and discards any unlabeled members as noise.

Coarse clusters of relative motion trajectories identified in the previous step are then individually evaluated in a refinement step.³³ First, trajectories generated from neighboring groups are identified using a position threshold of $\epsilon_{pos} = 5 \text{ km}$ and the same ϵ_{shape} for the normalized initial velocity vector. A convoy detection process then evaluates the i th sampled states from these neighboring relative motion trajectories. Each shape-based feature vector $\bar{f}_{v,i}$ is evaluated using ϵ_{shape} to identify neighboring samples. Next, the position feature vectors $\bar{f}_{p,i}$ are employed to compute the distance to the furthest sample point d_{max} . For each pair of sample positions, the heuristic $\epsilon_{pair} = \max(0.8d_{max}, 5\epsilon_{pos})$ is employed as a metric to determine if they are neighbors. This process is then repeated for all subsequent N_s sample points. If a set of relative trajectories meets all these criteria, they form a refined cluster. Coarse clusters may be subdivided into multiple refined clusters after position considerations, but if any sample point from a given trajectory is classified as noise, then the entire trajectory is discarded. This refinement step ensures that clustered departure and approach paths possess both similar geometry and position relative to the target spacecraft.

A relative motion primitive is selected to represent each refined cluster. These representatives are chosen by identifying the medoid of cluster $\mathcal{F}$,²⁴ equivalent to

$$\bar{f}_{medoid} = \underset{\bar{f}_i \in \mathcal{F}}{\operatorname{argmin}} \left(\sum_{j=1}^{|\mathcal{F}|} \|\bar{f}_{pos,i} - \bar{f}_{pos,j}\| \right)$$

The medoid is calculated using the position-based feature vector evaluated along all N_s samples of the trajectory, i.e., $\bar{f}_{pos,i} = [\bar{f}_{p,1}, \dots, \bar{f}_{p,N_s}]$.

Step 4: Aggregate local clusters into global clusters across periodic orbit

Relative motion behavior can be characterized for the entire periodic orbit by comparing local clusters associated with each reference target state and merging those with similar geometry into global clusters. This procedure is based on an approach developed by Bosanac¹⁵ to merge similar clusters from separate partitions. For each pair of neighboring nodes along a periodic orbit, only clusters composed of trajectories with the same number of sampled states are compared. Pairs of clusters are identified as candidates for merging using their motion primitives. For each representative, the $k = 3$ nearest neighbors in the adjacent node based on the entire shape-based feature vector $\bar{f}_v$ are determined to identify clusters with the most similar geometry. For each candidate pair of local clusters, their associated trajectories are clustered using the refined convoy detection process described previously. If any trajectories from these local groups are clustered together, then the two local pairs are merged to form a new global cluster.

Cluster aggregation is performed iteratively between the N th and $(N + 1)$ th node for the entire periodic orbit. For the first and second node, each merged pair of local clusters is assigned a label as a new global cluster. In subsequent iterations, merged pairs are added to an existing global cluster if the cluster in node N was previously merged with another in $N - 1$. The final step is a pairwise comparison between the final node evaluated and the first to connect all target states around the entire orbit. If any overlap exists between global clusters identified in the first nodes evaluated and those towards the end of the process, labels in the later nodes are updated to reflect their merger. This process ensures that global clusters are formed from local clusters associated with neighboring nodes with similar geometry and position relative to the target spacecraft.

RESULTS

Frenet Unit Axes for Periodic Orbits

A selection of members from families of planar periodic orbits are shown with their Frenet unit axes in Fig. 3. The nearly circular distant retrograde orbit (DRO) in Fig. 3a) demonstrates how the Frenet frame resembles conventional frames such as the LVLH frame for orbits more closely resembling a conic. In Fig.3b), the modified Frenet axes using signed curvature are consistently defined across three distinct lobes of a distant prograde orbit (DPO). The normal vector $+/- \hat{N}$ points directly away from the center of curvature in the outer regions at the top and bottom of the figure, consistent with the initial Frenet axes being defined for a state in the inner region. If a state

