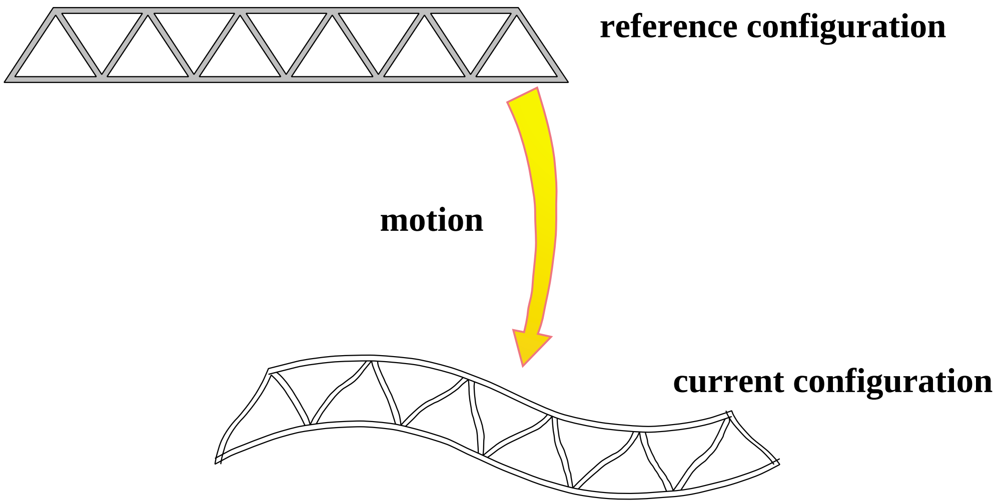
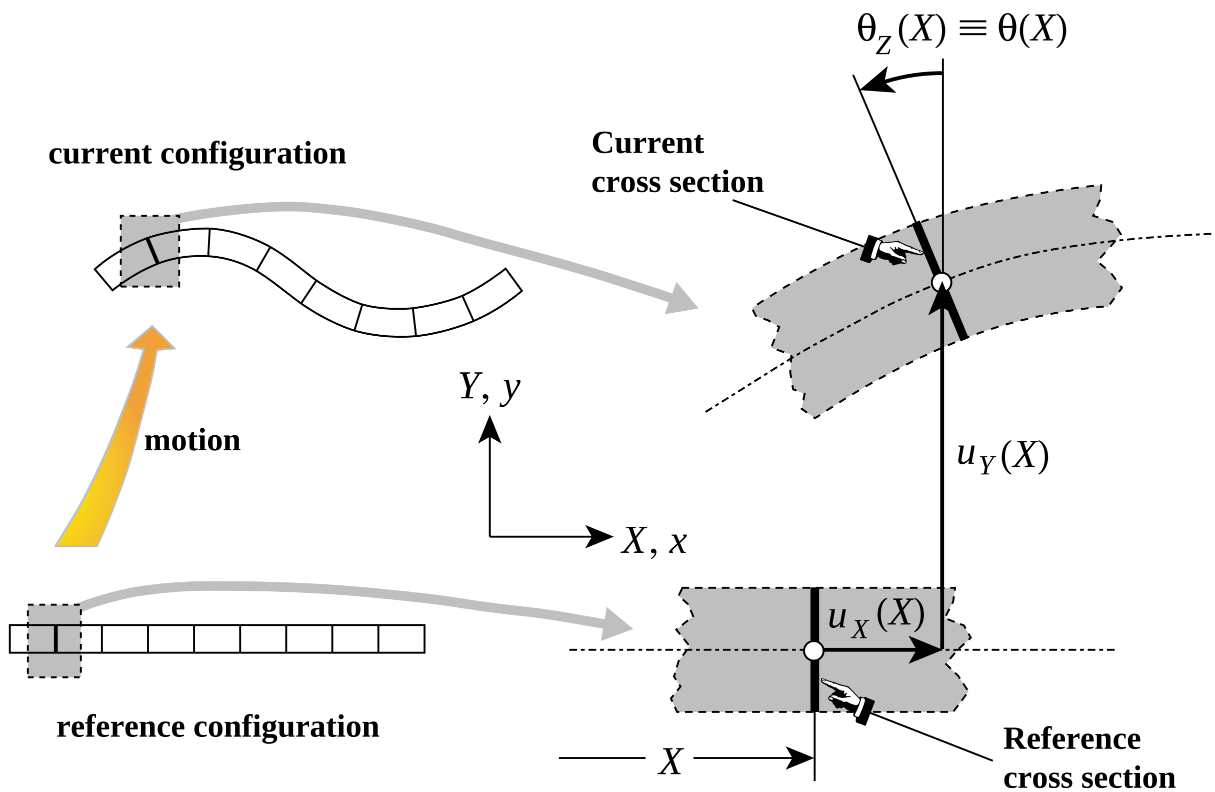
11

