D

Linear Algebra: Determinants, Inverses, Rank

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§D.1. Introduction

This Chapter discusses more specialized properties of matrices, such as determinants, inverses and rank. These apply only to *square* matrices unless extension to rectangular matrices is explicitly stated.

§D.2. Determinants

The *determinant* of a *square* matrix $\mathbf{A} = [a_{ij}]$ is a number denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, through which important properties such as singularity can be briefly characterized. This number is defined as the following function of the matrix elements:

$$|\mathbf{A}| = \det(\mathbf{A}) = \pm \prod a_{1j_1}\, a_{2j_2}\, \ldots\, a_{n\, j_n}, \tag{D.1}$$

where the column indices $j_1,\ j_2,\ \ldots\ j_n$ are taken from the set $\{1, 2, \ldots n\}$, with no repetitions allowed. The plus (minus) sign is taken if the permutation $(j_1\ j_2\ \ldots\ j_n)$ is even (odd).

Example D.1. For a 2×2 matrix,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{D.2}$$

Example D.2. For a 3×3 matrix,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \tag{D.3}$$

Remark D.1. The concept of determinant is not applicable to rectangular matrices or to vectors. Thus the notation $|\mathbf{x}|$ for a vector $\mathbf{x}$ can be reserved for its magnitude (as in Appendix A) without risk of confusion.

Remark D.2. Inasmuch as the product (D.1) contains $n!$ terms, the calculation of $|\mathbf{A}|$ from the definition is impractical for general matrices whose order exceeds 3 or 4. For example, if $n = 10$, the product (D.1) contains $10! = 3,628,800$ terms, each involving 9 multiplications, so over 30 million floating-point operations would be required to evaluate $|\mathbf{A}|$ according to that definition. A more practical method based on matrix decomposition is described in Remark D.3.

§D.2.1. Some Properties of Determinants

Some useful rules associated with the calculus of determinants are listed next.

I. Rows and columns can be interchanged without affecting the value of a determinant. Consequently

$$|\mathbf{A}| = |\mathbf{A}^T|. \tag{D.4}$$

II. If two rows, or two columns, are interchanged the sign of the determinant is reversed. For example:

$$\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}. \tag{D.5}$$

III. If a row (or column) is changed by adding to or subtracting from its elements the corresponding elements of any other row (or column) the determinant remains unaltered. For example:

$$\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = \begin{vmatrix} 3+1 & 4-2 \\ 1 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 1 & -2 \end{vmatrix} = -10. \tag{D.6}$$

IV. If the elements in any row (or column) have a common factor α then the determinant equals the determinant of the corresponding matrix in which $\alpha = 1$, multiplied by α. For example:

$$\begin{vmatrix} 6 & 8 \\ 1 & -2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 2 \times (-10) = -20. \tag{D.7}$$

V. When at least one row (or column) of a matrix is a linear combination of the other rows (or columns) the determinant is zero. Conversely, if the determinant is zero, then at least one row and one column are linearly dependent on the other rows and columns, respectively. For example, consider

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix}. \tag{D.8}$$

This determinant is zero because the first column is a linear combination of the second and third columns:

$$\text{column } 1 = \text{column } 2 + \text{column } 3. \tag{D.9}$$

Similarly, there is a linear dependence between the rows which is given by the relation

$$\text{row } 1 = \tfrac{7}{8} \text{ row } 2 + \tfrac{4}{5} \text{ row } 3. \tag{D.10}$$

VI. The determinant of an upper triangular or lower triangular matrix is the product of the main diagonal entries. For example,

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{vmatrix} = 3 \times 2 \times 4 = 24. \tag{D.11}$$

This rule is easily verified from the definition (D.1) because all terms vanish except $j_1 = 1$, $j_2 = 2, \ldots\, j_n = n$, which is the product of the main diagonal entries. Diagonal matrices are a particular case of this rule.

VII. The determinant of the product of two square matrices is the product of the individual determinants:

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|. \tag{D.12}$$

The proof requires the concept of triangular decomposition, which is covered in the Remark below. This rule can be generalized to any number of factors. One immediate application is to matrix powers: $|\mathbf{A}^2| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2$, and more generally $|\mathbf{A}^n| = |\mathbf{A}|^n$ for integer n.

VIII. The determinant of the transpose of a matrix is the same as that of the original matrix:

$$|\mathbf{A}^T| = |\mathbf{A}|. \tag{D.13}$$

This rule can be directly verified from the definition of determinant, and also as direct consequence of Rule I.