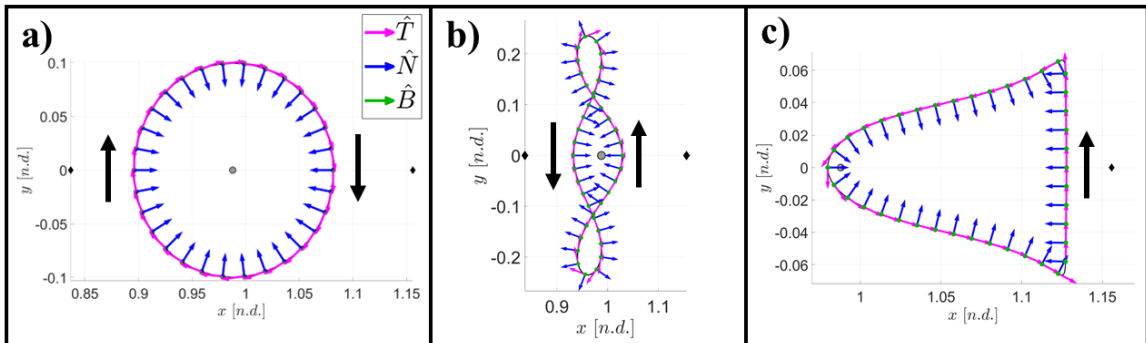


Figure 3. Frenet axes $\hat{T}$, $\hat{N}$, and $\hat{B}$ for planar periodic orbits: a) distant retrograde orbit, b) distant prograde orbit, and c) low prograde orbit.

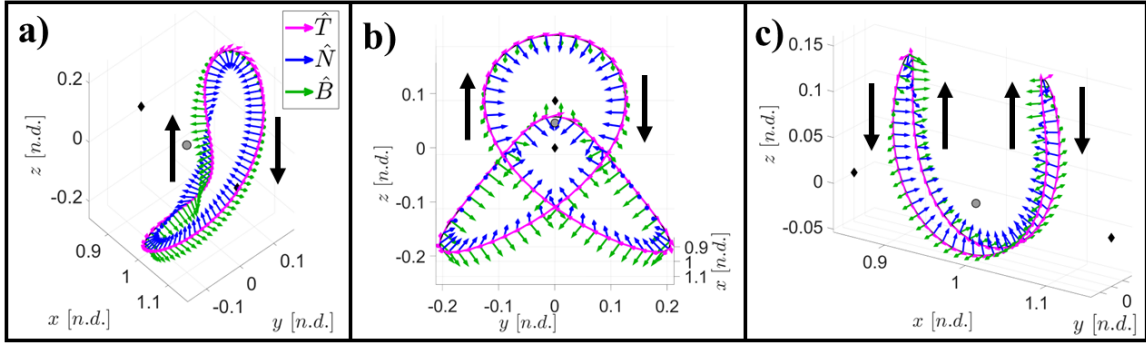


Figure 4. Frenet axes $\hat{T}$, $\hat{N}$, and $\hat{B}$ for spatial periodic orbits: a) L_2 axial orbit, b) dragonfly orbit, and c) butterfly orbit.

in one of the outer lobes had been used to initialize the Frenet frame, the orientation of $\hat{N}$ and $\hat{B}$ would be directed exactly opposite of those shown. Finally, Frenet axes for the low prograde orbit are shown in Fig. 3c), which features some of the uniquely challenging geometry seen in some types of periodic orbits. A low curvature section of the orbit at the right side of the figure is characterized by Frenet axes with a limited change in direction. This region is bordered by extreme curves on either end, resulting in the relative motion frame rapidly rotating in a corresponding manner. For all planar orbits, the binormal vector $\hat{B}$ is always oriented directly in the $+/-\hat{Z}$ direction depending on whether the orbit is prograde or retrograde.

Spatial orbit examples with Frenet unit vectors are plotted in Fig. 4. Significant torsion is evident in the L_2 axial orbit in Fig. 4a), resulting in the axes twisting in three-dimensional space. The butterfly orbit shown in Fig. 4b) is comprised of segments with a limited torsion joined by regions with abrupt turns. Both the $\hat{N}$ and $\hat{B}$ axes evolve rapidly in the location where the orbit passes closest to the Moon. Similar behavior is observed in the regions of the butterfly orbit in Fig. 4c) near the abrupt turns at the highest extension out of the xy -plane. The three-dimensional properties of spatial orbits precludes the use of signed curvature in propagating the unit vectors of the Frenet frame, but these examples do not feature any zero curvature points. Nevertheless, these Frenet axes evolve with the shape of each orbit to support describing relative motion in their vicinity.

Table 2. Local cluster statistics for chaser spacecraft trajectories departing and approaching a target at initial or final states at several points along a southern L_2 near-rectilinear halo orbit.