The TL Timoshenko Beam Element: Formulation

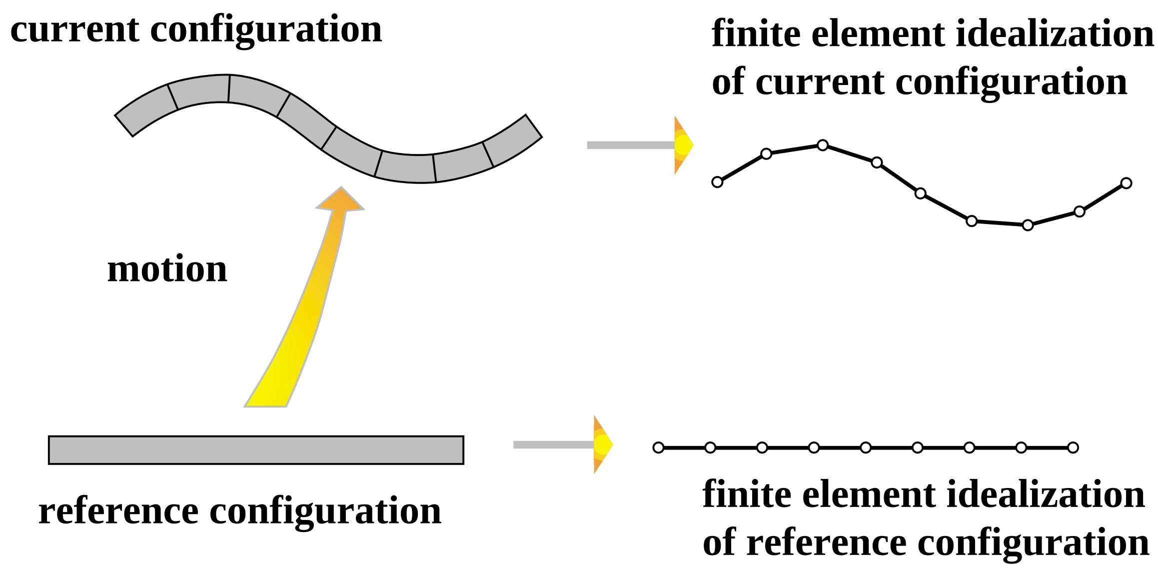
TL Beam Element Application



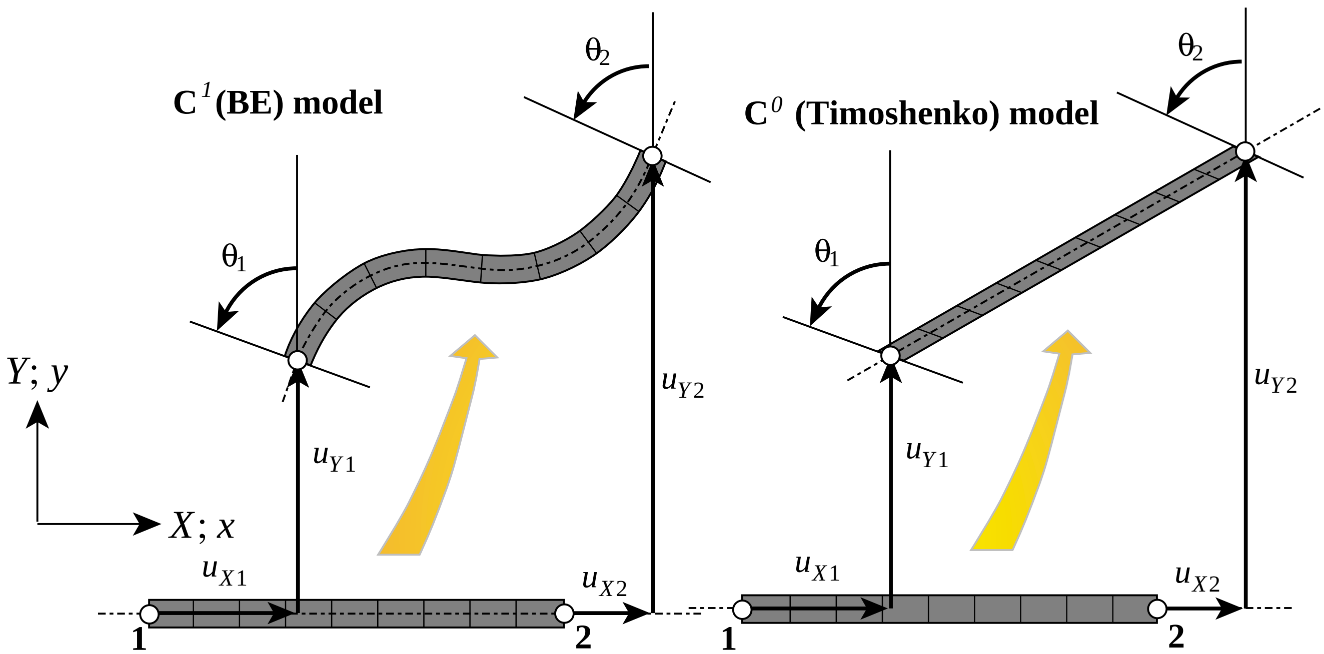
Beam Terminology



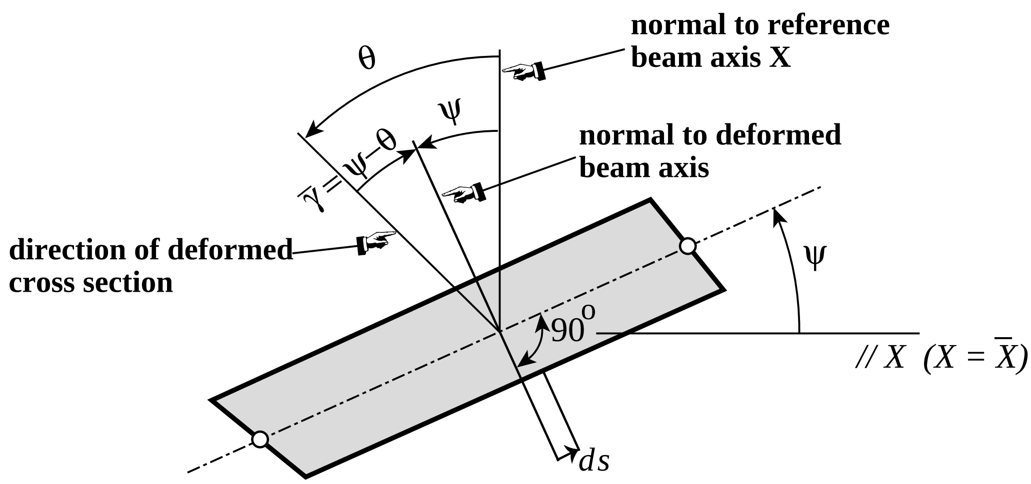
Reduction To One Dimensional Model