Remark D.3. Rules VI and VII are the key to the practical evaluation of determinants. Any square nonsingular matrix $\mathbf{A}$ (where the qualifier "nonsingular" is explained in §D.3) can be decomposed as the product of two triangular factors

$$\mathbf{A} = \mathbf{L}\mathbf{U}, \tag{D.14}$$

in which $\mathbf{L}$ is unit lower triangular and $\mathbf{U}$ is upper triangular. This is called a LU triangularization, LU factorization or LU decomposition. It can be carried out in $O(n^3)$ floating point operations. According to rule VII:

$$|\mathbf{A}| = |\mathbf{L}|\,|\mathbf{U}|. \tag{D.15}$$

According to rule VI, $|\mathbf{L}| = 1$ and $|\mathbf{U}| = u_{11}u_{22}\ldots u_{nn}$. The last operation requires only $O(n)$ operations. Thus the evaluation of $|\mathbf{A}|$ is dominated by the effort involved in computing the factorization (D.14). For $n = 10$, that effort is approximately $10^3 = 1000$ floating-point operations, compared to approximately 3×10^7 from the naive application of the definition (D.1), as noted in Remark D.2. Thus the LU-based method is roughly $30,000$ times faster for that modest matrix order, and the ratio increases exponentially for large n.

§D.2.2. Cramer's Rule

Cramer's rule provides a recipe for solving linear algebraic equations directly in terms of determinants. Let the simultaneous equations be as usual denoted as

$$\mathbf{A}\,\mathbf{x} = \mathbf{y}, \tag{D.16}$$

in which $\mathbf{A}$ is a given $n \times n$ matrix, $\mathbf{y}$ is a given $n \times 1$ vector, and $\mathbf{x}$ is the $n \times 1$ vector of unknowns. The explicit form of (D.16) is Equation (A.1) of Appendix A, with $n = m$.

The explicit solution for the components x_1, $x_2 \ldots$, x_n of $\mathbf{x}$ in terms of determinants is

$$x_1 = \frac{\begin{vmatrix} y_1 & a_{12} & a_{13} & \ldots & a_{1n} \\ y_2 & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ y_n & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix}}{|\mathbf{A}|}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & y_1 & a_{13} & \ldots & a_{1n} \\ a_{21} & y_2 & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & y_n & a_{n3} & \ldots & a_{nn} \end{vmatrix}}{|\mathbf{A}|}, \quad \ldots \tag{D.17}$$

The rule can be remembered as follows: in the numerator of the quotient for x_j, replace the j^{th} column of $\mathbf{A}$ by the right-hand side $\mathbf{y}$.

This method of solving simultaneous equations is known as *Cramer's rule*. Because the explicit computation of determinants is impractical for $n > 3$ as explained in Remark C.3, direct use of the rule has practical value only for $n = 2$ and $n = 3$ (it is marginal for $n = 4$). But such small-order systems arise often in finite element calculations at the *Gauss point level*; consequently implementors should be aware of this rule for such applications.

Example D.3. Solve the 3×3 linear system

$$\begin{bmatrix} 5 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}, \tag{D.18}$$

by Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 8 & 2 & 1 \\ 5 & 2 & 0 \\ 3 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{6}{6} = 1, \quad x_2 = \frac{\begin{vmatrix} 5 & 8 & 1 \\ 3 & 5 & 0 \\ 1 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{6}{6} = 1, \quad x_3 = \frac{\begin{vmatrix} 5 & 2 & 8 \\ 3 & 2 & 5 \\ 1 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{6}{6} = 1. \tag{D.19}$$

Example D.4. Solve the 2×2 linear algebraic system

$$\begin{bmatrix} 2+\beta & -\beta \\ -\beta & 1+\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \tag{D.20}$$

by Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 5 & -\beta \\ 0 & 1+\beta \end{vmatrix}}{\begin{vmatrix} 2+\beta & -\beta \\ -\beta & 1+\beta \end{vmatrix}} = \frac{5+5\beta}{2+3\beta}, \quad x_2 = \frac{\begin{vmatrix} 2+\beta & 5 \\ -\beta & 0 \end{vmatrix}}{\begin{vmatrix} 2+\beta & -\beta \\ -\beta & 1+\beta \end{vmatrix}} = \frac{5\beta}{2+3\beta}. \tag{D.21}$$

Remark D.4. Creamer's rule importance has grown in *symbolic computations* carried out by computer algebra systems. This happens when the entries of $\mathbf{A}$ and $\mathbf{y}$ are algebraic expressions. For example the example system (D.20). In such cases Cramer's rule may be competitive with factorization methods for up to moderate matrix orders, for example $n \leq 20$. The reason is that determinantal products may be simplified on the fly.