NRHO Target Node	Apolune	Mid-Apo./Peri.	Perilune	Mid-Peri./Apo.
# of Dep. Clusters	814	682	59	436
Median Dep. Cluster Size	19	17.5	17	21
Dep. Noise %	13.8%	6.9%	22.7%	6.2%
# of App. Clusters	818	444	62	686
Median App. Cluster Size	20	21	17	17
App. Noise %	13.9%	6.1%	22.6%	6.9%

Local Summaries of Relative Motion to/from a Periodic Orbit

To demonstrate the clustering procedure, an L_2 southern near-rectilinear halo orbit (NRHO) similar to the baseline design for the Lunar Gateway is selected as an example target orbit.¹ In addition to possessing a diverse range of relative motion behavior due to the differential between apolune and perilune radii, this orbit has also been extensively studied by the astrodynamics community.^{27,28} Specifically, $N_{node} = 60$ state vectors along the NRHO separated by intervals of equal arclength are sampled to serve as reference target states for departing or approaching relative motion trajectories. A set of chaser displacements are propagated, sampled, and grouped for each individual location to generate local clusters of relative trajectories, along with a motion primitive representative.

Local clustering results are summarized in Table 2 for target locations at apolune, perilune, and their midpoints. The highest quantity of clusters is observed in relative trajectories initialized near a target at apolune. This result indicates a high degree of geometric diversity; a consequence of the chaser spacecraft remaining near the target for a longer duration in this less dynamically sensitive region. At perilune, the opposite case is seen as a result of the close proximity of the Moon and its gravitational influence causing chaser spacecraft to depart or arrive in the target region more rapidly. This region is characterized by significantly fewer clusters comprised of far more trajectories, although this is not reflected in the median cluster size. A larger proportion of trajectories is also observed at perilune due to the lower quantity of curvature maxima events collected during sampling. At the two midpoints between apolune and perilune, intermediate values between these two extremes are observed. Similar clustering statistics are observed for both departing and approaching trajectories at apolune and perilune due to the symmetry of the NRHO, with results for departing and approaching trajectories mirrored at the midpoint between them. However, these values are not equivalent as a consequence of the asymmetry of the Fibonacci lattice approach used to generate displacements for the chaser spacecraft.

Each relative motion cluster is visualized using three individual plots, as seen in Fig. 5-7. The relative displacement conditions used to generate each set of trajectories are displayed in the left column, using red for paths departing the target vehicle and blue for approaching paths. Chaser spacecraft positions are depicted as dots on the surface of a $\|d\vec{r}\| = 10$ km rendezvous sphere centered at the target, with the initial or final velocity shown with a vector arrow. Clusters of relative motion trajectories produced by this process are visualized by plotting the representative motion primitive along with selected members in the Frenet frame using the same color scheme. Chaser positions at the critical range $\|\vec{r}_{c,out}\|$ for each are depicted as points at the furthest distance from the target. Representative primitives are plotted with a thicker and darker line to distinguish them. Markers along each primitive represent significant milestones along the associated periodic orbit, with apolune and perilune as cyan and magenta diamonds, with intermediate points between them shown with black crosses. Projections of the motion primitive along with these milestones appear at the bottom and back of each plot to aid visualization of a three-dimensional path. The final plot depicts the periodic orbit and associated Frenet unit axes. Common markers present in both the middle and right columns allow for significant milestones reached along the periodic orbit to be mapped directly to corresponding locations on the chaser relative trajectories. The black and white circles along the orbit indicate the approximate locations of both vehicles where the relative trajectory is respectively initiated near the target spacecraft and terminated at the outer boundary of the relative motion region.

Examples of departing and approaching clusters of relative motion trajectories initialized near a target spacecraft at apolune are shown in Fig. 5. The displacement conditions associated with the