Beam Mathematical Models

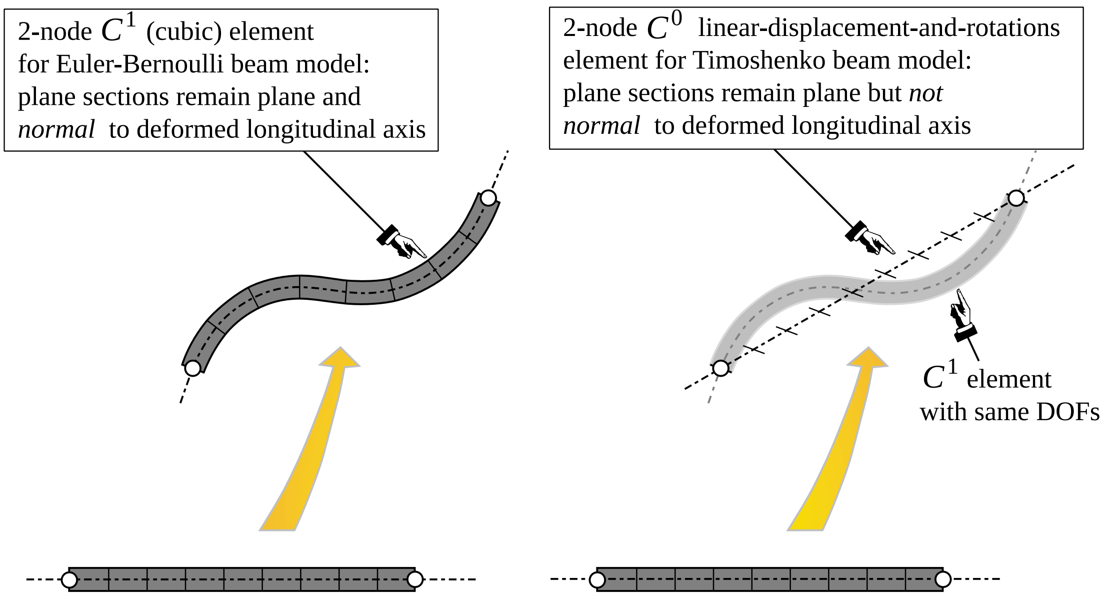


Timoshenko Beam Kinematics

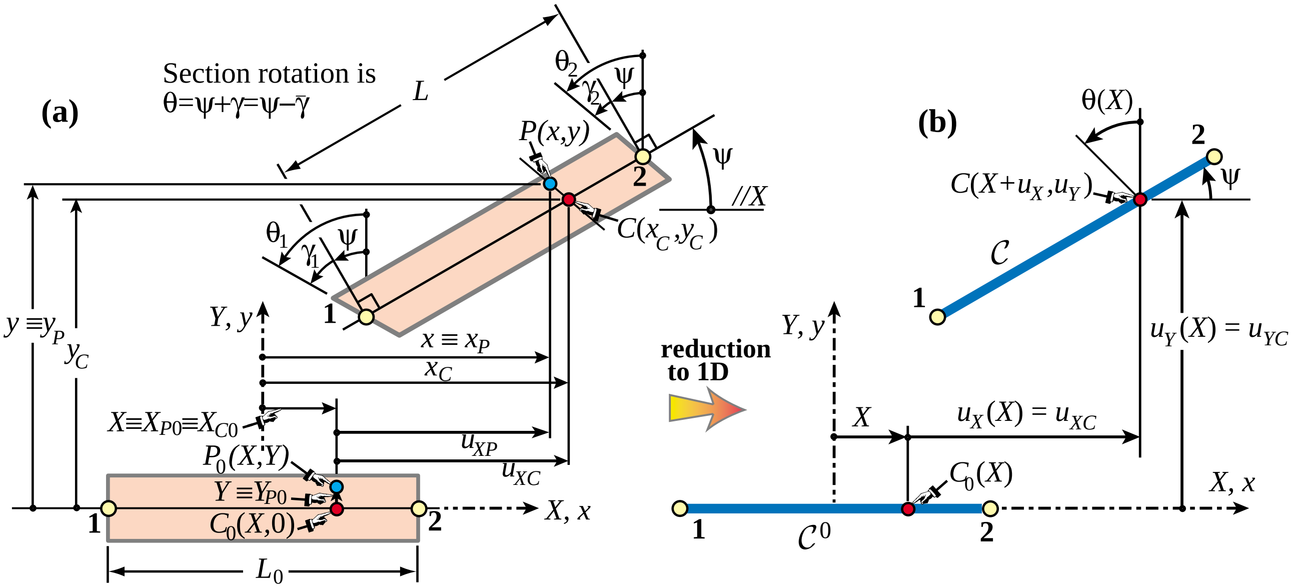


Note: in practice $\bar{\gamma} << \theta$**; typically** 0.1% **or less. Magnitude