§D.2.3. Homogeneous Systems

One immediate consequence of Cramer's rule is what happens if

$$y_1 = y_2 = \ldots = y_n = 0. \tag{D.22}$$

The linear equation systems with a null right hand side

$$\mathbf{Ax} = \mathbf{0}, \tag{D.23}$$

is called a *homogeneous system*. From the rule (D.17) we see that if $|\mathbf{A}|$ is nonzero, all solution components are zero, and consequently the only possible solution is the trivial one $\mathbf{x} = \mathbf{0}$. The case in which $|\mathbf{A}|$ vanishes is discussed in the next section.

§D.3. Singular Matrices, Rank

If the determinant $|\mathbf{A}|$ of a $n \times n$ square matrix $\mathbf{A} \equiv \mathbf{A}_n$ is zero, then the matrix is said to be *singular*. This means that at least one row and one column are linearly dependent on the others. If this row and column are removed, we are left with another matrix, say $\mathbf{A}_{n-1}$, to which we can apply the same criterion. If the determinant $|\mathbf{A}_{n-1}|$ is zero, we can remove another row and column from it to get $\mathbf{A}_{n-2}$, and so on. Suppose that we eventually arrive at an $r \times r$ matrix $\mathbf{A}_r$ whose determinant is nonzero. Then matrix $\mathbf{A}$ is said to have *rank* r, and we write $\text{rank}(\mathbf{A}) = r$.

If the determinant of $\mathbf{A}$ is nonzero, then $\mathbf{A}$ is said to be *nonsingular*. The rank of a nonsingular $n \times n$ matrix is equal to n.

Obviously the rank of $\mathbf{A}^T$ is the same as that of $\mathbf{A}$ since it is only necessary to transpose "row" and "column" in the definition.

The notion of rank can be extended to rectangular matrices as outlined in section §C.2.4 below. That extension, however, is not important for the material covered here.

Example D.5. The 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \tag{D.24}$$

has rank $r = 3$ because $|\mathbf{A}| = -3 \neq 0$.

Example D.6. The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \tag{D.25}$$

already used as an example in §C.1.1 is singular because its first row and column may be expressed as linear combinations of the others through the relations (D.9) and (D.10). Removing the first row and column we are left with a 2×2 matrix whose determinant is $2 \times 3 - (-1) \times (-1) = 5 \neq 0$. Consequently (D.25) has rank $r = 2$.

§D.3.1. Rank Deficiency

If the square matrix $\mathbf{A}$ is supposed to be of rank r but in fact has a smaller rank $\bar{r} < r$, the matrix is said to be *rank deficient*. The number $r - \bar{r} > 0$ is called the *rank deficiency*.

Example D.7. Suppose that the *unconstrained* master stiffness matrix $\mathbf{K}$ of a finite element has order n, and that the element possesses b independent rigid body modes. Then the expected rank of $\mathbf{K}$ is $r = n - b$. If the actual rank is less than r, the finite element model is said to be rank-deficient. This is usually undesirable.

Example D.8. An an illustration of the foregoing rule, consider the two-node, 4-DOF, Bernoulli-Euler plane beam element stiffness derived in Chapter 12:

$$\mathbf{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ symm & & & 4L^2 \end{bmatrix}, \tag{D.26}$$

in which EI and L are nonzero scalars. It may be verified that this 4×4 matrix has rank 2. The number of rigid body modes is 2, and the expected rank is $r = 4 - 2 = 2$. Consequently this model is rank sufficient.

§D.3.2. Rank of Matrix Sums and Products

In finite element analysis matrices are often built through sum and product combinations of simpler matrices. Two important rules apply to "rank propagation" through those combinations.

The rank of the product of two square matrices $\mathbf{A}$ and $\mathbf{B}$ cannot exceed the smallest rank of the multiplicand matrices. That is, if the rank of $\mathbf{A}$ is r_a and the rank of $\mathbf{B}$ is r_b,

$$\text{rank}(\mathbf{AB}) \leq \min(r_a, r_b). \tag{D.27}$$

Regarding sums: the rank of a matrix sum cannot exceed the sum of ranks of the summand matrices. That is, if the rank of $\mathbf{A}$ is r_a and the rank of $\mathbf{B}$ is r_b,

$$\text{rank}(\mathbf{A} + \mathbf{B}) \leq r_a + r_b. \tag{D.28}$$

§D.3.3. Singular Systems: Particular and Homogeneous Solutions

Having introduced the notion of rank we can now discuss what happens to the linear system (D.16) when the determinant of $\mathbf{A}$ vanishes, meaning that its rank is less than n. If so, (D.16) has either no solution or an infinite number of solutions. Cramer's rule is of limited or no help in this situation.