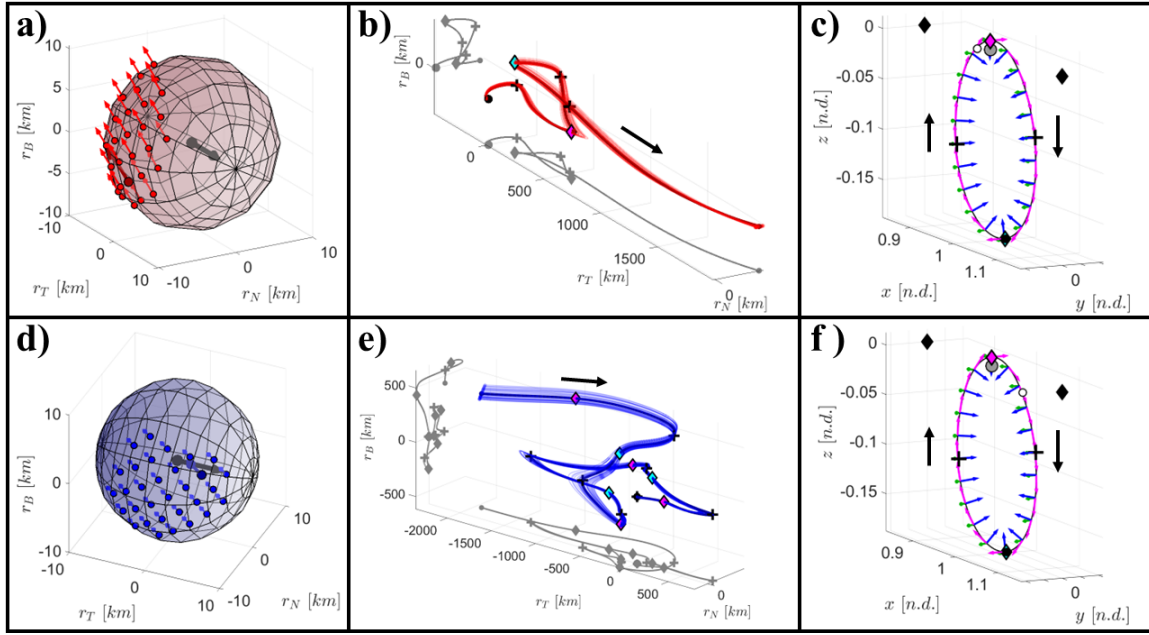


Figure 5. Clusters of relative motion trajectories a)-c) departing (red) and d)-f) approaching (blue) a target spacecraft at the apolune position of an NRHO.

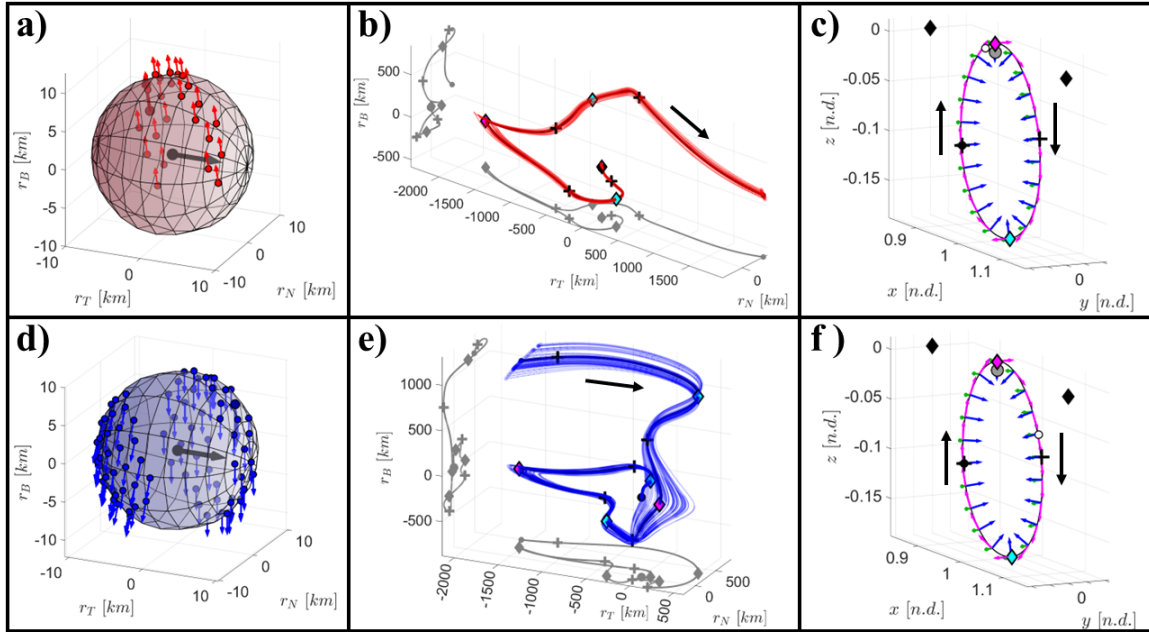


Figure 6. Clusters of relative motion trajectories a)-c) departing (red) and d)-f) approaching (blue) a target spacecraft at the midpoint of an NRHO.

departing and approaching clusters are both described by a contiguous region of position displacements along the rendezvous sphere with velocity displacements oriented parallel to one another. This is characteristic of many of the clusters observed at all locations on the periodic orbit, allowing for a direct mapping of initial or final chaser conditions to the resulting relative motion behavior.