of** $\bar{\gamma}$ **is grossly exaggerated in the figure for visualization convenience.**

Comparing The Two Models Over Individual Finite Element



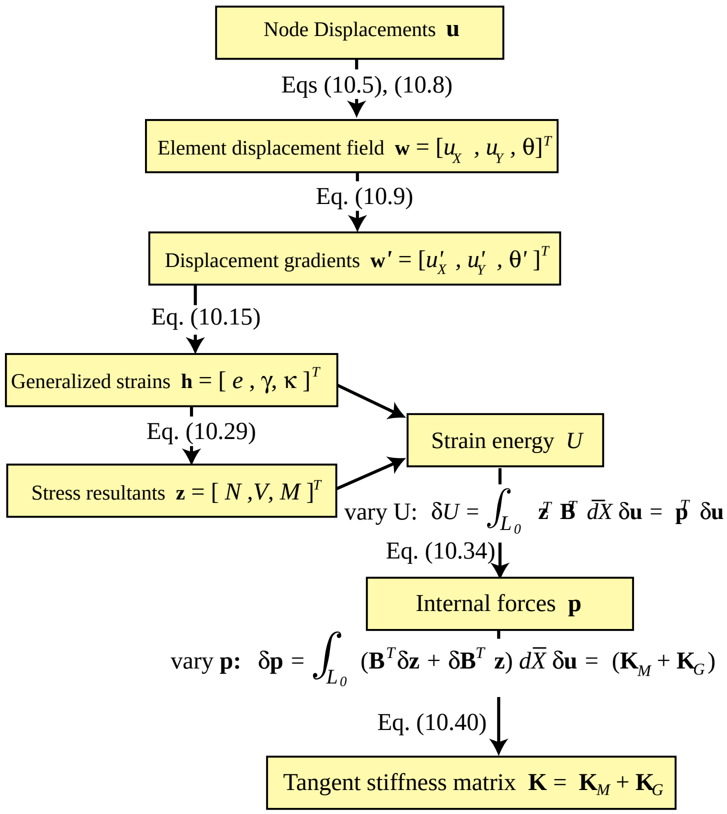
Element Kinematics: Reduction from 2D to 1D