To discuss this case further we note that if $|\mathbf{A}| = 0$ and the rank of $\mathbf{A}$ is $r = n - d$, where $d \geq 1$ is the *rank deficiency*, then there exist d nonzero independent vectors $\mathbf{z}_i$, $i = 1, \ldots d$ such that

$$\mathbf{A}\,\mathbf{z}_i = \mathbf{0}. \tag{D.29}$$

These d vectors, suitably orthonormalized, are called *null eigenvectors* of $\mathbf{A}$, and form a basis for its *null space*.

Let $\mathbf{Z}$ denote the $n \times d$ matrix obtained by collecting the $\mathbf{z}_i$ as columns. If $\mathbf{y}$ in (D.16) is in the *range* of $\mathbf{A}$, that is, there exists an nonzero $\mathbf{x}_p$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}_p$, its general solution is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \mathbf{x}_p + \mathbf{Z}\mathbf{w}, \tag{D.30}$$

where $\mathbf{w}$ is an arbitrary $d \times 1$ weighting vector. This statement can be easily verified by substituting this solution into $\mathbf{A}\mathbf{x} = \mathbf{y}$ and noting that $\mathbf{A}\mathbf{Z}$ vanishes.

The components $\mathbf{x}_p$ and $\mathbf{x}_h$ are called the *particular* and *homogeneous* portions respectively, of the total solution $\mathbf{x}$. (The terminology: *homogeneous solution* and *particular solution*, are often used.) If $\mathbf{y} = \mathbf{0}$ only the homogeneous portion remains.

If $\mathbf{y}$ is not in the range of $\mathbf{A}$, system (D.16) does not generally have a solution in the conventional sense, although least-square solutions can usually be constructed. The reader is referred to the many textbooks in linear algebra for further details.

§D.3.4. Rank of Rectangular Matrices

The notion of rank can be extended to rectangular matrices, real or complex, as follows. Let $\mathbf{A}$ be $m \times n$. Its *column range space* $\mathcal{R}(\mathbf{A})$ is the subspace spanned by $\mathbf{A}\mathbf{x}$ where $\mathbf{x}$ is the set of all complex n-vectors. Mathematically: $\mathcal{R}(\mathbf{A}) = \{\mathbf{A}\mathbf{x} : \mathbf{x} \in C^n\}$. The rank r of $\mathbf{A}$ is the dimension of $\mathcal{R}(\mathbf{A})$.

The *null space* $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$ is the set of n-vectors $\mathbf{z}$ such that $\mathbf{A}\mathbf{z} = \mathbf{0}$. The dimension of $\mathcal{N}(\mathbf{A})$ is $n - r$.

Using these definitions, the product and sum rules (D.27) and (D.28) generalize to the case of rectangular (but conforming) $\mathbf{A}$ and $\mathbf{B}$. So does the treatment of linear equation systems $\mathbf{A}\mathbf{x} = \mathbf{y}$ in which $\mathbf{A}$ is rectangular. Such systems often arise in the fitting of observation and measurement data.

In finite element methods, rectangular matrices appear in change of basis through congruential transformations, and in the treatment of multifreedom constraints.

§D.4. Matrix Inversion

The *inverse* of a square nonsingular matrix $\mathbf{A}$ is represented by the symbol $\mathbf{A}^{-1}$ and is defined by the relation

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}. \tag{D.31}$$

The most important application of the concept of inverse is the solution of linear systems. Suppose that, in the usual notation, we have

$$\mathbf{A}\,\mathbf{x} = \mathbf{y}. \tag{D.32}$$

Premultiplying both sides by $\mathbf{A}^{-1}$ we get the inverse relationship

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{y}. \tag{D.33}$$

More generally, consider the matrix equation for multiple (m) right-hand sides:

$$\underset{n\times n}{\mathbf{A}}\;\underset{n\times m}{\mathbf{X}} = \underset{n\times m}{\mathbf{Y}}, \tag{D.34}$$

which reduces to (D.32) for $m = 1$. The inverse relation that gives $\mathbf{X}$ as function of $\mathbf{Y}$ is

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}. \tag{D.35}$$

In particular, the solution of

$$\mathbf{A}\,\mathbf{X} = \mathbf{I}, \tag{D.36}$$

is $\mathbf{X} = \mathbf{A}^{-1}$. Practical methods for computing inverses are based on directly solving this equation; see Remark D.4.

§D.4.1. Explicit Computation of Inverses

The explicit calculation of matrix inverses is seldom needed in large matrix computations. But occasionally the need arises for the explicit inverse of small matrices that appear in element level computations. For example, the inversion of Jacobian matrices at Gauss points, or of constitutive matrices.