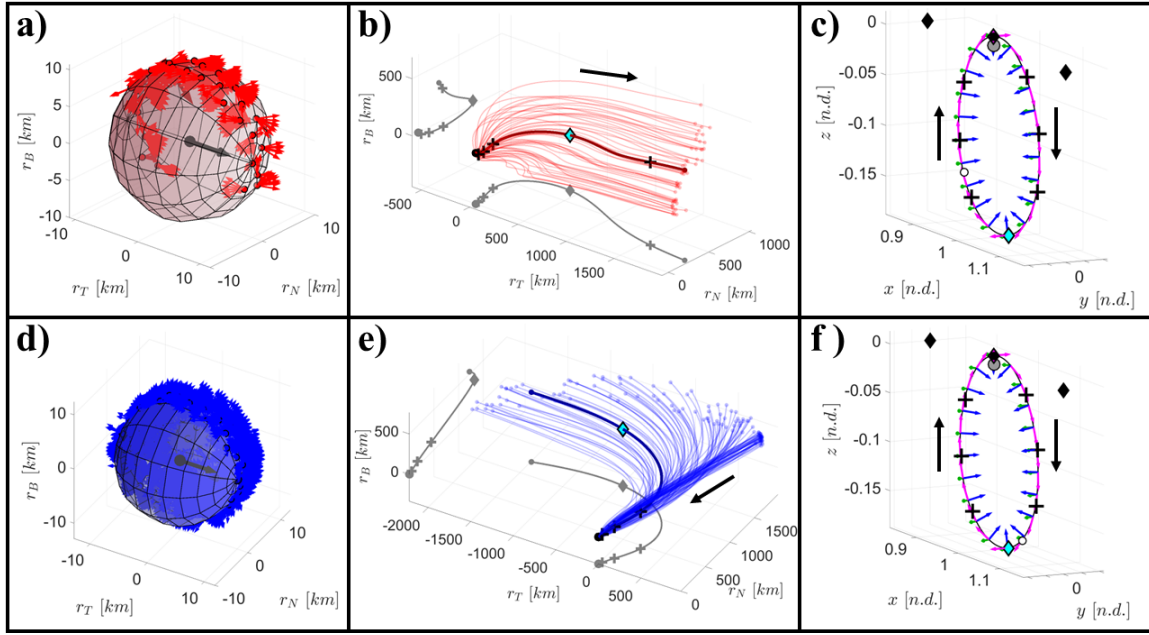


Figure 7. Clusters of relative motion trajectories a)-c) departing (red) and d)-f) approaching (blue) a target spacecraft at the perilune position of an NRHO.

For the cluster examples in the middle column, the majority of the motion relative to the target is oriented along the $\hat{T}$ axis, with smaller excursions in the $\hat{N}/\hat{B}$ axes. Use of the Frenet axes facilitates an intuitive understanding of relative motion with respect to the target vehicle. A local maximum range is reached in Fig. 5b) as both vehicles pass through perilune before the chaser reverses and reaches a local minima after reaching apolune. Similar behavior is observed in Fig. 5d) as the chaser spacecraft approaches the target vehicle over several orbital periods.

Clusters of relative motion trajectories departing and approaching a target vehicle at the midpoint between apolune and perilune along an NRHO are depicted in Fig. 6. Initial positions associated with the departing path in Fig. 6a) are once again connected as in the previous example, but are found along a narrow band across roughly one hemisphere of the rendezvous sphere. In Fig. 6d), the final positions reached by the chaser trajectories are divided into two separate regions with slightly diverging final velocities. Interestingly, the resulting approach trajectories in Fig. 6e) remain closely connected aside from one region near perilune where they fan out. Some of these trajectories pass in close proximity to the target vehicle, which could rule out this approach path based on trajectory design constraints for collision risk. For both clusters, range extrema with respect to the target vehicle are also correlated with apolune and perilune along the NRHO as in the previous example.

The larger clusters seen in Fig. 7 reflect the lower degree of geometric diversity present in relative motion trajectories initialized near perilune as a result of the rapidly changing gravitational dynamics present near the Moon. Both clusters are associated with a much larger array of chaser displacement states, with the final positions in 7d) covering half the rendezvous sphere and associated velocity displacements that are no longer nearly parallel. Departing chaser trajectories in Fig. 7b) follow a nearly direct path away from the vicinity of the target. The approach cluster in Fig. 7e) is composed of over 7,000 trajectories, all summarized with a single motion primitive. This example also demonstrates far more movement along the $\hat{N}$ direction than in previous cases. Note

the presence of additional cross markers in Fig. 7c) and Fig. 7f), placed at each eighth of the total periodic orbit arclength to accommodate trajectories propagated for less than a single orbital period.