Degrees of Freedom of Beam Element (they are model independent)

$$\mathbf{u} = \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_{X1} \\ f_{Y1} \\ f_{\theta 1} \\ f_{X2} \\ f_{Y2} \\ f_{\theta 2} \end{bmatrix}$$

Element Formulation Path



The Longest Trip Begins With The First Step

Element motion in Lagrangian description

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X + u_X - Y \sin\theta \\ u_Y + Y \cos\theta \end{bmatrix}$$

Extended displacement vector

$$\mathbf{w} = \begin{bmatrix} u_X(X) \\ u_Y(X) \\ \theta(X) \end{bmatrix}, \qquad \mathbf{w}' = \frac{d\mathbf{w}}{dX} = \begin{bmatrix} du_X/dX \\ du_Y/dX \\ d\theta/dX \end{bmatrix} = \begin{bmatrix} u'_X \\ u'_Y \\ \theta' \end{bmatrix}$$

Isoparametric Interpolation Of Extended Displacements

$$\mathbf{w} = \begin{bmatrix} u_X(X) \\ u_Y(X) \\ \theta(X) \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} = \mathbf{N}\,\mathbf{u},$$

$$\mathbf{w}' = \begin{bmatrix} u'_X \\ u'_Y \\ \theta' \end{bmatrix} = \frac{1}{L_0} \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} = \mathbf{N}'\,\mathbf{u} = \mathbf{G}\,\mathbf{u}.$$

Deformation Gradient and Displacement Gradient Tensors

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 + u'_X - Y\,\kappa\cos\theta & -\sin\theta \\ u'_Y - Y\,\kappa\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{G}_F = \mathbf{F} - \mathbf{I} = \begin{bmatrix} u'_X - Y\kappa\cos\theta & -\sin\theta \\ u'_Y - Y\kappa\sin\theta & \cos\theta - 1 \end{bmatrix}$$

Green-Lagrange Strains

Tensor form:

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{XX} & e_{XY} \\ e_{YX} & e_{YY} \end{bmatrix} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{G}_F + \mathbf{G}_F^T) + \tfrac{1}{2}\mathbf{G}_F^T\mathbf{G}_F$$

Tensor component form:

$$\underline{\mathbf{e}} = \tfrac{1}{2}\begin{bmatrix} 2(u'_X - Y\kappa\cos\theta) + (u'_X - Y\kappa\cos\theta)^2 + (u'_Y - Y\kappa\sin\theta)^2 & -(1+u'_X)\sin\theta + u'_Y\cos\theta \\ -(1+u'_X)\sin\theta + u'_Y\cos\theta & 0 \end{bmatrix}$$

These are geometrically exact, but unwieldy.
Simplifications are necessary - those are detailed in Chapter 10.

Simplified Strain Field

Axial strain is linearized with respect to displacements using a technique called polar decomposition. The result is

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{XX} \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} (1+u'_X)\cos\theta + u'_Y \sin\theta - Y\theta' - 1 \\ -(1+u'_X)\sin\theta + u'_Y \cos\theta \end{bmatrix} = \begin{bmatrix} e - Y\kappa \\ \gamma \end{bmatrix}$$

We can characterize axial strains, shear strains and bending strains

$$e = (1+u'_X)\cos\theta + u'_Y \sin\theta - 1 \qquad \gamma = -(1+u'_X)\sin\theta + u'_Y \cos\theta \qquad \kappa = \theta'$$