A general formula for elements of the inverse can be obtained by specializing Cramer's rule to (**?**). Let $\mathbf{B} = [b_{ij}] = \mathbf{A}^{-1}$. Then

$$b_{ij} = \frac{A_{ji}}{|\mathbf{A}|}, \tag{D.37}$$

in which A_{ji} denotes the so-called *adjoint* of entry a_{ij} of $\mathbf{A}$. The adjoint A_{ji} is defined as the determinant of the submatrix of order $(n-1)\times(n-1)$ obtained by deleting the j^{th} row and i^{th} column of $\mathbf{A}$, multiplied by $(-1)^{i+j}$.

This direct inversion procedure is useful only for small matrix orders, say 2 or 3. In the examples below the explicit inversion formulas for second and third order matrices are listed.

Example D.9. For order $n = 2$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}, \tag{D.38}$$

in which $|\mathbf{A}|$ is given by (D.2).

Example D.10. For order $n = 3$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{D.39}$$

where

$$\begin{aligned} b_{11} &= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, & b_{21} &= -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, & b_{31} &= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \\ b_{12} &= -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, & b_{22} &= \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, & b_{32} &= -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \\ b_{13} &= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, & b_{23} &= -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, & b_{33} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \end{aligned} \tag{D.40}$$

in which $|\mathbf{A}|$ is given by (D.3).

Example D.11.

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}^{-1} = -\frac{1}{8} \begin{bmatrix} 1 & -4 & 2 \\ -2 & 0 & 4 \\ -1 & 4 & -10 \end{bmatrix}. \tag{D.41}$$

If the order exceeds 3, the general inversion formula based on Cramer's rule becomes rapidly useless because it displays combinatorial complexity as noted in a previous Remark. For numerical work it is preferable to solve (D.36) after $\mathbf{A}$ is factored. Those techniques are described in detail in linear algebra books; see also Remark C.4.

§D.4.2. Some Properties of the Inverse

I. Assuming that $\mathbf{A}^{-1}$ exists, the The inverse of its transpose is equal to the transpose of the inverse:

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T, \tag{D.42}$$

because

$$(\mathbf{A}\mathbf{A}^{-1}) = (\mathbf{A}\mathbf{A}^{-1})^T = (\mathbf{A}^{-1})^T\, \mathbf{A}^T = \mathbf{I}. \tag{D.43}$$

II. The inverse of a symmetric matrix is also symmetric. Because of the previous rule, $(\mathbf{A}^T)^{-1} = \mathbf{A}^{-1} = (\mathbf{A}^{-1})^T$, hence $\mathbf{A}^{-1}$ is also symmetric.

III. The inverse of a matrix product is the reverse product of the inverses of the factors:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}. \tag{D.44}$$

This is easily verified by substituting both sides of (D.39) into (D.31). This property generalizes to an arbitrary number of factors.

IV. For a diagonal matrix $\mathbf{D}$ in which all diagonal entries are nonzero, $\mathbf{D}^{-1}$ is again a diagonal matrix with entries $1/d_{ii}$. The verification is straightforward.

V. If $\mathbf{S}$ is a block diagonal matrix:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{22} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{33} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{S}_{nn} \end{bmatrix} = \mathrm{diag}\,[\,\mathbf{S}_{ii}\,], \tag{D.45}$$

then the inverse matrix is also block diagonal and is given by

$$\mathbf{S}^{-1} = \begin{bmatrix} \mathbf{S}_{11}^{-1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{22}^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_{33}^{-1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{S}_{nn}^{-1} \end{bmatrix} = \mathrm{diag}\,[\,\mathbf{S}_{ii}^{-1}\,]. \tag{D.46}$$

VI. The inverse of an upper triangular matrix is also an upper triangular matrix. The inverse of a lower triangular matrix is also a lower triangular matrix. Both inverses can be computed in $O(n^2)$ floating-point operations.

Remark D.5. The practical numerical calculation of inverses is based on triangular factorization. Given a nonsingular $n \times n$ matrix $\mathbf{A}$, calculate its LU factorization $\mathbf{A} = \mathbf{LU}$, which can be obtained in $O(n^3)$ operations. Then solve the linear triangular systems:

$$\mathbf{UY} = \mathbf{I}, \qquad \mathbf{LX} = \mathbf{Y}, \tag{D.47}$$

and the computed inverse $\mathbf{A}^{-1}$ appears in $\mathbf{X}$. One can overwrite $\mathbf{I}$ with $\mathbf{Y}$ and $\mathbf{Y}$ with $\mathbf{X}$. The whole process can be completed in $O(n^3)$ floating-point operations. For symmetric matrices the alternative decomposition $\mathbf{A} = \mathbf{LDL}^T$, where $\mathbf{L}$ is unit lower triangular and $\mathbf{D}$ is diagonal, is generally preferred to save computing time and storage.