Global Summaries of Relative Motion to/from a Periodic Orbit

Prior to cluster aggregation, local clusters are generated for all $N_{node} = 60$ target states sampled from the southern L_2 NRHO. The number of local clusters are summarized for departing trajectories in Fig. 8a) and approaching trajectories in Fig. 9a) to characterize relative motion behavior in the vicinity of the periodic orbit. As Table 2 suggests, target states at apolune are associated with the highest number of identified clusters and those at perilune with the lowest number for both cases. While this total value evolves gradually between the two extremes, there is asymmetry as the portion of the orbit leading away from apolune in Fig. 8a) and toward apolune in Fig. 9a) both feature more local clusters than their corresponding opposite side. While gravitational dynamics near apolune promote a higher degree of geometric diversity in relative motion behavior, traversing towards the perilune region for departing trajectories or from it for approaching trajectories has a similar effect.

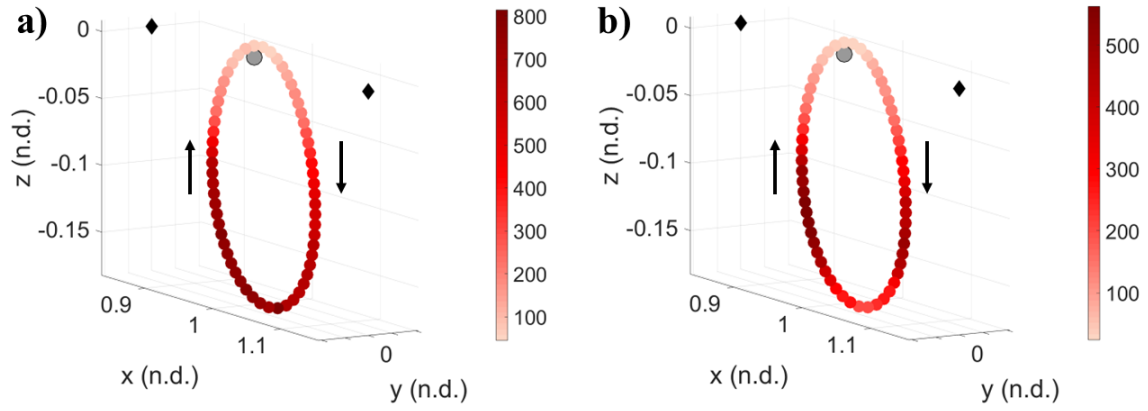


Figure 8. Summary of relative motion trajectories departing $N_{node} = 60$ initial target states along a southern L_2 NRHO showing a) total number of local clusters and b) total number of local clusters included in global clusters.

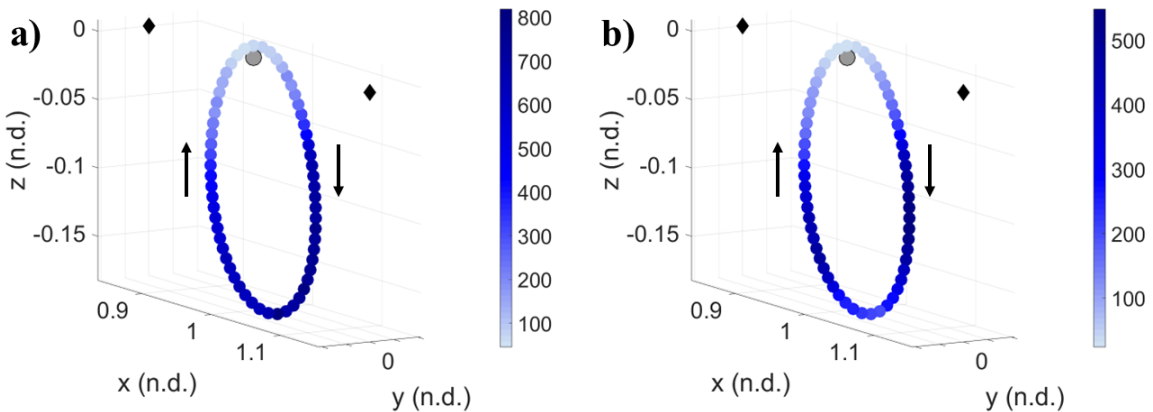


Figure 9. Summary of relative motion trajectories approaching $N_{node} = 60$ final target states along a southern L_2 NRHO showing a) total number of local clusters and b) total number of local clusters included in global clusters.