Collected in the generalized strain vector

$$\mathbf{h} = \begin{bmatrix} e \\ \gamma \\ \kappa \end{bmatrix}$$

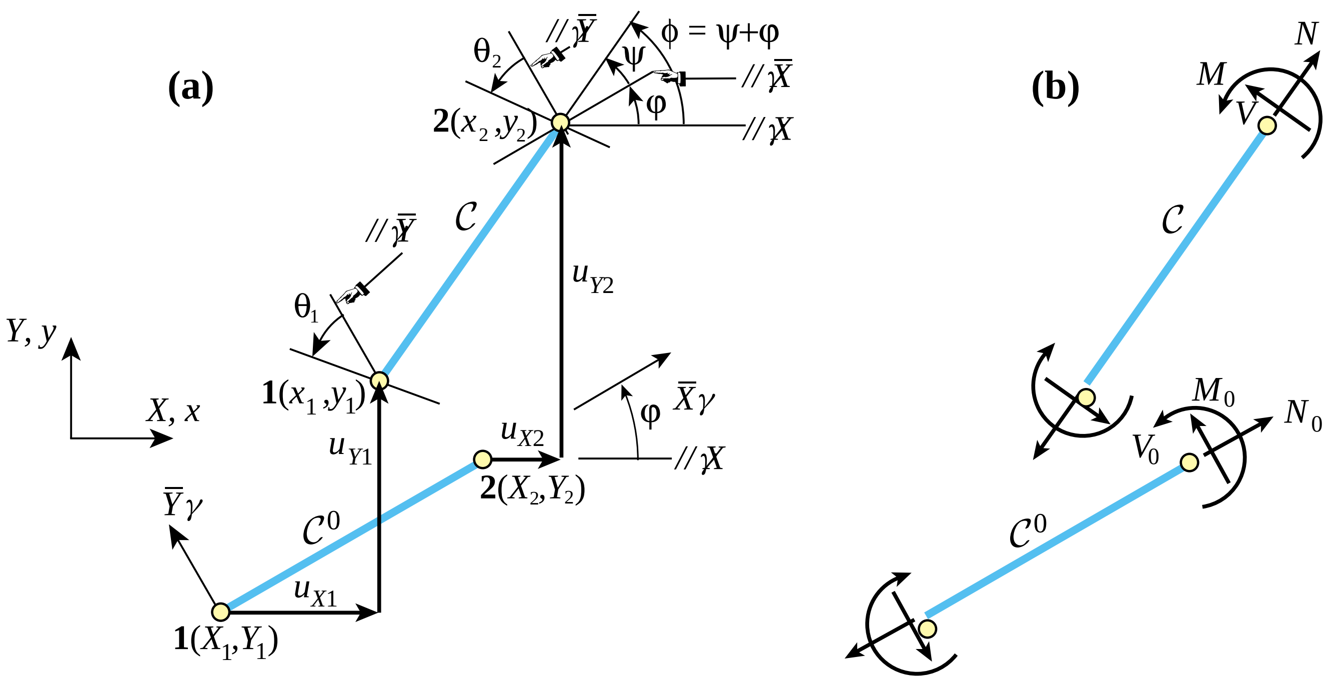
Geometrically Invariant Form of Axial and Shear Strains

$$1 + e = s' \cos(\theta - \psi) = \frac{L \cos \bar{\gamma}}{L_0}$$

$$\gamma = -s' \sin(\theta - \psi) = \frac{L \sin \bar{\gamma}}{L_0}$$

Using this invariant form we can pass to a beam element arbitrarily oriented in the reference configuration

Arbitrary Reference Configuration



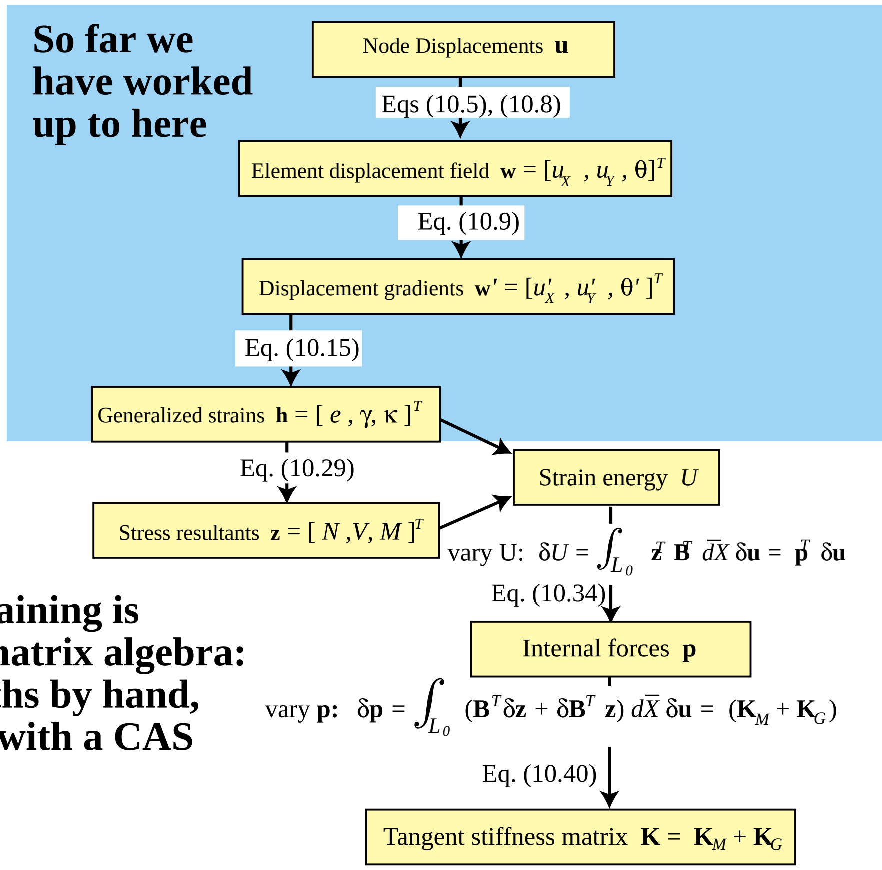
What Remains To Be Done

The remaining steps include:

PK2 stresses and stress resultants
Strain energy
Internal force
Tangent stiffness: material + geometric
Improving element performance

There is a lot of algebra left, even for this relatively simple element. A computer algebra system (CAS) can really help, especially to avoid algebra errors

Flowcharted Steps (again)



The remaining is
a lot of matrix algebra:
1-2 months by hand,
~1 week with a CAS

Energy and Internal Forces

Strain energy of beam element

$$U = \int_{L_0} (N^0 e + \tfrac{1}{2} E A_0 e^2)\, d\bar{X} + \int_{L_0} (V^0 \gamma + \tfrac{1}{2} G A_0 \gamma^2)\, d\bar{X} + \int_{L_0} (M^0 \kappa + \tfrac{1}{2} E I_0 \kappa^2)\, d\bar{X}$$

Vary to get internal force

$$\delta U = \int_{L_0} \big(N\,\delta e + V\,\delta\gamma + M\,\delta\kappa\big)\, d\bar{X} = \int_{L_0} \mathbf{z}^T\, \delta\mathbf{h}\, d\bar{X} = \int_{L_0} \mathbf{z}^T \mathbf{B}\; d\bar{X}\, \delta\mathbf{u}.$$

$$\mathbf{p} = \int_{L_0} \mathbf{B}^T\, \mathbf{z}\; d\bar{X}.$$

Evaluate by one point Gauss quadrature:

$$\mathbf{B}_m = \mathbf{B}|_{\xi=0} = \frac{1}{L_0} \begin{bmatrix} -c_m & -s_m & -\frac{1}{2} L_0 \gamma_m & c_m & s_m & -\frac{1}{2} L_0 \gamma_m \\ s_m & -c_m & \frac{1}{2} L_0 (1+e_m) & s_m & -c_m & \frac{1}{2} L_0 (1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{p} = L_0 \mathbf{B}_m^T \mathbf{z} = \begin{bmatrix} -c_m & -s_m & \frac{1}{2} L_0 \gamma_m & c_m & s_m & \frac{1}{2} L_0 \gamma_m \\ s_m & -c_m & -\frac{1}{2} L_0 (1+e_m) & -s_m & c_m & -\frac{1}{2} L_0 (1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} N \\ V \\ M \end{bmatrix}$$

Tangent Stiffness: Material Component

Vary internal force to get tangent stiffness

$$\delta\,\mathbf{p} = \int_{L_0} \left(\mathbf{B}^T\,\delta\mathbf{z} + \delta\mathbf{B}^T\,\mathbf{z}\right) d\bar{X} = (\mathbf{K}_M + \mathbf{K}_G)\,\delta\mathbf{u} = \mathbf{K}\,\delta\mathbf{u}.$$

Material stiffness matrix

$$\mathbf{K}_M = \int_{L_0} \mathbf{B}^T\mathbf{S}\mathbf{B}\,d\bar{X} \qquad \delta\mathbf{z} = \begin{bmatrix} \delta N \\ \delta V \\ \delta M \end{bmatrix} = \begin{bmatrix} EA_0 & 0 & 0 \\ 0 & GA_0 & 0 \\ 0 & 0 & EI_0 \end{bmatrix} \begin{bmatrix} \delta e \\ \delta\gamma \\ \delta\kappa \end{bmatrix} = \mathbf{S}\,\delta\mathbf{h}$$

Evaluate by one point Gauss quadrature:

$$\mathbf{K}_M = \int_{L_0} \mathbf{B}_m^T\mathbf{S}\mathbf{B}_m\,d\bar{X} = \mathbf{K}_M^a + \mathbf{K}_M^b + \mathbf{K}_M^s$$

Material Stiffness Subcomponents

Material stiffness subcomponents due to axial, bending and shear:

$$\mathbf{K}_M^a = \frac{E A_0}{L_0} \begin{bmatrix} c_m^2 & c_m s_m & -c_m \gamma_m L_0/2 & -c_m^2 & -c_m s_m & -c_m \gamma_m L_0/2 \\ c_m s_m & s_m^2 & -\gamma_m L_0 s_m/2 & -c_m s_m & -s_m^2 & -\gamma_m L_0 s_m/2 \\ -c_m \gamma_m L_0/2 & -\gamma_m L_0 s_m/2 & \gamma_m^2 L_0^2/4 & c_m \gamma_m L_0/2 & \gamma_m L_0 s_m/2 & \gamma_m^2 L_0^2/4 \\ -c_m^2 & -c_m s_m & c_m \gamma_m L_0/2 & c_m^2 & c_m s_m & c_m \gamma_m L_0/2 \\ -c_m s_m & -s_m^2 & \gamma_m L_0 s_m/2 & c_m s_m & s_m^2 & \gamma_m L_0 s_m/2 \\ -c_m \gamma_m L_0/2 & -\gamma_m L_0 s_m/2 & \gamma_m^2 L_0^2/4 & c_m \gamma_m L_0/2 & \gamma_m L_0 s_m/2 & \gamma_m^2 L_0^2/4 \end{bmatrix}$$

$$\mathbf{K}_M^b = \frac{E I_0}{L_0} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}_M^s = \frac{G A_0}{L_0} \begin{bmatrix} s_m^2 & -c_m s_m & -a_1 L_0 s_m/2 & -s_m^2 & c_m s_m & -a_1 L_0 s_m/2 \\ -c_m s_m & c_m^2 & c_m a_1 L_0/2 & c_m s_m & -c_m^2 & c_m a_1 L_0/2 \\ -a_1 L_0 s_m/2 & c_m a_1 L_0/2 & a_1^2 L_0^2/4 & a_1 L_0 s_m/2 & -c_m a_1 L_0/2 & a_1^2 L_0^2/4 \\ -s_m^2 & c_m s_m & a_1 L_0 s_m/2 & s_m^2 & -c_m s_m & a_1 L_0 s_m/2 \\ c_m s_m & -c_m^2 & -c_m a_1 L_0/2 & -c_m s_m & c_m^2 & -c_m a_1 L_0/2 \\ -a_1 L_0 s_m/2 & c_m a_1 L_0/2 & a_1^2 L_0^2/4 & a_1 L_0 s_m/2 & -c_m a_1 L_0/2 & a_1^2 L_0^2/4 \end{bmatrix}$$

in which $a_1 = 1 + e_m$.

Material Stiffness "Unlocking"

Eliminating "shear locking" by MacNeal's Residual Bending Flexibility (RBF) device:

GA_0 **is formally replaced by** $12EI_0/L_0^2$ **in shear material stiffness**

Bending and shear components merge into a modified bending stiffness:

$$\hat{\mathbf{K}}_M^b = \frac{EI}{L_0^3}\begin{bmatrix} 12s_m^2 & -12c_m s_m & 6a_1 L_0 s_m & -12s_m^2 & 12c_m s_m & 6a_1 L_0 s_m \\ -12c_m s_m & 12c_m^2 & -6c_m a_1 L_0 & 12c_m s_m & -12c_m^2 & -6c_m a_1 L_0 \\ 6a_1 L_0 s_m & -6c_m a_1 L_0 & a_2 L_0^2 & -6a_1 L_0 s_m & 6c_m a_1 L_0 & a_3 L_0^2 \\ -12s_m^2 & 12c_m s_m & -6a_1 L_0 s_m & 12s_m^2 & -12c_m s_m & -6a_1 L_0 s_m \\ 12c_m s_m & -12c_m^2 & 6c_m a_1 L_0 & -12c_m s_m & 12c_m^2 & 6c_m a_1 L_0 \\ 6a_1 L_0 s_m & -6c_m a_1 L_0 & a_3 L_0^2 & -6a_1 L_0 s_m & 6c_m a_1 L_0 & a_2 L_0^2 \end{bmatrix}$$

in which $a_1 = 1 + e_m\,, a_2 = 4 + 6e_m + 3e_m^2$ **and** $a_3 = 2 + 6e_m + 3e_m^2\,.$

Geometric Stiffness

The geometric stiffness comes from the variation of strain-displacement matrix B while stress resultants (axial forces, shear forces and bending moments) are kept fixed:

$$\mathbf{K}_G = \int_{L_0} \delta \mathbf{B}^T \, \mathbf{z} \; d\bar{X}$$

After tons of algebra (see Notes) one arrives at

$$\mathbf{K}_G = \int_{L_0} (\mathbf{W}_N N + \mathbf{W}_V V) \, d\bar{X} = \mathbf{K}_{GN} + \mathbf{K}_{GV}$$

which evaluated by one point Gauss rule gives the closed form

$$\mathbf{K}_G = \frac{N}{2} \begin{bmatrix} 0 & 0 & s_m & 0 & 0 & s_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ 0 & 0 & c_m & 0 & 0 & c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \end{bmatrix}$$

$$+ \frac{V}{2} \begin{bmatrix} 0 & 0 & c_m & 0 & 0 & c_m \\ 0 & 0 & s_m & 0 & 0 & s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \end{bmatrix}$$