§D.5. The Inverse of a Sum of Matrices

The formula for the inverse of a matrix product: $(\mathbf{A}\,\mathbf{B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$ is not too different from its scalar counterpart: $(ab)^{-1} = (1/a)(1/b) = (1/b)(1/a)$, except that factor order matters. On the other hand, formulas for matrix sum inverses in terms of the summands are considerably more involved, and there are many variants. We consider here the expression of $(\mathbf{A}+\mathbf{B})^{-1}$ where both $\mathbf{A}$ and $\mathbf{B}$ are square and $\mathbf{A}$ is nonsingular. We begin from the identity introduced by Henderson and Searle in their review article [349]:

$$(\mathbf{I}+\mathbf{P})^{-1} = (\mathbf{I}+\mathbf{P})^{-1}\,(\mathbf{I}+\mathbf{P}-\mathbf{P}) = \mathbf{I} - (\mathbf{I}+\mathbf{P})^{-1}\,\mathbf{P}, \tag{D.48}$$

in which $\mathbf{P}$ is a square matrix. Using (D.48) we may develop $(\mathbf{A}+\mathbf{B})^{-1}$ as follows:

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^{-1} &= \left(\mathbf{A}(\mathbf{I}+\mathbf{A}^{-1}\,\mathbf{B})^{-1}\right)^{-1} = (\mathbf{I}+\mathbf{A}^{-1}\,\mathbf{B})^{-1}\,\mathbf{A}^{-1}\mathbf{B} \\ &= \left(\mathbf{I} - (\mathbf{I}+\mathbf{A}^{-1}\,\mathbf{B})^{-1}\mathbf{A}^{-1}\right)\,\mathbf{A}^{-1} = \mathbf{A}^{-1} - (\mathbf{I}+\mathbf{A}^{-1}\,\mathbf{B})^{-1}\,\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{A}^{-1}. \end{aligned} \tag{D.49}$$

Here $\mathbf{B}$ may be singular. If $\mathbf{B} = \mathbf{0}$, it reduces to $\mathbf{A}^{-1} = \mathbf{A}^{-1}$. The check $\mathbf{B} = \beta\,\mathbf{A}$ also works. The last expression in (D.49) may be further transformed by matrix manipulations as

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^{-1} &= \mathbf{A}^{-1} - (\mathbf{I}+\mathbf{A}^{-1}\,\mathbf{B})^{-1}\,\mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{A}^{-1}\\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1}\,(\mathbf{I}+\mathbf{B}\,\mathbf{A}^{-1})^{-1}\,\mathbf{B}\,\mathbf{A}^{-1}\\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\,(\mathbf{I}+\mathbf{A}^{-1}\mathbf{B})^{-1}\,\mathbf{A}^{-1}\\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\,\mathbf{A}^{-1}(\mathbf{I}+\mathbf{B}\,\mathbf{A}^{-1})^{-1}.\end{aligned} \tag{D.50}$$

In all of these forms $\mathbf{B}$ may be singular (or null). If $\mathbf{B}$ is also invertible, the third expresion in (D.50) may be transformed to the a commonly used variant

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\,\mathbf{A}^{-1}. \tag{D.51}$$

The case of singular $\mathbf{A}$ may be handled using the notion of generalized inverses. This is a topic beyond the scope of this course, which may be studied, e.g., in the textbooks [78,121,617]. The special case of $\mathbf{B}$ being of low rank merges with the Sherman-Morrison and Woodbury formulas, covered below.

§D.6. The Sherman-Morrison and Related Formulas

The Sherman-Morrison formula gives the inverse of a matrix modified by a rank-one matrix. The Woodbury formula extends the Sherman-Morrison formula to a modification of arbitrary rank. In structural analysis these formulas are of interest for problems of *structural modifications*, in which a finite-element (or, in general, a discrete model) is changed by an amount expressable as a low-rank correction to the original model.

§D.6.1. The Sherman-Morrison Formula

Let $\mathbf{A}$ be a square $n \times n$ invertible matrix, whereas $\mathbf{u}$ and $\mathbf{v}$ are two n-vectors and β an arbitrary scalar. Assume that $\sigma = 1 + \beta\,\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$. Then

$$\left(\mathbf{A}+\beta\,\mathbf{u}\,\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\beta}{\sigma}\,\mathbf{A}^{-1}\,\mathbf{u}\,\mathbf{v}^T\,\mathbf{A}^{-1}. \tag{D.52}$$

When $\beta = 1$ this is called the Sherman-Morrison formula after [667]. (For a history of this remarkable expression and its extensions, which are quite important in many applications such as statistics and probability, see the review paper by Henderson and Searle cited previously.) Since any rank-one correction to $\mathbf{A}$ can be written as $\beta\mathbf{u}\mathbf{v}^T$, (D.52) gives the rank-one change to its inverse. The proof is by direct multiplication, as in Exercise D.5.