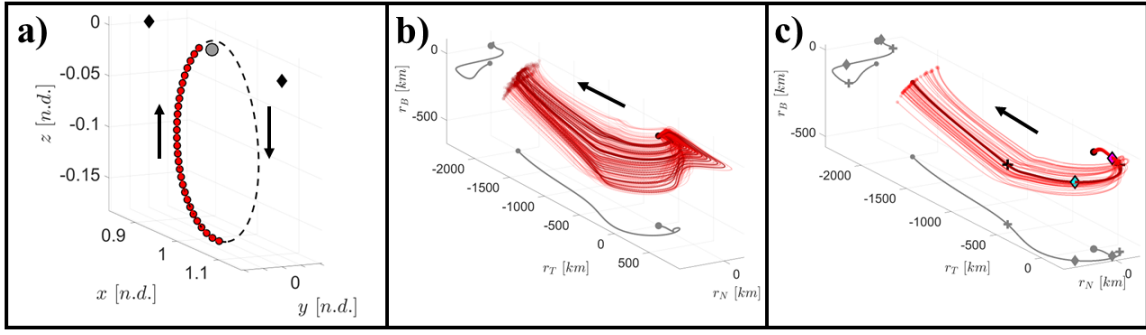


Figure 10. Aggregated global cluster of departing relative motion trajectories with a) nodes associated with merged local clusters, b) motion primitive representatives from each merged local cluster, and c) local cluster trajectories for medoid primitive.

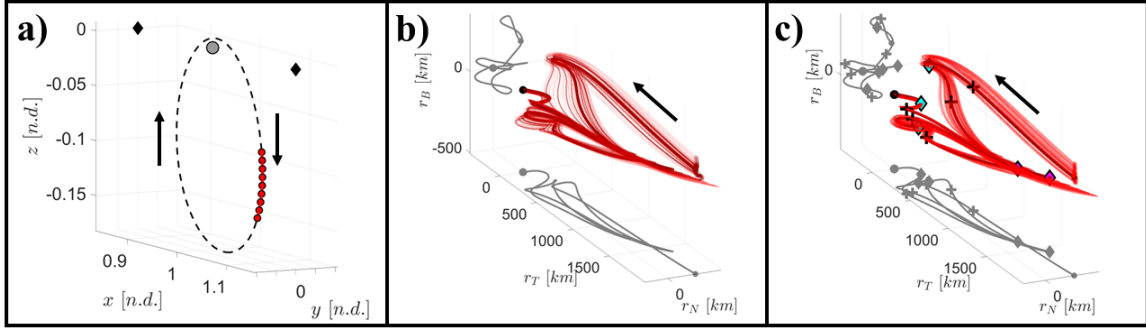


Figure 11. Aggregated global cluster of departing relative motion trajectories with a) nodes associated with merged local clusters, b) motion primitive representatives from each merged local cluster, and c) local cluster trajectories for medoid primitive.

The cluster aggregation procedure is then applied to merge local clusters from neighboring nodes with similar geometries to form nearly 4,000 global clusters for each set of relative motion trajectories. These results are summarized in Fig. 8b) for departing chaser paths and in 9b) for approaching paths by plotting the number of local clusters included in merged global clusters for each node. Target states near perilune have the fewest number of local clusters included in global clusters. However, regions adjacent to apolune possess more merged clusters than perilune despite a lower number of clusters than apolune itself for both sets of trajectories. More consistent dynamics across these segments may be producing more global clusters than the unique conditions at apolune. A similar asymmetry as seen for the total number of local clusters is also observed.

Examples of the merged global clusters composed of relative trajectories departing the target spacecraft are shown in Fig. 10 and Fig. 11. Global clusters of approaching paths are likewise displayed in Fig. 12 and Fig. 13. For all cases, the subfigure a) displays the corresponding nodes along the NRHO for each aggregated local cluster. Global clusters are summarized in subfigure b) by plotting the motion primitive of each member local cluster, with a new medoid representative shown in bold. Additional trajectories subsampled from each of the associated local clusters are plotted in a lighter shade to demonstrate the range of geometries present in each new global cluster. Finally, the entire local cluster associated with the medoid primitive is shown in subfigure c).

Composed of local clusters joined from nearly half the NRHO, the global cluster in Fig. 10

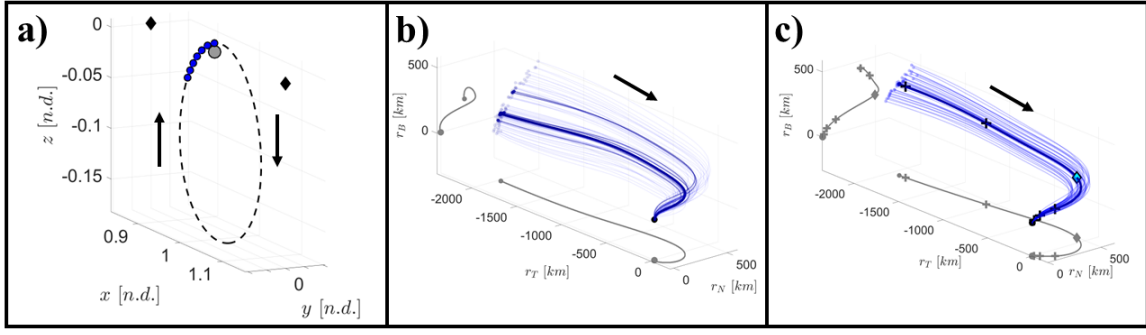


Figure 12. Aggregated global cluster of approaching relative motion trajectories with a) nodes associated with merged local clusters, b) motion primitive representatives from each merged local cluster, and c) local cluster trajectories for medoid primitive.

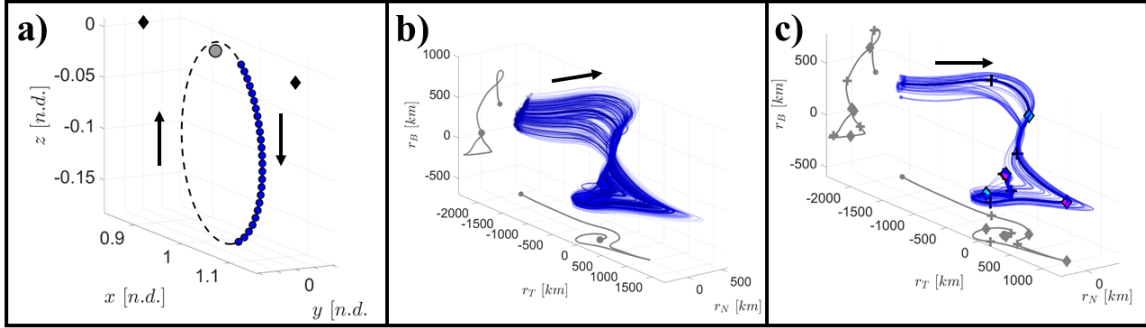


Figure 13. Aggregated global cluster of approaching relative motion trajectories with a) nodes associated with merged local clusters, b) motion primitive representatives from each merged local cluster, and c) local cluster trajectories for medoid primitive.

represents one of the larger examples found through aggregation. The motion primitives still bear a strong resemblance to each other despite originating from diverse dynamical regions, with the most significant difference being a small protrusion that gradually evolves between different local clusters. Otherwise, the range of geometries observed do not substantially differ from those seen within the local cluster in Fig. 10c). The global cluster in Fig. 11 is comprised of local clusters from a smaller region of nodes, but the geometry is characterized with a much greater degree of curvature. The region also roughly aligns with the segment of the NRHO featuring the densest number of merged departure trajectories as seen in Fig. 8. Compared to the representative local cluster in Fig. 10c), only one region just prior to the final curve demonstrates a significantly larger variance in geometry.

Local clusters are even merged in the dynamically sensitive region of the NRHO near the Moon, as seen in Fig. 12. The simpler geometry of these approach trajectories is a likely factor in the aggregation of this global cluster example. The representative primitives from each local cluster in Fig. 12b) line up closely with the volume spanned by the associated local cluster in Fig. 12c). The global cluster seen in Fig. 13 corresponds to a large region of the NRHO despite the relatively complex geometry of the merged trajectories. Only a slightly larger variance along the $\hat{B}$ direction is observed when comparing the region spanned by the representative primitives in Fig. 13b) to one of the associated local clusters in Fig. 13c).

CONCLUSION

This paper builds upon previous work adapting the use of motion primitives in astrodynamics applications for relative motion near periodic orbits in cislunar space. A set of relative equations of motion is presented to propagate relative trajectories in the Frenet axes. Using a southern L_2 near-rectilinear halo orbit as an example, a set of states along equal intervals of arclength are sampled as target spacecraft state vectors. Next, a set of relative displacements for a chaser spacecraft are populated to sufficiently span the design space around each target state. Relative departure and approach trajectories are generated by propagating these displacements forward and backward in time. These relative trajectories are concisely summarized by collecting state samples at maximum curvature events and assembling them into feature vectors. A clustering procedure is used to identify groups of departing and approaching trajectories with similar geometric characteristics for each individual target state, which are then represented by extracting a motion primitive for each cluster. These local clustering results are then aggregated to identify global groups of similar trajectories along the entire periodic orbit.

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