For practical computation of the change one solves the linear systems $\mathbf{A}\mathbf{a} = \mathbf{u}$ and $\mathbf{A}\mathbf{b} = \mathbf{v}$ for $\mathbf{a}$ and $\mathbf{b}$, using the known $\mathbf{A}^{-1}$. Compute $\sigma = 1 + \beta\mathbf{v}^T\mathbf{a}$. If $\sigma \neq 0$, the change to $\mathbf{A}^{-1}$ is the dyadic $-(\beta/\sigma)\mathbf{a}\mathbf{b}^T$.

§D.6.2. The Woodbury Formula

Let again $\mathbf{A}$ be a square $n \times n$ invertible matrix, whereas $\mathbf{U}$ and $\mathbf{V}$ are two $n \times k$ matrices with $k \leq n$ and β an arbitrary scalar. Assume that the $k \times k$ matrix $\boldsymbol{\Sigma} = \mathbf{I}_k + \beta\,\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U}$, in which $\mathbf{I}_k$ denotes the $k \times k$ identity matrix, is invertible. Then

$$\left(\mathbf{A} + \beta\mathbf{U}\mathbf{V}^T\right)^{-1} = \mathbf{A}^{-1} - \beta\,\mathbf{A}^{-1}\mathbf{U}\boldsymbol{\Sigma}^{-1}\mathbf{V}^T\mathbf{A}^{-1}. \tag{D.53}$$

This is called the Woodbury formula, after [810]. It reduces to (D.52) if $k = 1$, in which case $\boldsymbol{\Sigma} \equiv \sigma$ is a scalar. The proof is by direct multiplication.

§D.6.3. Formulas for Modified Determinants

Let $\widetilde{\mathbf{A}}$ denote the adjoint of $\mathbf{A}$. Taking the determinants from both sides of $\mathbf{A} + \beta\mathbf{u}\mathbf{v}^T$ one obtains

$$|\,\mathbf{A} + \beta\mathbf{u}\mathbf{v}^T\,| = |\,\mathbf{A}| + \beta\,\mathbf{v}^T\,\widetilde{\mathbf{A}}\,\mathbf{u}. \tag{D.54}$$

If $\mathbf{A}$ is invertible, replacing $\widetilde{\mathbf{A}} = |\mathbf{A}|\,\mathbf{A}^{-1}$ this becomes

$$|\mathbf{A} + \beta\,\mathbf{u}\mathbf{v}^T\,| = |\mathbf{A}|\,(1 + \beta\,\mathbf{v}^T\mathbf{A}^{-1}\,\mathbf{u}). \tag{D.55}$$

Similarly, one can show that if $\mathbf{A}$ is invertible, and $\mathbf{U}$ and $\mathbf{V}$ are $n \times k$ matrices,

$$|\,\mathbf{A} + \beta\,\mathbf{U}\mathbf{V}^T\,| = |\mathbf{A}|\;|\,\mathbf{I}_k + \beta\,\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U}|. \tag{D.56}$$

Notes and Bibliography

Much of the material summarized here is available in expanded form in linear algebra textbooks. For example, Bellman [68] and Strang [699].

For inverses of matrix sums, there are two SIAM Review articles: [325,349]. For an historical account of the topic and its close relation to the Schur complement, see the bibliography in Appendix P.

Exercises for Appendix D: Determinants, Inverses, Rank

EXERCISE D.1 If $\mathbf{A}$ is a square matrix of order n and c a scalar, show that $\det(c\mathbf{A}) = c^n \det \mathbf{A}$.

EXERCISE D.2 Let $\mathbf{u}$ and $\mathbf{v}$ denote real n-vectors normalized to unit length, so that $\mathbf{u}^T\mathbf{u} = 1$ and $\mathbf{v}^T\mathbf{v} = 1$, and let $\mathbf{I}$ denote the $n \times n$ identity matrix. Show that

$$\det(\mathbf{I} - \mathbf{u}\mathbf{v}^T) = 1 - \mathbf{v}^T\mathbf{u} \tag{ED.1}$$

EXERCISE D.3 Let $\mathbf{u}$ denote a real n-vector normalized to unit length, so that $\mathbf{u}^T\mathbf{u} = 1$ and $\mathbf{I}$ denote the $n \times n$ identity matrix. Show that

$$\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T \tag{ED.2}$$

is orthogonal: $\mathbf{H}^T\mathbf{H} = \mathbf{I}$, and idempotent: $\mathbf{H}^2 = \mathbf{H}$. This matrix is called a *elementary Hermitian*, a *Householder matrix*, or a *reflector*. It is a fundamental ingredient of many linear algebra algorithms; for example the QR algorithm for finding eigenvalues.

EXERCISE D.4 The *trace* of a $n \times n$ square matrix $\mathbf{A}$, denoted $\textbf{trace}(\mathbf{A})$ is the sum $\sum_{i=1}^{n} a_{ii}$ of its diagonal coefficients. Show that if the entries of $\mathbf{A}$ are real,

$$\textbf{trace}(\mathbf{A}^T\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 \tag{ED.3}$$

EXERCISE D.5 Prove the Sherman-Morrison formula (D.53) by direct matrix multiplication.

EXERCISE D.6 Prove the Sherman-Morrison formula (D.53) for $\beta = 1$ by considering the following block bordered system

$$\begin{bmatrix} \mathbf{A} & \mathbf{U} \\ \mathbf{V}^T & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \end{bmatrix} \tag{ED.4}$$

in which $\mathbf{I}_k$ and $\mathbf{I}_n$ denote the identy matrices of orders k and n, respectively. Solve (D.56) two ways: eliminating first $\mathbf{B}$ and then $\mathbf{C}$, and eliminating first $\mathbf{C}$ and then $\mathbf{B}$. Equate the results for $\mathbf{B}$.

EXERCISE D.7 Show that the eigenvalues of a real symmetric square matrix are real, and that the eigenvectors are real vectors.

EXERCISE D.8 Let the n real eigenvalues λ_i of a real $n \times n$ symmetric matrix $\mathbf{A}$ be classified into two subsets: r eigenvalues are nonzero whereas $n - r$ are zero. Show that $\mathbf{A}$ has rank r.

EXERCISE D.9 Show that if $\mathbf{A}$ is p.d., $\mathbf{Ax} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$.

EXERCISE D.10 Show that for any real $m \times n$ matrix $\mathbf{A}$, $\mathbf{A}^T\mathbf{A}$ exists and is nonnegative.

EXERCISE D.11 Show that a triangular matrix is normal if and only if it is diagonal.

EXERCISE D.12 Let $\mathbf{A}$ be a real orthogonal matrix. Show that all of its eigenvalues λ_i, which are generally complex, have unit modulus.

EXERCISE D.13 Let $\mathbf{A}$ and $\mathbf{T}$ be real $n \times n$ matrices, with $\mathbf{T}$ nonsingular. Show that $\mathbf{T}^{-1}\mathbf{AT}$ and $\mathbf{A}$ have the same eigenvalues. (This is called a similarity transformation in linear algebra).

EXERCISE D.14 (Tough) Let $\mathbf{A}$ be $m \times n$ and $\mathbf{B}$ be $n \times m$. Show that the nonzero eigenvalues of $\mathbf{AB}$ are the same as those of $\mathbf{BA}$ (Kahan).

EXERCISE D.15 Let $\mathbf{A}$ be real skew-symmetric, that is, $\mathbf{A} = -\mathbf{A}^T$. Show that all eigenvalues of $\mathbf{A}$ are purely imaginary or zero.

EXERCISE D.16 Let $\mathbf{A}$ be real skew-symmetric, that is, $\mathbf{A} = -\mathbf{A}^T$. Show that $\mathbf{U} = (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})$, called a Cayley transformation, is orthogonal.

EXERCISE D.17 Let $\mathbf{P}$ be a real square matrix that satisfies

$$\mathbf{P}^2 = \mathbf{P}. \tag{ED.5}$$

Such matrices are called *idempotent*, and also *orthogonal projectors*. Show that all eigenvalues of $\mathbf{P}$ are either zero or one.

EXERCISE D.18 The necessary and sufficient condition for two square matrices to commute is that they have the same eigenvectors.

EXERCISE D.19 A matrix whose elements are equal on any line parallel to the main diagonal is called a Toeplitz matrix. (They arise in finite difference or finite element discretizations of regular one-dimensional grids.) Show that if $\mathbf{T}_1$ and $\mathbf{T}_2$ are any two Toeplitz matrices, they commute: $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$. Hint: do a Fourier transform to show that the eigenvectors of any Toeplitz matrix are of the form $\{e^{i\omega nh}\}$; then apply the previous Exercise.